The Parcel Method in a Baroclinic Atmosphere

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ABSTRACT

The parcel method of investigating the susceptibility of the atmosphere to particular disturbances involves postulating parcel displacements in an undisturbed environment and deducing the likelihood of such a displacement by estimating the kinetic energy change that would result. On the convective scale, such a method has provided realistic and practical criteria for the existence and intensity of such motions. On the larger scale, theoreticians tend to reject such arguments as heuristic at best.

This paper is an attempt to reconsider the role of parcel theory on different length scales and, in particular, in the application of symmetric instability ideas to frontal rainbands. It is concluded that parcel theory has to be applied with care but that in some situations it can provide a useful—though partial—insight into atmospheric disturbances.

1. Introduction

In parcel theory the motion of a continuous atmosphere is replaced by that of the parcel of interest and the undisturbed environment through which it moves. This implies the neglect of the perturbation pressure forces acting on the parcel. Also, the surrounding air has to allow the parcel to pass without being affected itself; thus, a source of ambient air replaces the parcel at its original location and ambient air disappears at the parcel's new location. Clearly, this model can only be a good representation of nature in special circumstances. The parcel method appears to have been introduced into meteorology by Solberg (1936).

Having accepted these assumptions it is imagined that the parcel is given a displacement and at its new location the resultant force acting is examined to determine whether the parcel will continue to move away from or back toward its original location. This analysis then forms the basis of determining whether a given flow is stable or unstable to such displacements. Notice that the resultant force is computed, consistent with the assumptions, neglecting the pressure gradient force. Furthermore, the parcel, if unstable, can be imagined to move until it reaches an equilibrium level, or point, where the resultant force changes to being unfavorable to further displacement. There may in practice be a range of initial displacement vectors which allow continued displacement and a maximization principle can be invoked to determine the preferred vector. Over the total trajectory the kinetic energy equation can be used to determine the parcel kinetic energy in terms of the structure of the flow through which it has been moved. This parcel kinetic energy will be a maximum for the preferred displacement. The use of kinetic energy acquired by the parcel is in sharp distinction with the use of global energetics for example in the general circulation or in the life cycle of a baroclinic wave. Here, kinetic energy is a local concept, being the integrated resultant force along the parcel trajectory or, equivalently, the total work done on the parcel by the buoyancy and inertial forces.

An obvious feature of a local model such as parcel displacement is that no account is made of the boundary conditions that any geophysical flow must satisfy. In effect, the unstable displacements described by parcel theory will only occur in the atmosphere if the boundary conditions can be accommodated. As will be seen the predictions of parcel theory, which take no explicit account of such conditions, are often remarkably realistic. The example of baroclinic instability is a case in point, where nonlocal dynamics are evidently crucial in a normal mode analysis but where the simple application of parcel theory can give rather accurate predictions over a wide class of unstable flows.

A form of parcel theory has been used in small-scale convection studies with some success. The familiar notion of the positive area on a thermodynamic diagram arises from this application of the theory. On the synoptic scale, Eady (1949) and Green (1960) used the idea of exchanging parcels in an undisturbed environment to illuminate the origin of the energy which the former had shown could be released in baroclