1. Classification *INPE-COM.4/RPE*
C.D.U.: 533.9

2. Period

4. Distribution Criterion

internal [ ]

external [X]

3. Key Words (selected by the author)

WAVE PROPAGATION
WARM PLASMA

5. Report Nº *INPE-1547-RPE/067*

6. Date *August, 1979*

7. Revised by

René A. Medrano-B

9. Authorized by

Nelson de Jesus Parada
Director

10. Sector *DCE/DGA/GIO* Code

11. Nº of Copies *10*

14. Nº of Pages *57*

15. Price

8. Title and Sub-title

WAVES IN WARM PLASMAS

12. Authorship *J. A. Bittencourt*

13. Signature of the responsible Bittencourt

16. Summary/Notes

This chapter analyses the problem of wave propagation in a warm electron gas and in a warm, fully ionized plasma, considering the cases when there is no magnetic field present, as well as when the plasma is immersed in an externally applied magnetic field. The various possible modes of wave propagation are described and special emphasis is given to those aspects which are related with the pressure gradient terms in the equations of motion for each particle species. A summary of the various modes of wave propagation in a fully ionized, warm plasma, is presented in the last section of this chapter.

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WAVES IN WARM PLASMAS

1. INTRODUCTION

In the previous chapter we have analyzed the wave propagation problem in a cold plasma. Now we want to extend the theory developed in the previous chapter to include the pressure-gradient term in the momentum equation. We shall consider the cases of wave propagation in a warm electron gas (in which ion motion is ignored) and in a fully ionized warm plasma (considering only one ion species), in the absence as well as in the presence of an externally applied magnetic field.

2. WAVES IN A FULLY IONIZED ISOTROPIC WARM PLASMA

2.1 - Derivation of the equations for the electron and ion velocities

We consider now a fully ionized warm plasma having only one ion species, with no externally applied magnetic field ($\underset{\sim}{B}_0 = 0$). To analyse the problem of wave propagation in this case we start by writing down the equations of conservation of mass and of momentum for the electrons and the ions,

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \underset{\sim}{u}_\alpha) = 0 \tag{2.1}$$

$$m_\alpha \frac{D\underset{\sim}{u}_\alpha}{Dt} = q_\alpha (\underset{\sim}{E} + \underset{\sim}{u}_\alpha \times \underset{\sim}{B}) - \frac{1}{n_\alpha} \underset{\sim}{\nabla} p_\alpha - m_\alpha \nu_{\alpha\beta} (\underset{\sim}{u}_\alpha - \underset{\sim}{u}_\beta) \tag{2.2}$$

where for the electrons $\alpha = e$, $\beta = i$, and for the ions $\alpha = i$, $\beta = e$. These equations are complemented by the following adiabatic energy equation for each species ($\alpha = e, i$),

$$p_\alpha n_\alpha^{-\gamma} = \text{constant} \tag{2.3}$$

where $\gamma = 1 + 2/N$ is the adiabatic constant and N denotes the number of degrees of freedom. Applying the $\underset{\sim}{\nabla}$ operator to (2.3) and using the ideal gas law $p_\alpha = n_\alpha k_B T_\alpha$, we can rewrite (2.3) in the form

$$\underset{\sim}{\nabla} p_\alpha = \gamma k_B T_\alpha \underset{\sim}{\nabla} n_\alpha \tag{2.4}$$

We restrict our attention to *small-amplitude* waves in order to linearize the equations, and assume that

$$n_\alpha (\underset{\sim}{r}, t) = n_0 + n'_\alpha \exp(i\underset{\sim}{k}\cdot\underset{\sim}{r} - i\omega t) \qquad |n'_\alpha| \ll n_0 \tag{2.5}$$

$$\underset{\sim}{u}_\alpha (\underset{\sim}{r}, t) = \underset{\sim}{u}_\alpha \exp(i\underset{\sim}{k}\cdot\underset{\sim}{r} - i\omega t) \qquad u_\alpha \ll |\omega/k| \tag{2.6}$$

$$\underset{\sim}{E} (\underset{\sim}{r}, t) = \underset{\sim}{E} \exp(i\underset{\sim}{k}\cdot\underset{\sim}{r} - i\omega t) \tag{2.7}$$

$$\underset{\sim}{B}\ (\underset{\sim}{r},\ t) = \underset{\sim}{B} \exp\ (i\underset{\sim}{k}\cdot\underset{\sim}{r} - i\omega t) \tag{2.8}$$

Using these expressions in (2.1), and neglecting second order terms, we find

$$\frac{n_\alpha'}{n_o} = \frac{\underset{\sim}{k}\cdot\underset{\sim}{u}_\alpha}{\omega} \tag{2.9}$$

Similarly, we obtain for (2.2), after the substitution of $\underset{\sim}{\nabla} p_\alpha$ from (2.4) and linearizing,

$$- i\,\omega\,\underset{\sim}{u}_\alpha = \frac{q_\alpha}{m_\alpha}\,\underset{\sim}{E} - V_{S\alpha}^2\ i\underset{\sim}{k}\ \left(\frac{n_\alpha'}{n_o}\right) - \nu_{\alpha\beta}\ (\underset{\sim}{u}_\alpha - \underset{\sim}{u}_\beta) \tag{2.10}$$

where $V_{S\alpha} = (\gamma k_B T_\alpha / m_\alpha)^{1/2}$ is the adiabatic sound speed for the particles of type α.

Substituting (2.9) into (2.10), and multiplying by $i\omega$, we obtain the following equation involving the variables $\underset{\sim}{u}_\alpha$, $\underset{\sim}{u}_\beta$ and $\underset{\sim}{E}$,

$$\omega^2\ \underset{\sim}{u}_\alpha = i\omega\ \frac{q_\alpha}{m_\alpha}\ \underset{\sim}{E} + V_{S\alpha}^2\ \underset{\sim}{k}\ (\underset{\sim}{k}\cdot\underset{\sim}{u}_\alpha) - i\omega\ \nu_{\alpha\beta}\ (\underset{\sim}{u}_\alpha - \underset{\sim}{u}_\beta) \tag{2.11}$$

The relationship between the electric field, and the electron and ion velocities, can be obtained from Maxwell curl equations with harmonic variations of $\underset{\sim}{E}$ and $\underset{\sim}{B}$, according to (2.7) and (2.8),

$$\underset{\sim}{k} \times \underset{\sim}{E} = \omega \underset{\sim}{B} \tag{2.12}$$

$$i \underset{\sim}{k} \times \underset{\sim}{B} = \mu_0 \underset{\sim}{J} - \frac{i\omega}{c^2} \underset{\sim}{E} \tag{2.13}$$

and the linearized expression for the plasma current density,

$$\underset{\sim}{J} = n_0 \sum_\alpha q_\alpha \underset{\sim}{u}_\alpha = n_0 e (\underset{\sim}{u}_i - \underset{\sim}{u}_e). \tag{2.14}$$

Combining (2.12), (2.13) and (2.14) we find

$$\underset{\sim}{E}_\ell = \frac{ien_0}{\omega\varepsilon_0} (\underset{\sim}{u}_{e\ell} - \underset{\sim}{u}_{i\ell}) \tag{2.15}$$

$$\underset{\sim}{E}_t = \frac{ien_0}{\omega\varepsilon_0} \frac{(\underset{\sim}{u}_{et} - \underset{\sim}{u}_{it})}{(1 - k^2c^2/\omega^2)} \tag{2.16}$$

where the subscripts ℓ and t indicate components *longitudinal* and *transverse*, respectively, with respect to the direction of the wave propagation vector $\underset{\sim}{k}$ (see Fig. 1 of Chapter 16).

Substituting (2.15) and (2.16) into (2.11), and writing this equation for each type of particles (electrons and ions), we have the following set of coupled equations for the *longitudinal* components of the electron and ion velocities,

$$\underset{\sim}{u}_{e\ell}\ (\omega^2 - \omega_{pe}^2 - k^2\ V_{se}^2 + i\omega\ \nu_{ei}) + \underset{\sim}{u}_{i\ell}\ (\omega_{pe}^2 - i\omega\ \nu_{ei}) = 0 \qquad (2.17)$$

$$\underset{\sim}{u}_{e\ell}\ (\omega_{pi}^2 - i\omega\ \nu_{ie}) + \underset{\sim}{u}_{i\ell}\ (\omega^2 - \omega_{pi}^2 - k^2\ V_{si}^2 + i\omega\ \nu_{ie}) = 0 \qquad (2.18)$$

and for the *transverse* components,

$$\underset{\sim}{u}_{et} \left[\omega^2 - \frac{\omega_{pe}^2}{(1 - k^2c^2/\omega^2)} + i\omega\ \nu_{ei}\right] + \underset{\sim}{u}_{it} \left[\frac{\omega_{pe}^2}{(1 - k^2c^2/\omega^2)} - i\omega\ \nu_{ei}\right] = 0 \qquad (2.19)$$

$$\underset{\sim}{u}_{et} \left[\frac{\omega_{pi}^2}{(1 - k^2c^2/\omega^2)} - i\omega\ \nu_{ie}\right] + \underset{\sim}{u}_{it} \left[\omega^2 - \frac{\omega_{pi}^2}{(1 - k^2c^2/\omega^2)} + i\omega\ \nu_{ie}\right] = 0 \qquad (2.20)$$

Note that the effect of the pressure gradient term appears only on the longitudinal component of the motion and, consequently, the transverse modes of propagation are the same ones as in the cold plasma model, but with the motion of the ions included.

2.2 - Longitudinal waves

In what follows, in order to simplify the algebra, we shall neglect collisions ($\nu_{ei} = \nu_{ie} = 0$). In order to have *longitudinal* *waves* ($\underset{\sim}{u}_{e\ell} \neq 0$; $\underset{\sim}{u}_{i\ell} \neq 0$), the determinant of the coefficients in the system of Eqs. (2.17) and (2.18) must vanish. This condition gives

$$(\omega^2 - \omega_{pe}^2 - k^2 V_{se}^2)(\omega^2 - \omega_{pi}^2 - k^2 V_{si}^2) - \omega_{pe}^2 \omega_{pi}^2 = 0 \qquad (2.21)$$

Multiplying the terms within parenthesis, this equation can be recast into the form

$$k^4 (V_{se}^2 V_{si}^2) + k^2 \left[\omega_{pe}^2 V_{si}^2 + \omega_{pi}^2 V_{se}^2 - \omega^2 (V_{se}^2 + V_{si}^2)\right]$$

$$+ \omega^2 (\omega^2 - \omega_{pe}^2 - \omega_{pi}^2) = 0 \qquad (2.22)$$

Note that in the special case of the cold plasma model, in which the pressure gradient terms are ignored (i. e., $V_{se} = V_{si} = 0$), (2.22) gives $\omega^2 = \omega_{pe}^2 + \omega_{pi}^2$, which corresponds to the longitudinal plasma oscillations when the motion of both the electrons and the ions are taken into account. Eq. (2.22) has two roots for k^2, so that there are *two longitudinal modes* of propagation. One of these is termed the longitudinal *electron plasma wave* and the other is the longitudinal *ion plasma wave*. These plasma modes are electrostatic in character, and contain all the charge accumulation and no magnetic field, whereas the transverse electro magnetic mode contains the entire magnetic field and has no charge accumulation.

Although it is not difficult to obtain the two exact solutions for k^2 from (2.22), it is more convenient to analyze it for some special cases which emphasize the role played by the inclusion of ion motion and the pressure gradient terms.

For this purpose, let us first rewrite (2.22) for the case when ion motion is not taken into account, which becomes

$$- k^2 V_{se}^2 \omega^2 + \omega^2 (\omega^2 - \omega_{pe}^2) = 0 \tag{2.23}$$

or

$$\omega^2 = \omega_{pe}^2 + k^2 V_{se}^2 \tag{2.24}$$

Now, $V_{se}^2 = \gamma k_B T_e / m_e$, and since for a plane wave the compression is one-dimensional, we have $\gamma = 3$, so that

$$\omega^2 = \omega_{pe}^2 + (3 k_B T_e / m_e) k^2 \tag{2.25}$$

This equation is known as the *Bohm-Gross dispersion relation* for the longitudinal *electron plasma* wave. This relation shows a reflection point ($k = 0$) for $\omega = \omega_{pe}$. For very high frequencies ($\omega >> \omega_{pe}$) the phase velocity is $\omega/k = V_{se}$, which represents an electron acoustic wave.

Next, let us include the motion of the ions but under the assumption that its temperature is such that $V_{si} = 0$. Then, (2.22) simplifies to

$$k^2 V_{se}^2 (\omega_{pi}^2 - \omega^2) + \omega^2 (\omega^2 - \omega_{pe}^2 - \omega_{pi}^2) = 0 \qquad (2.26)$$

At very high frequencies ($\omega >> \omega_{pe}$) we still have $\omega/k = V_{se}$, but now (2.26) shows a reflection point ($k = 0$) at $\omega = (\omega_{pe}^2 + \omega_{pi}^2)^{1/2}$.

Finally, let us analyze (2.22) in the limits of high and low frequencies. From the definitions of ω_{pe} and V_{si}, we have

$$\omega_{pe}^2 V_{si}^2 = (\frac{T_i}{T_e}) \omega_{pi}^2 V_{se}^2 \qquad (2.27)$$

Therefore, (2.22) can be rewritten as

$$k^4 V_{se}^2 V_{si}^2 + k^2 \left[\omega_{pi}^2 V_{se}^2 (1 + \frac{T_i}{T_e}) - \omega^2 (V_{se}^2 + V_{si}^2) \right] +$$

$$+ \omega^2 (\omega^2 - \omega_{pe}^2 - \omega_{pi}^2) = 0 \qquad (2.28)$$

For *high frequencies*, such that $\omega^2 >> \omega_{pi}^2 (1 + T_i/T_e)$, (2.28) becomes

$$k^4 V_{se}^2 V_{si}^2 - k^2 \omega^2 (V_{se}^2 + V_{si}^2) + \omega^2 (\omega^2 - \omega_{pe}^2 - \omega_{pi}^2) = 0 \qquad (2.29)$$

Further, considering $V_{se}^2 \omega^2 >> V_{si}^2 (\omega_{pe}^2 + \omega_{pi}^2)$, or equivalently $\omega^2 >> \omega_{pi}^2 (T_i/T_e) (1 + m_e/m_i)$, a condition which also satisfies

$\omega^2 \gg \omega_{pi}^2 \ (1 + T_i/T_e)$, we can add the term $k^2 \ V_{si}^2 \ (\omega_{pe}^2 + \omega_{pi}^2)$ to the left-hand side of (2.29) and rearrange this equation in the following approximate form

$$(k^2 \ V_{si}^2 - \omega^2) \ (k^2 \ V_{se}^2 - \omega^2 + \omega_{pe}^2 + \omega_{pi}^2) \cong 0 \tag{2.30}$$

From this equation we see that, for high frequencies $[\omega^2 \gg \omega_{pi}^2 \ (1 + T_i/T_e)]$ the dispersion relation for the longitudinal *ion plasma wave* is

$$\omega^2 = k^2 \ V_{si}^2 \tag{2.31}$$

while, for the *electron plasma wave*, the dispersion relation is

$$\omega^2 = \omega_{pe}^2 + \omega_{pi}^2 + k^2 \ V_{se}^2 \tag{2.32}$$

Next, for *low frequencies*, such that $\omega^2 \ll \omega_{pi}^2 \ (1 + T_i/T_e)$, (2.28) becomes

$$k^4 \ V_{se}^2 \ V_{si}^2 + k^2 \ V_{se}^2 \ \omega_{pi}^2 \ (1 + \frac{T_i}{T_e}) - \omega^2 \ \omega_{pe}^2 = 0 \tag{2.33}$$

Multiplying this equation by $-\omega^2/(\omega_{pe}^2 \ k^4)$, assuming $k \neq 0$, it can be rewritten as

$$(\frac{\omega}{k})^4 - (\frac{\omega}{k})^2 \ V_{se}^2 \ \frac{\omega_{pi}^2}{\omega_{pe}^2} \ (1 + \frac{T_i}{T_e}) - V_{se}^2 \ V_{si}^2 \ \frac{\omega^2}{\omega_{pe}^2} = 0 \tag{2.34}$$

Since we are considering low frequencies, $\omega^2 << \omega_{pi}^2\ (1 + T_i/T_e)$, and as long as (ω/k) is not much larger than V_{si}, the last term in the left hand side of (2.34) can be neglected as compared to the second one. Therefore, (2.34) gives, for low frequencies,

$$(\frac{\omega}{k})^2 = V_{se}^2 \frac{\omega_{pi}^2}{\omega_{pe}^2} (1 + \frac{T_i}{T_e}) \tag{2.35}$$

Using the relation (2.27), this equation can be rewritten in the form

$$\omega^2 = k^2\ V_{sp}^2 \tag{2.36}$$

where

$$V_{sp}^2 = \gamma\ k_B\ (T_e + T_i)\ /\ m_i \tag{2.37}$$

which is known as the *plasma sound speed*. It can be verified that the other root of (2.33) gives an evanescent wave at very low frequencies.

A plot of phase velocity versus frequency for the longitudinal waves is shown in Fig. 1. The longitudinal waves with phase velocities equal to V_{se} or V_{si} at high frequencies represent, respectively, acoustic oscillations due to the electrons and due to the ions. The low frequency wave travelling at the plasma sound speed, V_{sp}, represents an acoustic oscillation of *both* the electrons and the ions. This low frequency wave is known as the *ion-acoustic wave*.

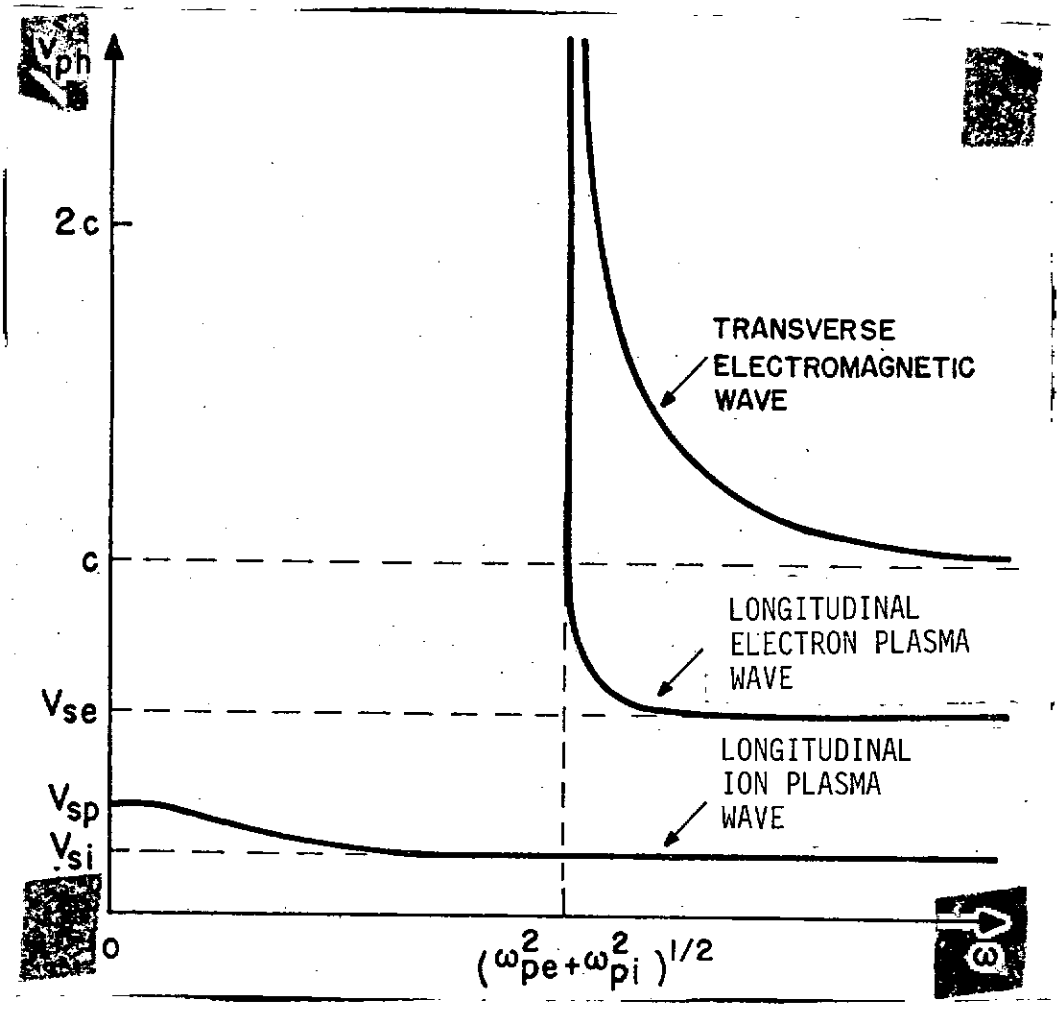

Fig. 1 - Phase velocity, as a function of frequency, for waves in a fully ionized, isotropic ($\underset{\sim}{B}_o = 0$) warm plasma. (The curves for the longitudinal waves also hold for propagation in the direction of $\underset{\sim}{B}_o$, when $\underset{\sim}{B}_o \neq 0$).

2.3 - Transverse wave

For the existence of a *transverse mode* of propagation ($\underset{\sim}{u}_{et} \neq 0$; $\underset{\sim}{u}_{it} \neq 0$) the determinant of the coefficients in the system of Eqs. (2.19) and (2.20) must vanish. Neglecting collisions ($\nu_{ei} = \nu_{ie} = 0$), we find

$$(\omega^2 - \frac{\omega_{pe}^2}{1-k^2c^2/\omega^2})\ (\omega^2 - \frac{\omega_{pi}^2}{1-k^2c^2/\omega^2}) - \frac{\omega_{pe}^2\ \omega_{pi}^2}{(1-k^2c^2/\omega^2)^2} = 0 \qquad (2.38)$$

which simplifies to

$$k^2\ c^2 = \omega^2 - (\omega_{pe}^2 + \omega_{pi}^2) \qquad (2.39)$$

This equation is similar to the dispersion relation (16.4.12) for the propagation of transverse waves in a cold isotropic plasma, except that the reflection point is now $(\omega_{pe}^2 + \omega_{pi}^2)^{1/2}$ as a consequence of the inclusion of ion motion. A plot of phase velocity as a function of frequency for the dispersion relation (2.39) is also shown in Fig. 1. A dispersion plot in terms of ω as a function of k is displayed in Fig. 2 for the three modes of propagation.

In summary there are *three modes of wave propagation* in a warm fully ionized isotropic plasma (as compared to only one mode in the case of a cold isotropic plasma). They are the *transverse electromagnetic mode* (also present in the case of a cold plasma), the *longitudinal electron plasma mode* and the *longitudinal ion plasma mode*.

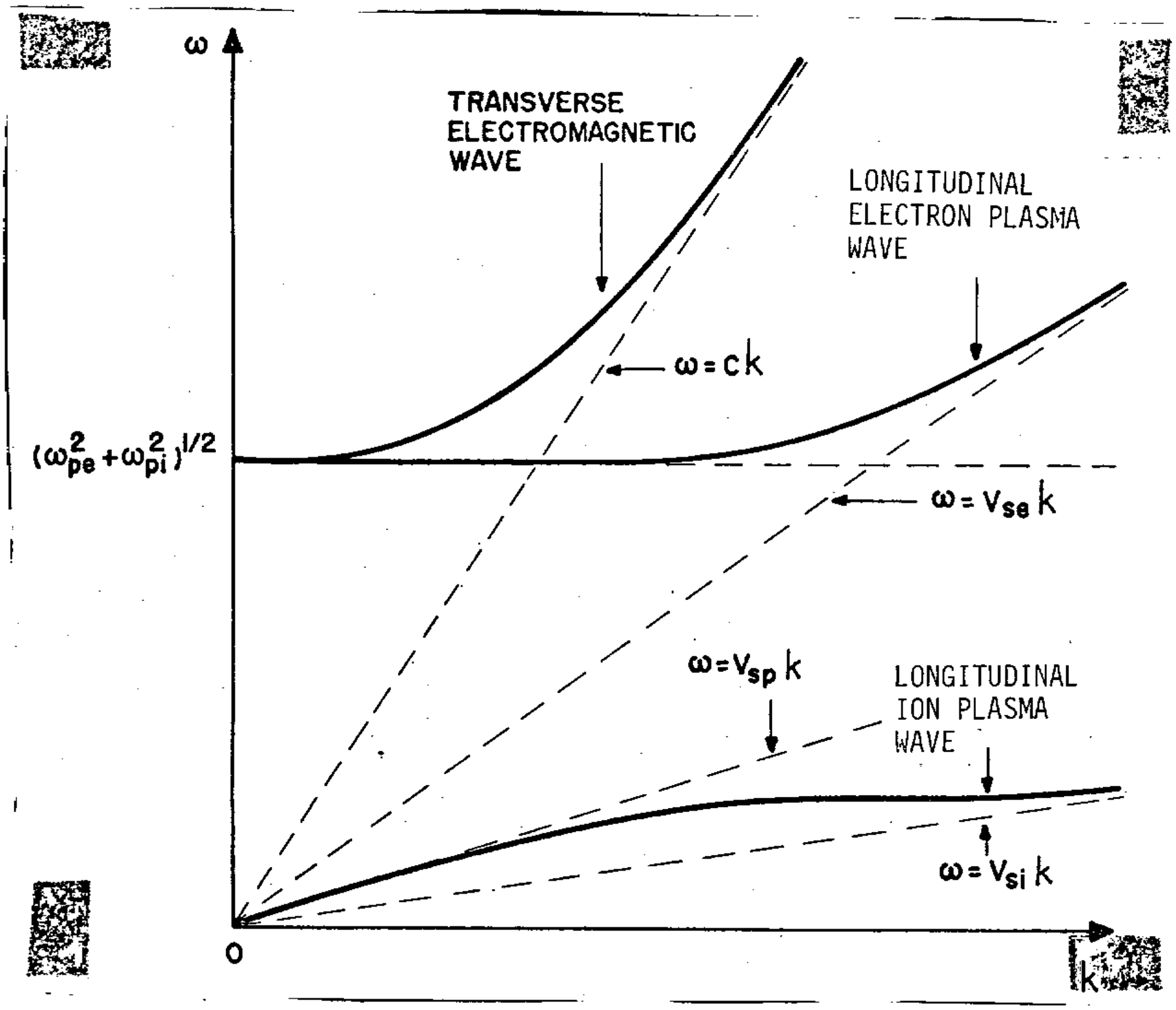

Fig. 2 - Dispersion relation for the three modes of wave propagation in a warm isotropic fully ionized plasma.

3. BASIC EQUATIONS FOR WAVES IN A WARM MAGNETOPLASMA

The basic equations for the study of wave propagation in a warm fully ionized magnetoplasma are (2.1), (2.2) and (2.3). Proceeding in the same manner as in the previous section, but now considering an externally applied uniform magnetostatic field, $\underset{\sim}{B}_0$, we obtain, in place of (2.11),

$$\omega^2 \underset{\sim}{u}_\alpha = i\omega \frac{q_\alpha}{m_\alpha} (\underset{\sim}{E} + \underset{\sim}{u}_\alpha \times \underset{\sim}{B}_o) + V_{S\alpha}^2 \underset{\sim}{k} (\underset{\sim}{k}\cdot\underset{\sim}{u}_\alpha) - i\omega\, \nu_{\alpha\beta} (\underset{\sim}{u}_\alpha - \underset{\sim}{u}_\beta) \quad (3.1)$$

This equation is complemented by (2.15) and (2.16) or, equivalently, by

$$\underset{\sim}{k} \times (\underset{\sim}{k} \times \underset{\sim}{E}) + \frac{\omega^2}{c^2} \underset{\sim}{E} = - \frac{i\omega\, e n_o}{c^2\, \varepsilon_o} (\underset{\sim}{u}_i - \underset{\sim}{u}_e) \quad (3.2)$$

If we choose a Cartesian coordinate system, such that the z-axis is along $\underset{\sim}{B}_o$ and $\underset{\sim}{k}$ is in the x-z plane (Fig. 3), we have

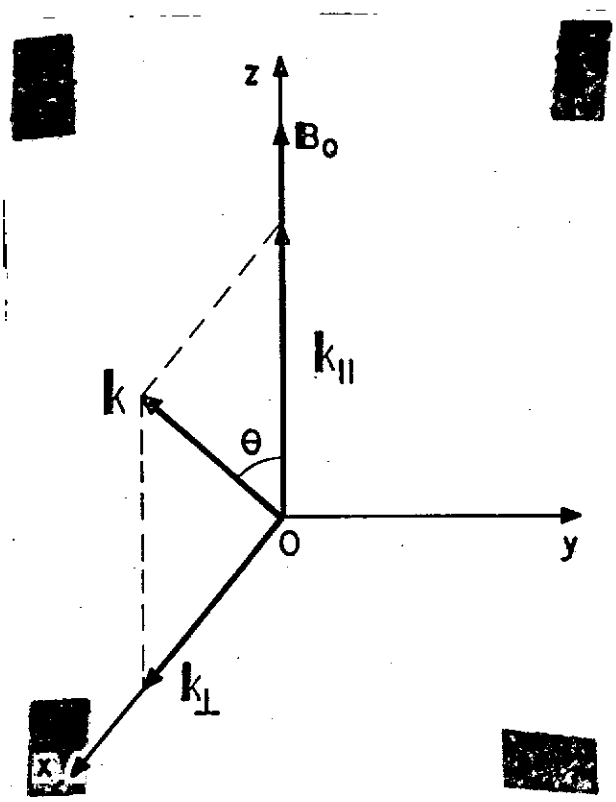

Fig. 3 - Cartesian coordinate system chosen with $\underset{\sim}{B}_o$ along the z-axis and $\underset{\sim}{k}$ in the x-z plane.

$$\underset{\sim}{B}_o = B_o \, \hat{\underset{\sim}{z}} \tag{3.3}$$

$$\underset{\approx}{k} = \underset{\approx}{k}_{\parallel} + \underset{\approx}{k}_{\perp} = k \sin\Theta \, \hat{\underset{\sim}{x}} + k \cos\Theta \, \hat{\underset{\sim}{z}} \tag{3.4}$$

and consequently (3.1) and (3.2) become, respectively, [see Eqs. (16.5.10) and (16.5.5)]

$$\omega^2 \underset{\sim}{u}_\alpha - i\omega \frac{q_\alpha}{m_\alpha} B_0 \left(u_{\alpha y} \hat{\underset{\sim}{x}} - u_{\alpha x} \hat{\underset{\sim}{y}}\right) - V_{s\alpha}^2 k^2 (\sin\Theta \, u_{\alpha x} + \cos\Theta \, u_{\alpha z}) \cdot$$

$$\cdot (\sin\Theta \, \hat{\underset{\sim}{x}} + \cos\Theta \, \hat{\underset{\sim}{z}}) + i\omega \, \nu_{\alpha\beta} \left(\underset{\sim}{u}_\alpha - \underset{\sim}{u}_\beta\right) = i\omega \frac{q_\alpha}{m_\alpha} \underset{\sim}{E} \tag{3.5}$$

and

$$\underset{\approx}{a} \cdot \underset{\sim}{E} = - \frac{ien_o}{\omega\varepsilon_o} \left(\underset{\sim}{u}_i - \underset{\sim}{u}_e\right) \tag{3.6}$$

where the components of the dyad $\underset{\approx}{a}$, which represents the operator $[(c^2/\omega^2) \, \underset{\sim}{k} \times \cdot (\underset{\sim}{k} \times \ldots) + (\ldots)]$, can be arranged in matrix form as

$$\underset{\approx}{a} = \begin{pmatrix} 1 - \frac{k^2c^2}{\omega^2} \cos^2\Theta & 0 & \frac{k^2c^2}{\omega^2} \sin\Theta \cos\Theta \\ 0 & 1 - \frac{k^2c^2}{\omega^2} & 0 \\ \frac{k^2c^2}{\omega^2} \sin\Theta \cos\Theta & 0 & 1 - \frac{k^2c^2}{\omega^2} \sin^2\Theta \end{pmatrix} \tag{3.7}$$

With this matrix definition of $\underset{\approx}{a}$, the dot product in Eq. (3.6) can be thought of as a matrix product between $\underset{\approx}{a}$ and the vector column $\underset{\sim}{E}$. Taking the inverse of the matrix associated with $\underset{\approx}{a}$ (assuming a non-vanishing determinant of its elements) and multiplying (3.6) by $(\underset{\approx}{a})^{-1}$, we obtain

$$\underset{\sim}{E} = - \frac{ien_o}{\omega\varepsilon_o} (\underset{\approx}{a})^{-1} \cdot (\underset{\sim}{u}_i - \underset{\sim}{u}_e) \tag{3.8}$$

since $(\underset{\approx}{a})^{-1} \cdot \underset{\approx}{a} = \underset{\approx}{1}$, where $\underset{\approx}{1}$ is the unit dyad.

Eq. (3.8) can be used to replace $\underset{\sim}{E}$ in Eq. (3.5). For the electrons we take $\alpha = e$ and $\beta = i$ in Eq. (3.5), whereas for the ions $\alpha = i$ and $\beta = e$. We obtain, therefore, a system of six equations with the six unknowns $\underset{\sim}{u}_{\alpha j}$ (with $j = x, y, z$, and $\alpha = e, i$). The requirement that the determinant of its coefficients be equal to zero gives the dispersion relation.

4. WAVES IN A WARM ELECTRON GAS IN A MAGNETIC FIELD

In view of the complexity of the algebra involved, we shall initially consider the simple case of a gas of electrons immersed in an externally applied magnetic field, neglecting for the moment the macroscopic motion of the ions ($\underset{\sim}{u}_i = 0$).

4.1 - Derivation of the dispersion relation

From Eq. (3.5) we obtain for the electrons (taking $\underset{\sim}{u}_i = 0$)

$$\underset{\sim}{u}_e + i \frac{\omega_{ce}}{\omega} (u_{ey} \,\hat{\underset{\sim}{x}} - u_{ex} \,\hat{\underset{\sim}{y}}) - \frac{V_{se}^2 \, k^2}{\omega^2} (\sin\Theta \, u_{ex} + \cos\Theta u_{ez})(\sin\Theta \,\hat{\underset{\sim}{x}} +$$

$$+ \cos\Theta \,\hat{\underset{\sim}{z}}) + i \frac{\nu_e}{\omega} \underset{\sim}{u}_e = \frac{\omega_{pe}^2}{\omega^2} (\underset{\approx}{a})^{-1} \cdot \underset{\sim}{u}_e \qquad (4.1)$$

Using the notation introduced in Eqs. (16.5.14), (16.5.15) and (16.5.16), Eq. (4.1) can be rewritten in the form

$$U \underset{\sim}{u}_e + (- \frac{V_{se}^2 \, k^2}{\omega^2} \sin^2\Theta u_{ex} + i \, Y u_{ey} - \frac{V_{se}^2 \, k^2}{\omega^2} \sin\Theta \cos\Theta u_{ez}) \,\hat{\underset{\sim}{x}} -$$

$$- i\,Y\,u_{ex}\,\hat{\underset{\sim}{y}} + \left(- \frac{V_{se}^2\,k^2}{\omega^2}\sin\Theta\cos\Theta\,u_{ex} - \frac{V_{se}^2\,k^2}{\omega^2}\cos^2\Theta\,u_{ez}\right)\hat{\underset{\sim}{z}} =$$

$$= X\,(\underset{\approx}{a})^{-1}\cdot\underset{\sim}{u}_e \tag{4.2}$$

Defining a dyad $\underset{\approx}{b}$ through the matrix

$$\underset{\approx}{b} = \begin{pmatrix} \left(U - \dfrac{V_{se}^2\,k^2}{\omega^2}\sin^2\Theta\right) & i\,Y & - \dfrac{V_{se}^2\,k^2}{\omega^2}\sin\Theta\cos\Theta \\ -\,i\,Y & U & 0 \\ - \dfrac{V_{se}^2\,k^2}{\omega^2}\sin\Theta\cos\Theta & 0 & \left(U - \dfrac{V_{se}^2\,k^2}{\omega^2}\cos^2\Theta\right) \end{pmatrix} \tag{4.3}$$

equation (4.2) becomes

$$\left[\underset{\approx}{b} - X\,(\underset{\approx}{a})^{-1}\right]\cdot\underset{\sim}{u}_e = 0 \tag{4.4}$$

This equation is of the form $\underset{\approx}{c}\cdot\underset{\sim}{u}_e = 0$, with $\underset{\approx}{c} \equiv \underset{\approx}{b} - X\,(\underset{\approx}{a})^{-1}$. A nontrivial solution exists only if the determinant of the matrix $\underset{\approx}{c}$ vanishes. Therefore, in order to have nontrivial solutions ($\underset{\sim}{u}_e \neq 0$) we must have

$$\det\left[\underset{\approx}{b} - X\,(\underset{\approx}{a})^{-1}\right] = 0 \tag{4.5}$$

This condition gives the dispersion relation for wave propagation in a warm electron gas immersed in a magnetic field.

In order to simplify matters, in the two following subsections we examine the dispersion relation (4.5) for the special cases of propagation parallel and perpendicular to the magnetic field.

4.2 - Wave propagation along the magnetic field

For the case of propagation *along* the magnetic field ($\underset{\sim}{k} \parallel \underset{\sim}{B}_0$) we have $\underset{\sim}{k} = k\,\hat{\underset{\sim}{z}}$ and $\Theta = 0^0$, so that (3.7) and (4.3) simplify to

$$\underset{\approx}{a} = \begin{pmatrix} (1 - k^2c^2/\omega^2) & 0 & 0 \\ 0 & (1 - k^2c^2/\omega^2) & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (4.6)$$

$$\underset{\approx}{b} = \begin{pmatrix} U & iY & 0 \\ -iY & U & 0 \\ 0 & 0 & (U - V_{se}^2\, k^2/\omega^2) \end{pmatrix} \qquad (4.7)$$

Therefore, the determinant (4.5) becomes

$$\begin{vmatrix} U - \dfrac{X}{1 - k^2c^2/\omega^2} & iY & 0 \\ -iY & U - \dfrac{X}{1 - k^2c^2/\omega^2} & 0 \\ 0 & 0 & U - \dfrac{V_{se}^2\ k^2}{\omega^2} - X \end{vmatrix} = 0 \tag{4.8}$$

Which gives the following dispersion relations for *transverse waves* ($u_{ex} \neq 0$; $u_{ey} \neq 0$),

$$U - \frac{X}{1 - k^2c^2/\omega^2} = \pm\ Y \tag{4.9}$$

and for a *longitudinal wave* ($u_{ez} \neq 0$),

$$U - \frac{V_{se}^2\ k^2}{\omega^2} - X = 0 \tag{4.10}$$

Note that in this case the z component of Eq. (4.4) is uncoupled from the x and y components, so that the longitudinal mode is independent of the transverse modes.

Eq. (4.9) yields the following expressions corresponding, respectively, to the "plus" and "minus" signs,

$$\frac{k^2c^2}{\omega^2} = 1 - \frac{X}{U - Y} \tag{4.11}$$

$$\frac{k^2c^2}{\omega^2} = 1 - \frac{X}{U + Y} \tag{4.12}$$

These dispersion relations correspond, respectively, to the *right* and *left circularly polarized waves* (RCP and LCP) obtained in section 6, of Chapter 16 [see Eqs. (16.6.6) and (16.6.8)], for *transverse waves* in a cold plasma.

For the longitudinal wave, substituting $U = 1 + i\nu_e/\omega$ and $X = \omega_{pe}^2/\omega^2$ in (4.10), the dispersion relation becomes

$$\omega^2 + i\nu_e \ \omega = \omega_{pe}^2 + k^2 \ V_{se}^2 \tag{4.13}$$

Hence, as compared to the cold plasma model, instead of the longitudinal oscillation at ω_{pe} (present in the cold plasma) there is in this case an additional mode of propagation, known as the *electron plasma wave*. Neglecting collisions ($\nu_e = 0$), (4.13) becomes the same dispersion relation as obtained in section 2 [Eq. (2.24)] for waves in an isotropic warm plasma. Hence, for propagation along the magnetic field, the longitudinal electron plasma wave is not affected by the presence of the magnetic field.

In summary, there are three modes of propagation in a warm electron gas for $\underset{\sim}{k}$ parallel to the magnetic field: the transverse *RCP* and *LCP waves*, and the longitudinal *electron plasma wave*. The addition of the pressure gradient term, in the equation of motion for the electrons, has no effect on the transverse waves. A plot of phase velocity versus frequency for these three modes is displayed in Fig. 4. The corresponding $\omega(k)$ dispersion plot is shown in Fig. 5.

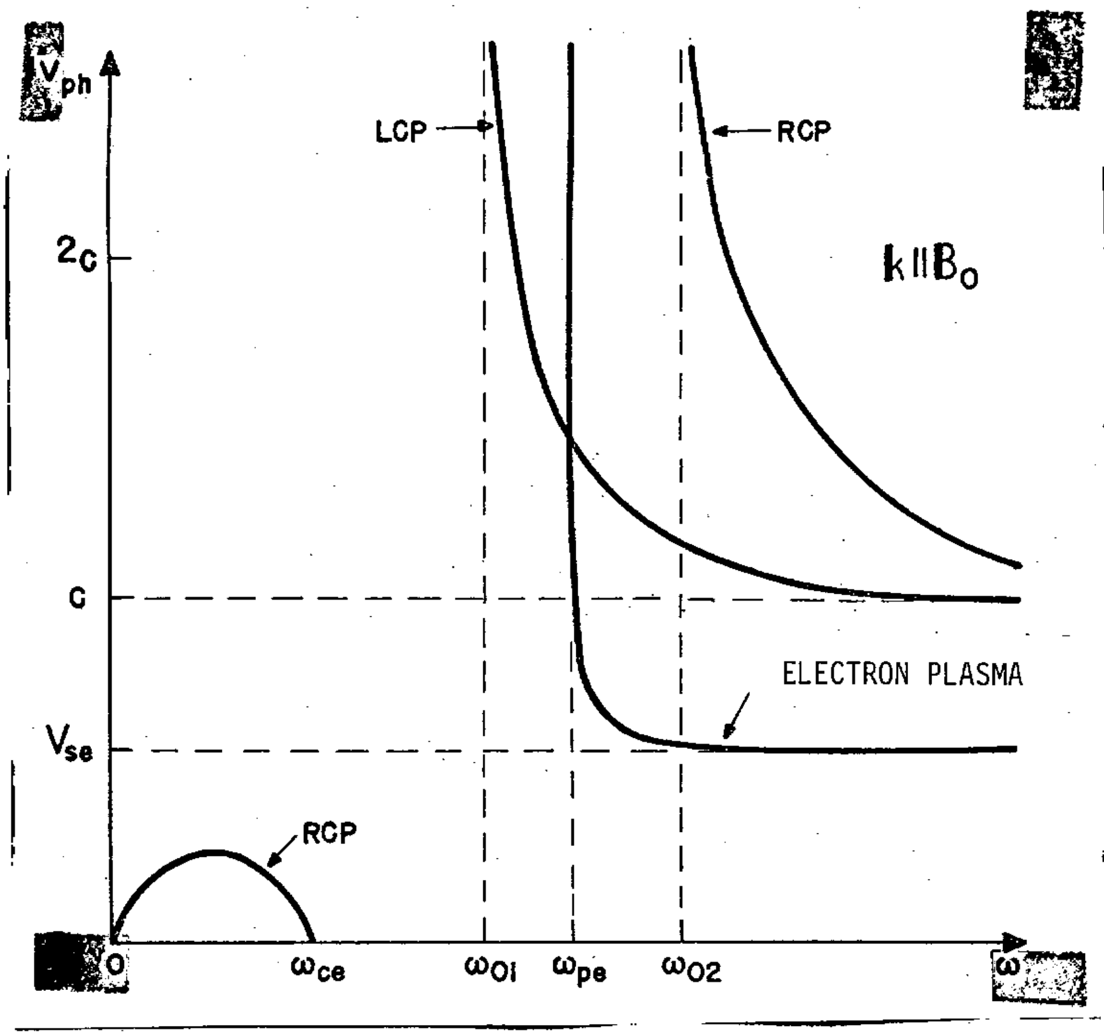

Fig.4 - Phase velocity, as a function of frequency, for waves propagating in the direction of $\underset{\sim}{B}_0$ in a warm electron gas.

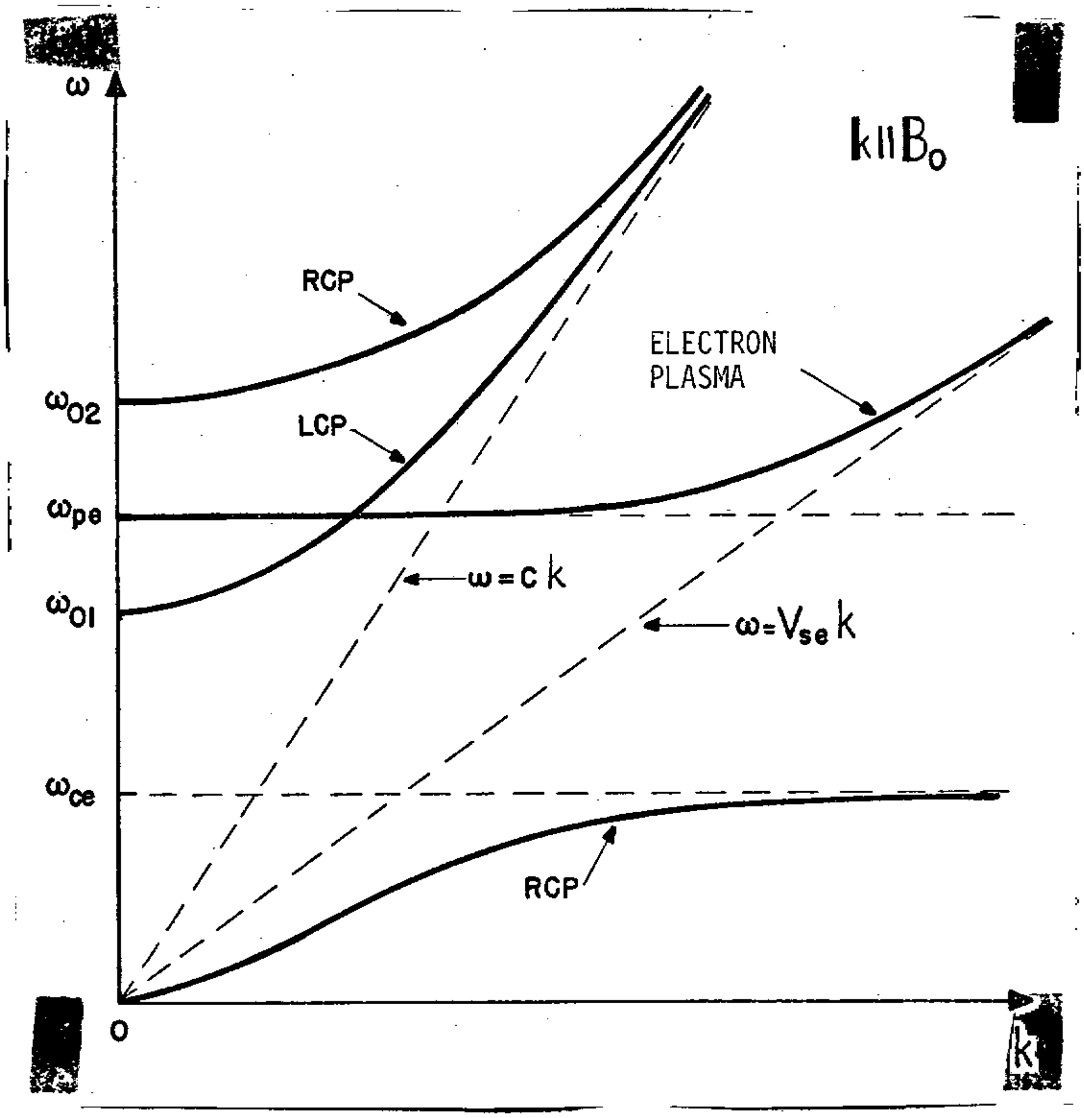

Fig. 5 - Dispersion plot for waves propagating parallel to $\underset{\sim}{B}_0$ in a warm electron gas.

4.3 - Wave propagation normal to the magnetic field

For the case of propagation *across* the magnetic field ($\underset{\sim}{k} \perp \underset{\sim}{B}_o$) we have $\underset{\sim}{k} = k\,\underset{\sim}{\hat{x}}$ and $\Theta = 90^0$, so that (3.7) and (4.3) simplify to

$$\underset{\approx}{a} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1 - k^2c^2/\omega^2) & 0 \\ 0 & 0 & (1 - k^2c^2/\omega^2) \end{pmatrix} \qquad (4.11)$$

$$\underset{\approx}{b} = \begin{pmatrix} (U - V_{se}^2\ k^2/\omega^2) & iY & 0 \\ -iY & U & 0 \\ 0 & 0 & U \end{pmatrix} \qquad (4.15)$$

From these expressions it is clear that the z component of (4.4) is uncoupled from the x and y components. Thus, in order to have a *transverse wave* oscillating along the z-axis ($u_{ez} \neq 0$), we must have from the z-component of (4.4),

$$U - \frac{X}{(1 - k^2c^2/\omega^2)} = 0 \tag{4.16}$$

or

$$\frac{k^2c^2}{\omega^2} = 1 - \frac{X}{U} \tag{4.17}$$

which is the familiar dispersion relation for the transverse *ordinary wave* (the electric field of the wave oscillates in the same direction as $\underset{\sim}{B}_0$) found in section 7, of Chapter 16 [see Eq. (16.7.4)].

From (4.4), (4.14) and (4.15) it is clear that the equations for u_{ex} and u_{ey} are coupled. Therefore, in order to have nontrivial solutions (*longitudinal* wave for $u_{ex} \neq 0$ and *transverse* wave for $u_{ey} \neq 0$) we must require the determinant formed with the coefficients of the x and y components of (4.4) to vanish, that is,

$$\begin{vmatrix} U - V_{se}^2\, k^2/\omega^2 - X & i\,Y \\ & \\ -\,i\,Y & U - X/(1 - k^2c^2/\omega^2) \end{vmatrix} = 0 \qquad (4.18)$$

This determinant gives, neglecting collisions ($\nu_e = 0;\ U = 1$),

$$(\omega^2 - V_{se}^2 k^2 - \omega_{pe}^2)\ (\omega^2 - \frac{\omega_{pe}^2}{1 - k^2c^2/\omega^2}) - \omega^2\ \omega_{ce}^2 = 0 \qquad (4.19)$$

Expanding this expression, and rearranging, we get

$$k^4\ (c^2\ V_{se}^2) - k^2\ \left[V_{se}^2\ (\omega^2 - \omega_{pe}^2) + c^2\,(\omega^2 - \omega_{pe}^2 - \omega_{ce}^2)\right] + (\omega^2 - \omega_{pe}^2)^2 -$$

$$-\,\omega^2\ \omega_{ce}^2 = 0 \qquad (4.20)$$

This dispersion relation is quadratic in k^2, so that there will be two values of k^2 as a function of ω, that is, *two modes of propagation*. Since, generally, we have $V_{se} << c$, the first term within brackets, in the left-hand side of (4.20), can be neglected as compared to the other. With this approximation, (4.20) becomes,

$$k^4\ (c^2\ V_{se}^2) - k^2c^2\ (\omega^2 - \omega_{pe}^2 - \omega_{ce}^2\ ) + (\omega^2 - \omega_{pe}^2)^2 - \omega^2\ \omega_{ce}^2 = 0 \quad (4.21)$$

Although it is not difficult to obtain the exact solution of this equation, it is more instructive to analyze it for some special limiting cases. First, let us obtain the approximate solution of (4.21) in the region where $\omega^2 \gg k^2 V_{se}^2$, that is, when the term $k^4 c^2 V_{se}^2$ is much smaller than any of the others. For k^2 positive this condition implies in phase velocities much larger than V_{se} and, for this reason, it will be referred to as the *high phase velocity* limit. With this condition, (4.21) reduces to

$$- k^2c^2 (\omega^2 - \omega_{pe}^2 - \omega_{ce}^2) + (\omega^2 - \omega_{pe}^2)^2 - \omega^2 \omega_{ce}^2 = 0 \qquad (4.22)$$

or

$$k^2c^2 = \frac{(\omega^2 + \omega\omega_{ce} + \omega_{pe}^2)\ (\omega^2 - \omega\omega_{ce} - \omega_{pe}^2)}{(\omega^2 - \omega_{pe}^2 - \omega_{ce}^2)} \ ; \quad (\omega^2 \gg k^2 V_{se}^2) \qquad (4.23)$$

This equation is exactly the same dispersion relation found in section 7, of Chapter 16 [Eq. (16.7.7)], for the *transverse extraordinary wave* in a cold plasma, except that now the condition $\omega^2 \gg k^2 V_{se}^2$ must be satisfied for (4.23) to be applicable.

Next, let us obtain the approximate solution of (4.21) in the region where $\omega^2 \ll k^2c^2$. For k^2 positive this condition implies in phase velocities much smaller than the velocity of light and, for this reason, it will be referred to as the *low phase velocity* limit. Thus, for $\omega^2 \ll k^2c^2$, (4.21) reduces to

$$k^4 (c^2 V_{se}^2) - k^2c^2 (\omega^2 - \omega_{pe}^2 - \omega_{ce}^2) = 0 \tag{4.24}$$

or

$$\omega^2 = \omega_{pe}^2 + \omega_{ce}^2 + k^2 V_{se}^2 \quad ; \qquad (\omega^2 << k^2c^2) \tag{4.25}$$

When $\underset{\sim}{B}_0 = 0$ (i. e., $\omega_{ce} = 0$) this equation becomes identical to the dispersion relation for the longitudinal *electron plasma wave* [see Eq. (2.24)]. It is a valid solution for (4.21) only under the condition $\omega^2 << k^2c^2$.

Fig. 6 displays the phase velocity as a function of frequency for the *transverse ordinary mode* [Eq. (4.17)] and for the two modes described by Eq. (4.20). Note that, of these last two modes, one is a purely transverse *extraordinary wave*, while the other is partially transverse (electromagnetic *extraordinary wave* in the high phase velocity limit) and partially longitudinal (*electron plasma wave* in the low phase velocity limit). In this last mode, the transition from a basically transverse electromagnetic wave to a basically longitudinal electron plasma wave occurs in the frequency region where the phase velocity lies between c and V_{se}. The corresponding $\omega(k)$ dispersion plot is shown in Fig. 7.

4.4 - Wave propagation in an arbitrary direction

For propagation in an arbitrary direction with respect to the magnetic field, the dispersion relation is obtained from Eq. (4.5) with the dyads $\underset{\approx}{a}$ and $\underset{\approx}{b}$ as given by Eqs. (3.7) and (4.3). For an arbitrary angle between 0° and 90°, we expect the phase velocity versus frequency curves to lie between those of Figs. 4 and 6. Therefore, instead of getting involved in the cumbersome algebra behind (4.5), we present only the dispersion curves of

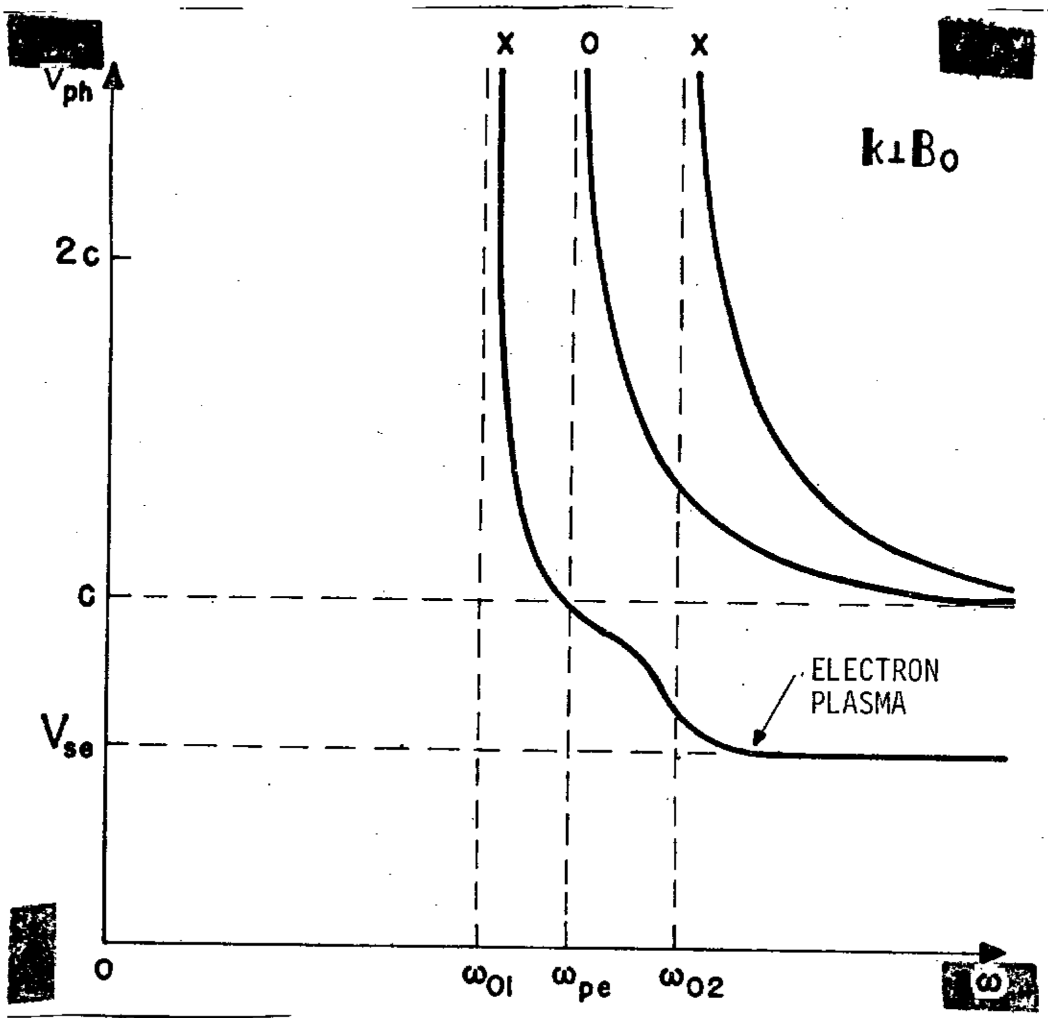

Fig. 6 - Phase velocity as a function of frequency for waves propagating perpendicular to $\underset{\sim}{B}_0$ in a warm electron gas.

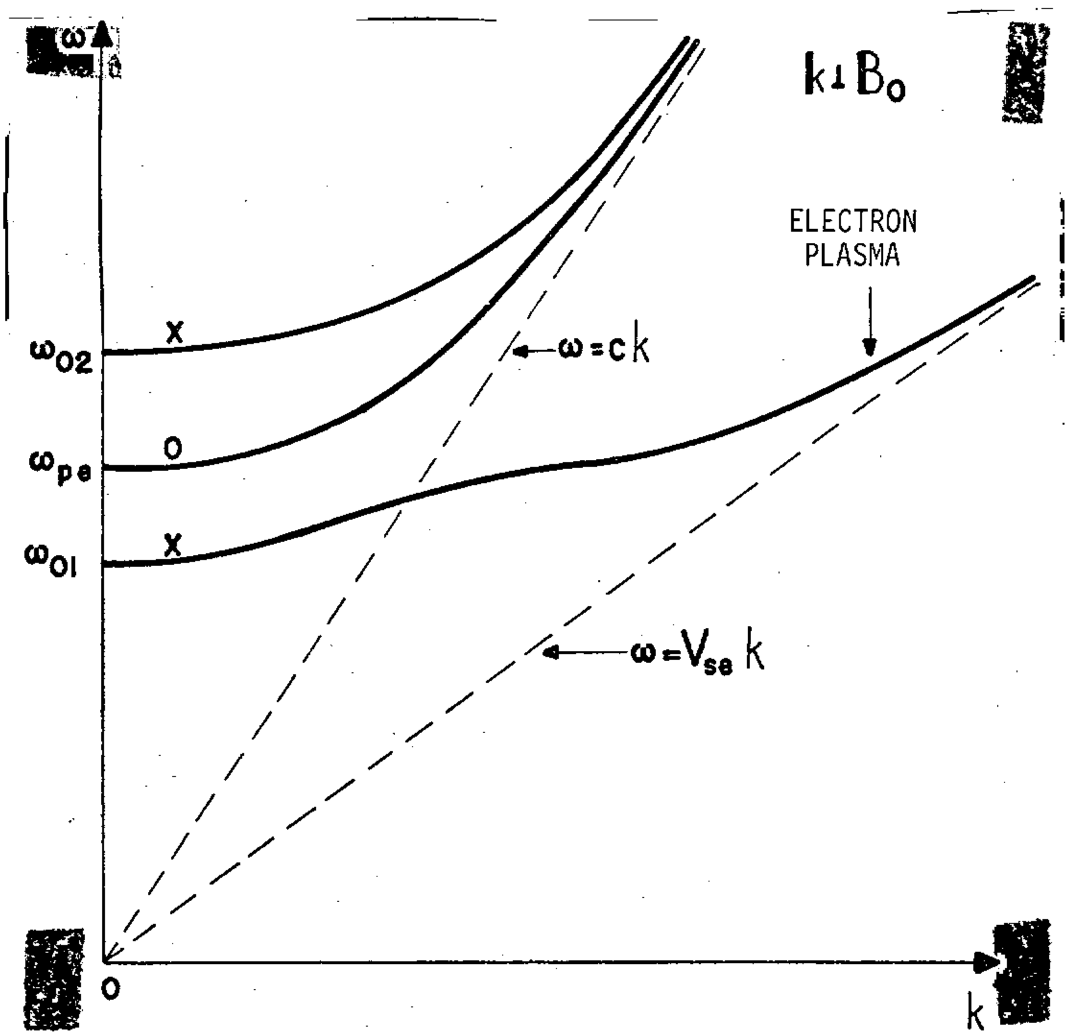

Fig. 7 - Dispersion plot for waves propagating perpendicular to $\underset{\sim}{B}_0$ in a warm electron gas.

Fig. 8, in which the shaded area illustrates how the transition occurs from $\Theta = 0^0$ to $\Theta = 90^0$. It can be easily verified that the only resonance which exists for an arbitrary angle is at the frequency $\omega = \omega_{ce} \cos \Theta$. The reflection points for any angle of propagation occur at the frequencies ω_{01}, ω_{pe} and ω_{02}.

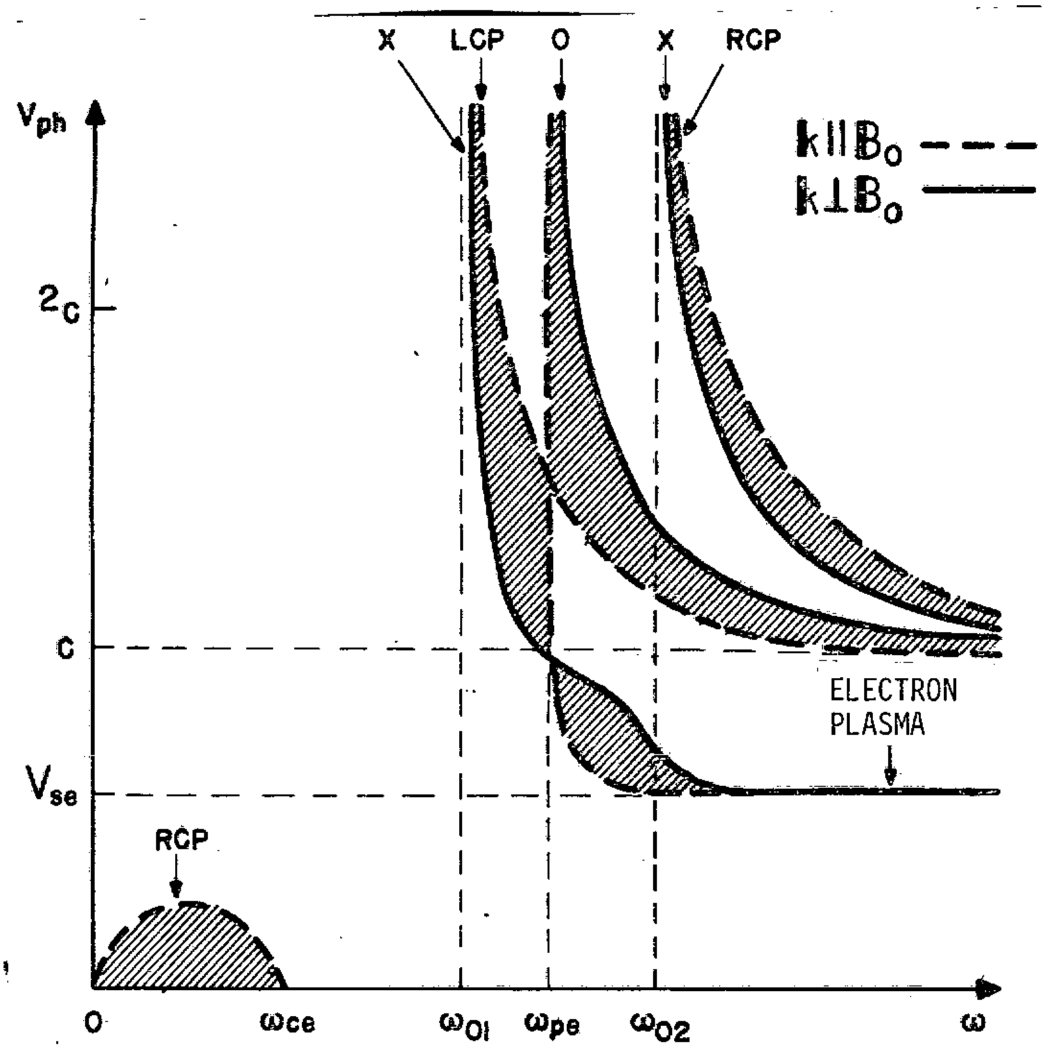

Fig. 8 - Phase velocity versus frequency for wave propagation in a warm electron gas immersed in a magnetic field.

5. WAVES IN A FULLY IONIZED WARM MAGNETOPLASMA

We consider now the propagation of plane waves in a fully ionized warm plasma having only one ion species, immersed in an externally applied uniform magnetostatic field.

5.1 - Derivation of the dispersion relation

The equation of motion for the *electrons* is, from (3.5),

$$\omega^2 \underset{\sim}{u}_e + i\omega\omega_{ce} (u_{ey} \underset{\sim}{\hat{x}} - u_{ex} \underset{\sim}{\hat{y}}) -$$

$$- V_{se}^2 k^2 (\sin\Theta\, u_{ex} + \cos\Theta\, u_{ez}) (\sin\Theta\, \underset{\sim}{\hat{x}} + \cos\Theta\, \underset{\sim}{\hat{z}}) +$$

$$+ i\omega\, \nu_{ei} (\underset{\sim}{u}_e - \underset{\sim}{u}_i) = - i\omega \frac{e}{m_e} \underset{\sim}{E} \tag{5.1}$$

and for the ions,

$$\omega^2 \underset{\sim}{u}_i - i\omega\omega_{ci} (u_{iy} \underset{\sim}{\hat{x}} - u_{ix} \underset{\sim}{\hat{y}}) -$$

$$- V_{si}^2 k^2 (\sin\Theta\, u_{ix} + \cos\Theta\, u_{iz}) (\sin\Theta\, \underset{\sim}{\hat{x}} + \cos\Theta\, \underset{\sim}{\hat{z}}) +$$

$$+ i\omega\, \nu_{ie} (\underset{\sim}{u}_i - \underset{\sim}{u}_e) = i\omega \frac{e}{m_i} \underset{\sim}{E} \tag{5.2}$$

Eqs. (5.1) and (5.2), involving the variables $\underset{\sim}{u}_e$, $\underset{\sim}{u}_i$ and $\underset{\sim}{E}$, are complemented by (3.6),

$$\underset{\approx}{a} \cdot \underset{\sim}{E} = \frac{ie\, n_o}{\omega\, \varepsilon_o} (\underset{\sim}{u}_e - \underset{\sim}{u}_i) \tag{5.3}$$

where the dyad $\underset{\approx}{a}$ is defined according to (3.7).

Eqs. (5.1) and (5.2) can be written, respectively, in compact form, as

$$\underset{\approx}{b}_e \cdot \underset{\sim}{u}_e = - i \frac{e}{\omega m_e} \underset{\sim}{E} - i \frac{\nu_{ei}}{\omega} (\underset{\sim}{u}_e - \underset{\sim}{u}_i) \tag{5.4}$$

and

$$\underset{\approx}{b}_i \cdot \underset{\sim}{u}_i = i \frac{e}{\omega m_i} \underset{\sim}{E} - i \frac{\nu_{ie}}{\omega} (\underset{\approx}{u}_i - \underset{\approx}{u}_e) \tag{5.5}$$

where the dyads $\underset{\approx}{b}_e$ and $\underset{\approx}{b}_i$ are appropriately defined by

$$\underset{\approx}{b}_e = \begin{pmatrix} (1 - \frac{V_{se}^2 k^2}{\omega^2} \sin^2 \Theta) & i Y_e & - \frac{V_{se}^2 k^2}{\omega^2} \sin \Theta \cos \Theta \\ - i Y_e & 1 & 0 \\ - \frac{V_{se}^2 k^2}{\omega^2} \sin \Theta \cos \Theta & 0 & (1 - \frac{V_{se}^2 k^2}{\omega^2} \cos^2 \Theta) \end{pmatrix} \tag{5.6}$$

$$\underset{\approx}{b}_i = \begin{pmatrix} (1 - \frac{V_{si}^2 k^2}{\omega^2} \sin^2 \Theta) & - i Y_i & - \frac{V_{si}^2 k^2}{\omega^2} \sin \Theta \cos \Theta \\ i Y_i & 1 & 0 \\ - \frac{V_{si}^2 k^2}{\omega^2} \sin \Theta \cos \Theta & 0 & (1 - \frac{V_{si}^2 k^2}{\omega^2} \cos^2 \Theta) \end{pmatrix} \tag{5.7}$$

where $Y_e = \omega_{ce}/\omega$ and $Y_i = \omega_{ci}/\omega$. Multiplying Eqs. (5.4) and (5.5), respectively, by the inverse matrices corresponding to $\underset{\approx}{b}_e$ and $\underset{\approx}{b}_i$, we get

$$\underset{\sim}{u}_e = - i \frac{e}{\omega m_e} (\underset{\approx}{b}_e)^{-1} \cdot \underset{\sim}{E} - i \frac{\nu_{ei}}{\omega} (\underset{\approx}{b}_e)^{-1} \cdot (\underset{\sim}{u}_e - \underset{\sim}{u}_i) \tag{5.8}$$

$$\underset{\sim}{u}_i = i \frac{e}{\omega m_i} (\underset{\approx}{b}_i)^{-1} \cdot \underset{\sim}{E} + i \frac{\nu_{ie}}{\omega} (\underset{\approx}{b}_i)^{-1} \cdot (\underset{\sim}{u}_e - \underset{\sim}{u}_i) \tag{5.9}$$

Subtracting (5.9) from (5.8), and rearranging, yields

$$\left[\underset{\approx}{1} + i \frac{\nu_{ei}}{\omega} (\underset{\approx}{b}_e)^{-1} + i \frac{\nu_{ie}}{\omega} (\underset{\approx}{b}_i)^{-1}\right] \cdot (\underset{\sim}{u}_e - \underset{\sim}{u}_i) + i \frac{e}{\omega} \left[\frac{1}{m_e} (\underset{\approx}{b}_e)^{-1} + \right.$$
$$\left. + \frac{1}{m_i} (\underset{\approx}{b}_i)^{-1}\right] \cdot \underset{\sim}{E} = 0 \tag{5.10}$$

Combining (5.10) and (5.3) to eliminate the variable $(\underset{\sim}{u}_e - \underset{\sim}{u}_i)$, results in the following equation involving only the electric field vector

$$\left[\underset{\approx}{1} + i \frac{\nu_{ei}}{\omega} (\underset{\approx}{b}_e)^{-1} + i \frac{\nu_{ie}}{\omega} (\underset{\approx}{b}_i)^{-1}\right] \cdot (\underset{\approx}{a} \cdot E) - \left[\frac{\omega_{pe}^2}{\omega^2} (\underset{\approx}{b}_e)^{-1} - \right.$$
$$\left. - \frac{\omega_{pi}^2}{\omega^2} (\underset{\approx}{b}_i)^{-1} \right] \cdot \underset{\sim}{E} = 0 \tag{5.11}$$

or

$$\left\{ \left[\underset{\approx}{1} + i \frac{\nu_{ei}}{\omega} (\underset{\approx}{b}_e)^{-1} + i \frac{\nu_{ie}}{\omega} (\underset{\approx}{b}_i)^{-1} \right] \cdot \underset{\approx}{a} - X_e (\underset{\approx}{b}_e)^{-1} - \right.$$

$$\left. - X_i (\underset{\approx}{b}_i)^{-1} \right\} \cdot \underset{\sim}{E} = 0 \tag{5.12}$$

where $X_e = \omega_{pe}^2/\omega^2$ and $X_i = \omega_{pi}^2/\omega^2$.

As before, the dispersion relation is obtained by setting the determinant of the 3 x 3 matrix in (5.12) equal to zero, that is,

$$\det \left\{ \left[\underset{\approx}{1} + i \frac{\nu_{ei}}{\omega} (\underset{\approx}{b}_e)^{-1} + i \frac{\nu_{ie}}{\omega} (\underset{\approx}{b}_i)^{-1} \right] \cdot \underset{\approx}{a} - X_e (\underset{\approx}{b}_e)^{-1} \right.$$

$$\left. - X_i (\underset{\approx}{b}_i)^{-1} \right\} = 0 \tag{5.13}$$

If collisions are neglected ($\nu_{ei} = \nu_{ie} = 0$), (5.13) simplifies to

$$\det \left[\underset{\approx}{a} - X_e (\underset{\approx}{b}_e)^{-1} - X_i (\underset{\approx}{b}_i)^{-1} \right] = 0 \tag{5.14}$$

In the following subsections, in order to simplify the algebra involved, we shall neglect collisions and analyze the problem using Eq. (5.14).

5.2 - Wave propagation along the magnetic field

For $\Theta = 0^0 (\underset{\sim}{k} \parallel \underset{\sim}{B}_o)$ we have from (3.7), (5.6) and (5.7), respectively,

$$\underset{\approx}{a} = \begin{pmatrix} (1 - k^2c^2/\omega^2) & 0 & 0 \\ 0 & (1 - k^2c^2/\omega^2) & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (5.15)$$

$$\underset{\approx}{b}_e = \begin{pmatrix} 1 & i\,Y_e & 0 \\ -i\,Y_e & 1 & 0 \\ 0 & 0 & (1 - V_{se}^2\ k^2/\omega^2) \end{pmatrix} \qquad (5.16)$$

$$\underset{\approx}{b}_i = \begin{pmatrix} 1 & -i\,Y_i & 0 \\ i\,Y_i & 1 & 0 \\ 0 & 0 & (1 - V_{si}^2\ k^2/\omega^2) \end{pmatrix} \qquad (5.17)$$

The inverse of the matrices (5.16) and (5.17) are, respectively,

$$(\underset{\approx}{b}_e)^{-1} = \begin{pmatrix} \dfrac{1}{(1 - Y_e^2)} & \dfrac{-i\,Y_e}{(1 - Y_e^2)} & 0 \\[2ex] \dfrac{i\,Y_e}{(1 - Y_e^2)} & \dfrac{1}{(1 - Y_e^2)} & 0 \\[2ex] 0 & 0 & \dfrac{1}{(1 - V_{se}^2\ k^2/\omega^2)} \end{pmatrix} \tag{5.18}$$

$$(\underset{\approx}{b}_i)^{-1} = \begin{pmatrix} \dfrac{1}{(1 - Y_i^2)} & \dfrac{i\,Y_i}{(1 - Y_i^2)} & 0 \\[2ex] \dfrac{-i\,Y_i}{(1 - Y_i^2)} & \dfrac{1}{(1 - Y_i^2)} & 0 \\[2ex] 0 & 0 & \dfrac{1}{(1 - V_{si}^2\ k^2/\omega^2)} \end{pmatrix} \tag{5.19}$$

Substituting the matrices (5.15), (5.18) and (5.19), into (5.12), and setting $\nu_{ei} = \nu_{ie} = 0$, we obtain

$$\begin{pmatrix} A_1 & A_2 & 0 \\ -A_2 & A_1 & 0 \\ 0 & 0 & A_3 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0 \qquad (5.20)$$

where

$$A_1 = 1 - \frac{k^2\ c^2}{\omega^2} - \frac{X_i}{(1 - Y_i^2)} - \frac{X_e}{(1 - Y_e^2)} \qquad (5.20a)$$

$$A_2 = - \frac{i\, X_i\, Y_i}{(1 - Y_i^2)} + \frac{i\, X_e\, Y_e}{(1 - Y_e^2)} \qquad (5.20b)$$

$$A_3 = 1 - \frac{X_i}{(1 - V_{si}^2\ k^2/\omega^2)} - \frac{X_e}{(1 - V_{se}^2\ k^2/\omega^2)} \qquad (5.20c)$$

It is clear from this matrix equation that the *longitudinal* component of the electric field (E_z) is uncoupled from the *transverse* components (E_x and E_y). Therefore, for longitudinal waves to exist ($E_z \neq 0$), the coefficient of E_z in (5.20) must be equal to zero, which gives the following dispersion relation for *longitudinal waves*,

$$1 - \frac{X_i}{(1 - V_{si}^2 k^2/\omega^2)} - \frac{X_e}{(1 - V_{se}^2 k^2/\omega^2)} = 0 \qquad (5.21)$$

This dispersion relation can be rearranged in the following form

$$k^4 \, (V_{se}^2 \, V_{si}^2) + k^2 \left[\omega_{pe}^2 \, V_{si}^2 + \omega_{pi}^2 \, V_{se}^2 - \omega^2 (V_{se}^2 + V_{si}^2) \right] +$$

$$+ \, \omega^2 \, (\omega^2 - \omega_{pe}^2 - \omega_{pi}^2) = 0 \tag{5.22}$$

which is identical to Eq. (2.22). Therefore, since it is a quadratic equation in k^2, there are in general *two longitudinal modes* of propagation. Note that these two longitudinal modes, propagating along $\underset{\sim}{B}_o$, are not affected by the magnetic field strength. This dispersion relation has already been analyzed in section 2, where it was shown that the two longitudinal modes are the *electron plasma wave* and the *ion plasma wave*.

The dispersion relation for transverse waves ($E_x \neq 0$; $E_y \neq 0$) are seen, from (5.20), to be given by

$$\left[1 - \frac{k^2 c^2}{\omega^2} - \frac{X_i}{(1 - Y_i^2)} - \frac{X_e}{(1 - Y_e^2)} \right]^2 - \left[\frac{X_i \, Y_i}{(1 - Y_i^2)} - \frac{X_e \, Y_e}{(1 - Y_e^2)} \right]^2 = 0 \tag{5.23}$$

Using the notation

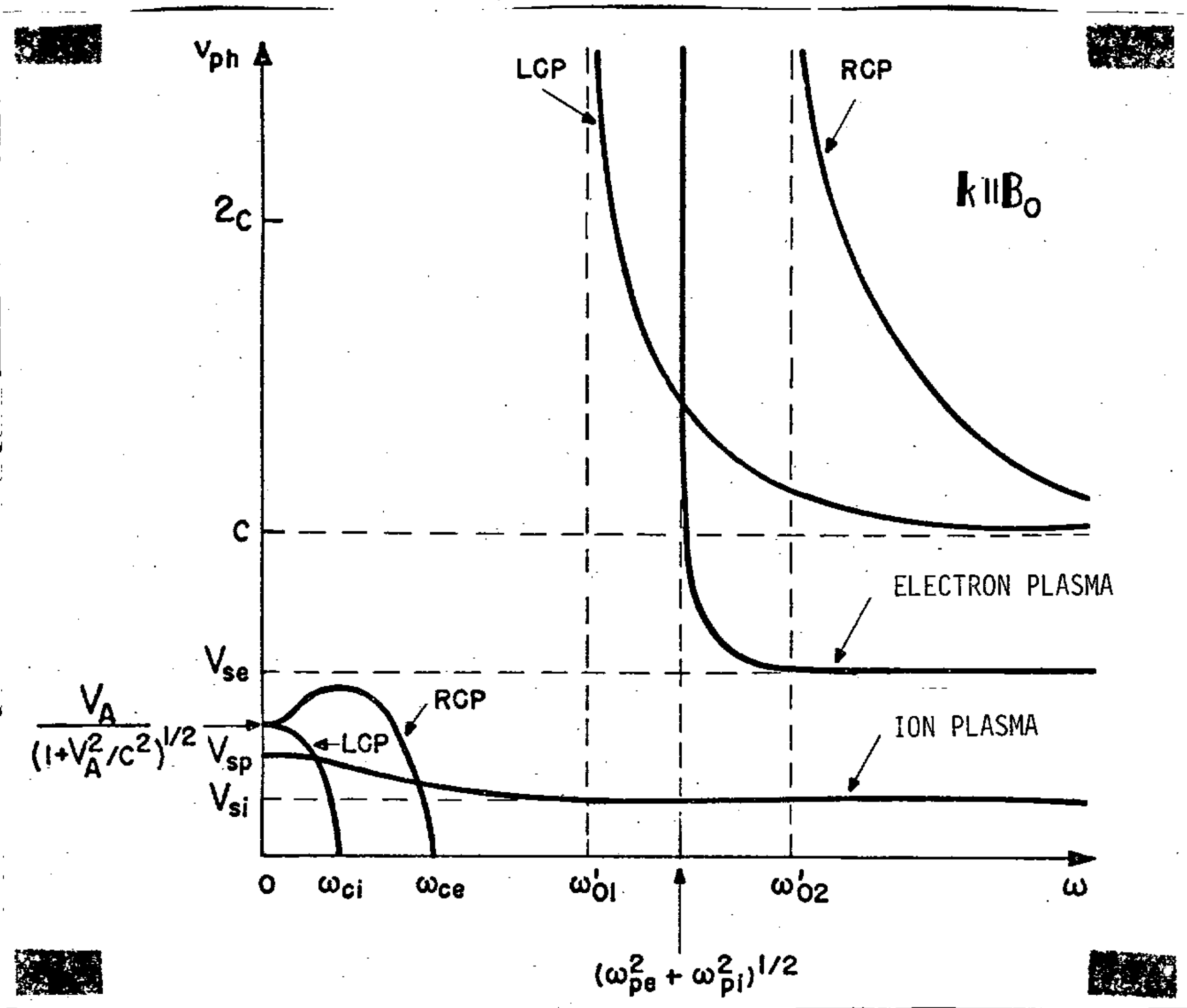

Fig. 9 - Phase velocity as a function of frequency for plane waves travelling along the magnetic field in a warm fully ionized magnetoplasma.

$$S = 1 - \frac{X_i}{(1 - Y_i^2)} - \frac{X_e}{(1 - Y_e^2)} \qquad (5.24)$$

$$D = \frac{X_i\ Y_i}{(1 - Y_i^2)} - \frac{X_e\ Y_e}{(1 - Y_e^2)} \qquad (5.25)$$

and letting

$$R = S + D \qquad (5.26)$$

$$L = S - D \qquad (5.27)$$

then (5.23) becomes

$$\left(\frac{k^2c^2}{\omega^2} - R\right)\left(\frac{k^2c^2}{\omega^2} - L\right) = 0 \qquad (5.28)$$

There are, therefore, *two transverse modes* that propagate along the magnetic field with dispersion relations given by

$$\left(\frac{k^2c^2}{\omega^2}\right)_R = R \qquad (5.29)$$

and

$$\left(\frac{k^2c^2}{\omega^2}\right)_L = L \qquad (5.30)$$

From the x-component of (5.20) we have

$$\frac{E_y}{E_x} = \frac{(S - k^2c^2/\omega^2)}{i\,D} \tag{5.31}$$

so that, using (5.29), we obtain

$$\left(\frac{E_y}{E_x}\right)_R = i \tag{5.32}$$

whereas, using (5.30),

$$\left(\frac{E_y}{E_x}\right)_L = -\,i \tag{5.33}$$

Therefore, the dispersion relation $(k^2c^2/\omega^2) = R$ corresponds to a *right-hand circularly polarized wave,* and $(k^2c^2/\omega^2)_L = L$ to a *left-hand circularly polarized wave.*

The phase velocity as a function of frequency for propagation along $\underset{\sim}{B}_0$ is shown in Fig. 9. The reflection points at ω'_{01} and ω'_{02} are not exactly the same ones given by equations (16.6.13) and (16.6.14), but are slightly different as a result of the inclusion of ion motion. Also, because ion motion has been taken into account, besides the resonance at $\omega = \omega_{ce}$ for the RCP wave, there is also a resonance at $\omega = \omega_{ci}$ for the LCP wave.

In the very low-frequency limit, the phase velocities of the RCP and LCP waves tend to $V_A/(1+V_A^2/c^2)^{1/2}$, instead of going to zero as in the case of the cold plasma model. This result can be seen as follows. For very low frequency waves such that

$$\omega \ll \omega_{ci} \tag{5.34}$$

we obtain, using (5.24) and (5.25),

$$R = L = 1 + \frac{\omega_{pe}^2}{\omega_{ci}\,\omega_{ce}} \qquad (\omega \ll \omega_{ci}) \tag{5.35}$$

Therefore, using the definitions of ω_{pe}, ω_{ci} and ω_{ce}, the dispersion relation for the RCP and LCP waves becomes

$$\frac{k^2c^2}{\omega^2} = 1 + \frac{n_o\, m_i}{\varepsilon_o\, B_o^2} \tag{5.36}$$

The average mass density is $\rho = n_o\,(m_e + m_i) \simeq n_o\, m_i$, and since $\varepsilon_o = 1/(\mu_o\, c^2)$, (5.36) can be rewritten as

$$\frac{k^2c^2}{\omega^2} = 1 + \frac{c^2\,\mu_o\,\rho}{B_o^2} \tag{5.37}$$

or

$$\frac{k^2c^2}{\omega^2} = 1 + \frac{c^2}{V_A^2} \tag{5.38}$$

where $V_A = (B_0^2/\mu_o\ \rho)^{1/2}$ is the Alfvēn velocity, defined in (15.1.4). Thus, from (5.38), in the very low-frequency limit the phase velocity of both transverse waves is given by

$$v_{ph} = \frac{\omega}{k} = \frac{V_A}{(1+V_A^2/c^2)^{1/2}} \tag{5.39}$$

Note that, for plasmas in which $V_A^2 << c^2$ (weak B_o field or high density), (5.39) reduces to $v_{ph} = V_A$. This very low-frequency limit corresponds to the Alfvēn wave discussed in Chapter 15.

5.3 - <u>Wave propagation normal to the magnetic field</u>

Considering now $\underset{\sim}{k} \perp \underset{\sim}{B}_o$, we set $\theta = 90^o$ in Eqs. (3.7), (5.6) and (5.7), to obtain,

$$\underset{\approx}{a} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1 - k^2c^2/\omega^2) & 0 \\ 0 & 0 & (1 - k^2c^2/\omega^2) \end{pmatrix} \tag{5.40}$$

$$\underset{\approx}{b}_e = \begin{pmatrix} (1 - V_{se}^2 k^2/\omega^2) & i\,Y_e & 0 \\ -i\,Y_e & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.41}$$

$$\underset{\approx}{b}_i = \begin{pmatrix} (1 - V_{si}^2 k^2/\omega^2) & -i\,Y_i & 0 \\ i\,Y_i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.42}$$

Taking the inverse of the matrices in (5.41) and (5.42), we obtain for (5.12)(neglecting collisions),

$$\begin{pmatrix} S_{II} & -i\,D_I & 0 \\ i\,D_I & (S_I - k^2c^2/\omega^2) & 0 \\ 0 & 0 & (P - k^2c^2/\omega^2) \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0 \tag{5.43}$$

where

$$S_I = 1 - \frac{X_i\ (1 - k^2V_{si}^2/\omega^2)}{(1 - Y_i^2 - k^2V_{si}^2/\omega^2)} - \frac{X_e\ (1 - k^2V_{se}^2/\omega^2)}{(1 - Y_e^2 - k^2V_{se}^2/\omega^2)} \tag{5.44}$$

$$S_{II} = 1 - \frac{X_i}{(1 - Y_i^2 - k^2 V_{si}^2/\omega^2)} - \frac{X_e}{(1 - Y_e^2 - k^2 V_{se}^2/\omega^2)} \tag{5.45}$$

$$D_I = \frac{X_i \; Y_i}{(1 - Y_i^2 - k^2 V_{si}^2/\omega^2)} - \frac{X_e \; Y_e}{(1 - Y_e^2 - k^2 V_{se}^2/\omega^2)} \tag{5.46}$$

$$P = 1 - X_i - X_e$$

From (5.43) it it clear that E_z is uncoupled from the electric field components E_x and E_y so that the *ordinary mode* (the *transverse* mode which has $E_z \neq 0$ and is not affected by the presence of the magnetostatic field) has the dispersion relation

$$k^2 c^2/\omega^2 = P \tag{5.48}$$

or

$$k^2 c^2 = \omega^2 \; (\omega_{pe}^2 + \omega_{pi}^2) \tag{5.49}$$

which is the same expression obtained in (2.39).

The modes involving the field components E_x and E_y (longitudinal for $E_x \neq 0$ and transverse for $E_y \neq 0$) are seen, from (5.43), to be coupled, and have the following dispersion relation

$$S_{II} \; (S_I - k^2 c^2/\omega^2) - D_I^2 = 0 \tag{5.50}$$

Substituting the expressions for S_I, S_{II} and D_I into (5.50), results in a cubic equation in k^2, showing that in general there are three modes of propagation. A detailed analysis of this dispersion relation shows that these three modes of propagation are the *partially transverse extraordinary wave*, the *longitudinal electron plasma wave* and the *longitudinal ion plasma wave*.

Fig. 10 shows the phase velocity as a function of frequency for the four modes of propagation in a direction normal to the magnetic field. The basic points to be noted in this plot are: (1) the presence of the reflection points at $(\omega_{pe}^2 + \omega_{pi}^2)^{1/2}$, ω'_{01} and ω'_{02}; (2) the transition from a basically longitudinal (electron plasma) wave to a basically transverse electromagnetic (extraordinary) wave, in the frequency region where the phase velocity lies between V_{se} and c; and (3) in the very low-frequency limit the phase velocity of the ion-acoustic wave tends to $\left[(V_A^2 + V_{sp}^2) / (1 + V_A^2/c^2)\right]^{1/2}$

5.4 - Wave propagation in an arbitrary direction

For arbitrary directions of propagation the dispersion relation is given by (5.14). Since a detailed analysis of this dispersion relation is a rather non-instructive and tedious affair, we shall content ourselves by merely

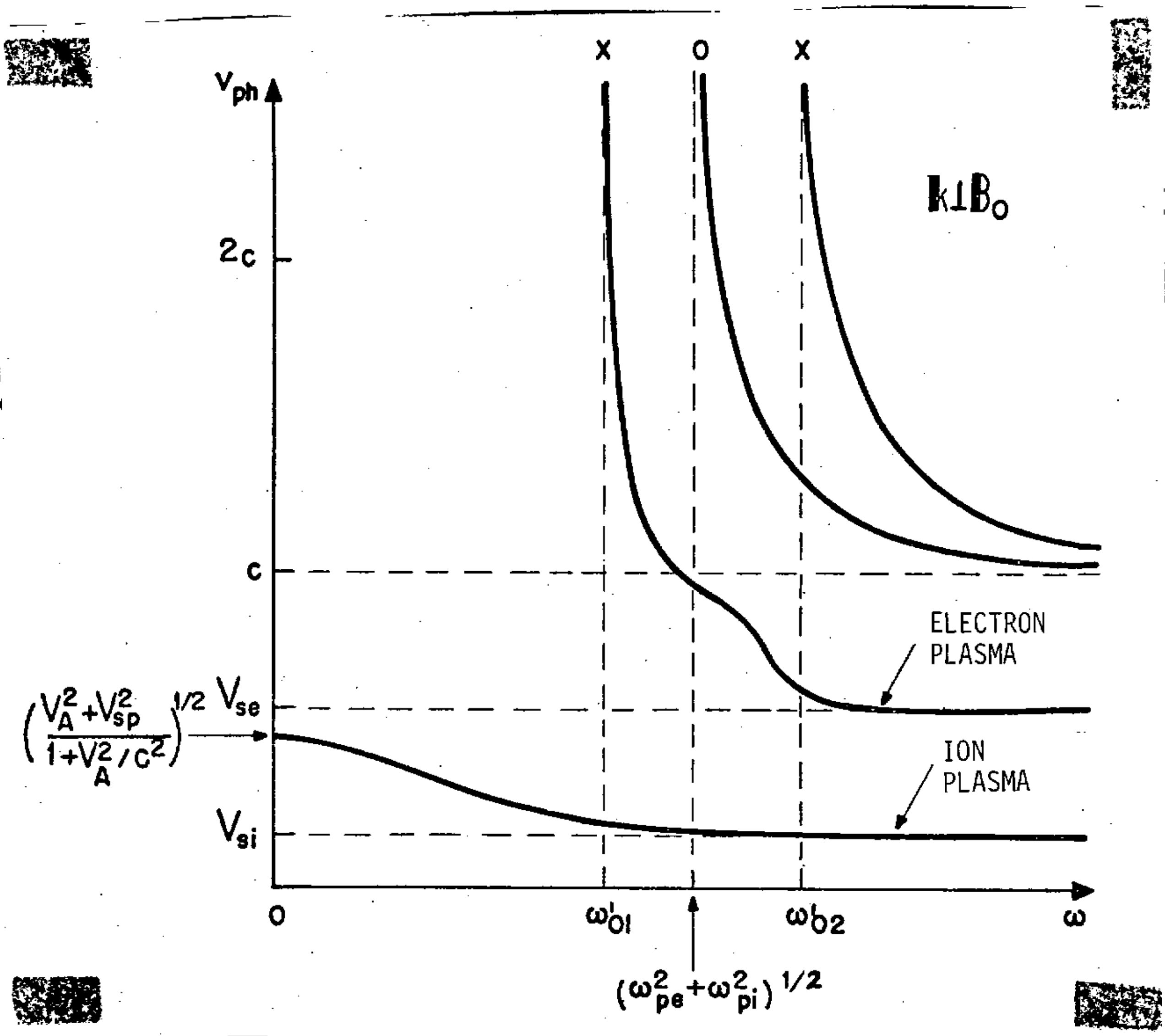

Fig. 10 - Phase velocity as a function of frequency for waves propagating in a direction normal to the magnetic field in a warm fully ionized magnetoplasma.

presenting the plot of phase velocity versus frequency in Fig. 11, in which the shaded areas give an indication of how the curves evolve from $\Theta = 0^0$ to $\Theta = 90^0$.

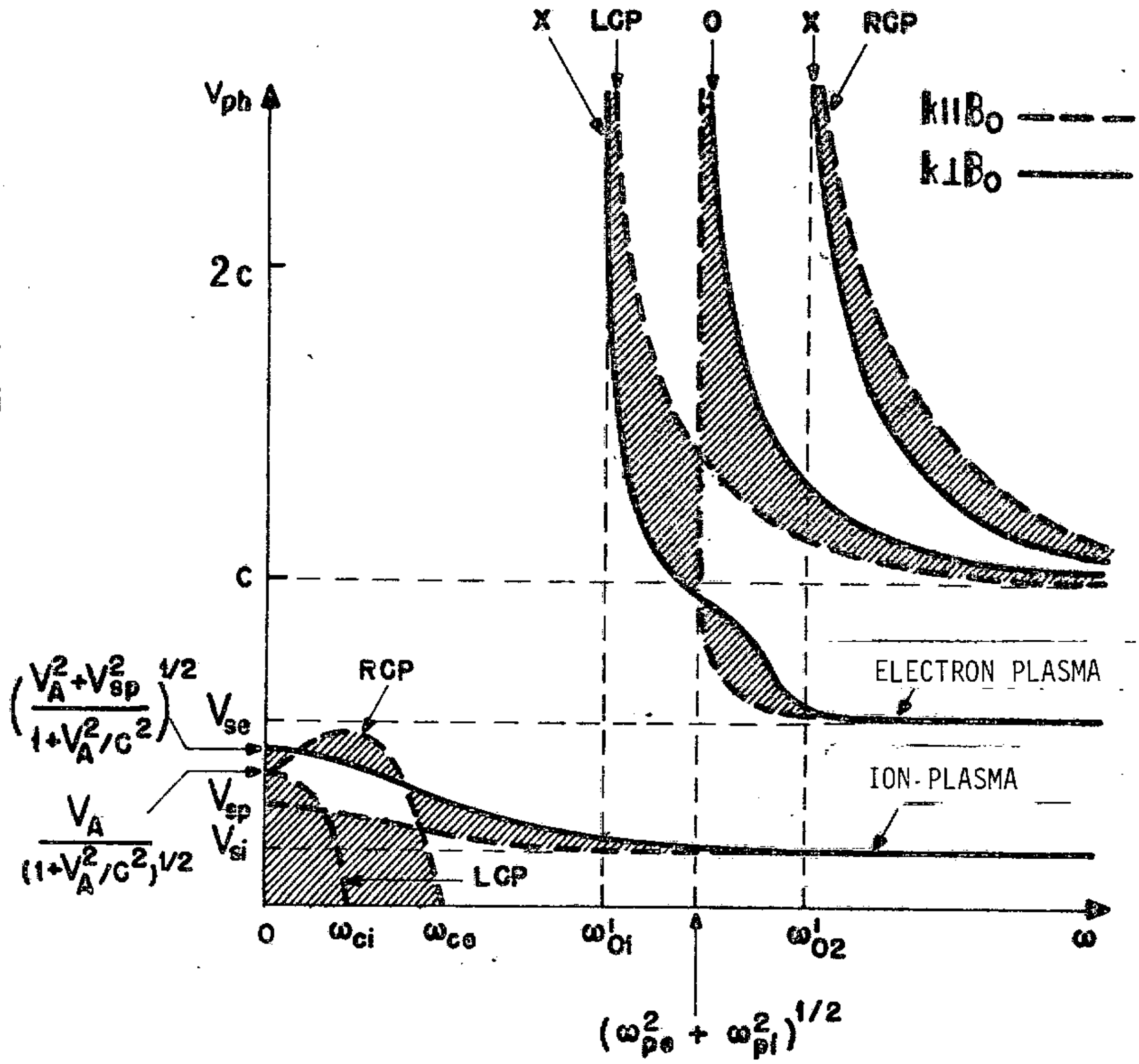

Fig. 11 - Phase velocity as a function of frequency for wave propagation in a warm fully ionized magnetoplasma.

6. SUMMARY

The modes for wave propagation in a *warm* fully ionized plasma can be summarized as follows:

$\underset{\sim}{B}_0 = 0$:

Transverse electromagnetic wave

Longitudinal electron plasma wave

Longitudinal ion plasma wave

$\underset{\sim}{k} \parallel \underset{\sim}{B}_0$:

Transverse right-hand circularly polarized wave

Transverse left-hand circularly polarized wave

Longitudinal electron plasma wave

Longitudinal ion plasma wave

$\underset{\sim}{k} \perp \underset{\sim}{B}_0$:

Transverse ordinary wave

Partially transverse extraordinary wave

Longitudinal electron plasma wave

Longitudinal ion plasma wave

For the case of a warm electron gas, in which the motion of the ions is ignored, the longitudinal ion plasma mode is absent. For the case of a cold plasma, both the ion plasma and the electron-plasma modes are absent. Note that for $\underset{\sim}{k} \perp \underset{\sim}{B}_0$ the electron plasma mode and the extraordinary mode are coupled.

PROBLEMS

17.1 - Show that one of the roots of the dispersion relation (2.33), at very low frequencies, corresponds to an evanescent wave.

17.2 - Make a plot analogous to Fig. 8 for wave propagation in a warm electron gas immersed in a magnetic field, but in terms of ω as a function of the real part of k.

17.3 - Show that the reflection points ω'_{01} and ω'_{02}, for the LCP and RCP waves propagating along $\underset{\sim}{B}_0$ in a fully ionized warm plasma (see Fig. 9) are given, respectively, by

$$\omega'_{01} = \frac{1}{2} \{ - (\omega_{ce} - \omega_{ci}) + [(\omega_{ce} + \omega_{ci})^2 + 4\omega_{pe}^2]^{1/2} \}$$

$$\omega'_{02} = \frac{1}{2} \{ (\omega_{ce} - \omega_{ci}) + [(\omega_{ce} + \omega_{ci})^2 + 4\omega_{pe}^2]^{1/2} \}$$

Compare these expressions with Eqs. (16.6.13) and (16.6.14).

17.4 - Starting from Eqs. (5.12), (5.40), (5.41) and (5.42) provide all the necessary steps to obtain Eq. (5.43).

17.5 - Obtain a cubic equation in k^2, from Eq. (5.50), and analyse the dispersion relations for these three modes of wave propagation across $\underset{\sim}{B}_o$ in a fully ionized warm plasma.

17.6 - Make plots analogous to Figs. 9, 10 and 11 for wave propagation in a fully ionized warm plasma, but in terms of ω as function of the real part of k.

17.7 - Show that the resonances in a warm fully ionized magnetoplasma, neglecting collisions, occur at the frequencies $\omega = \omega_{ce} \cos \theta$ and $\omega = \omega_{ci} \cos \theta$.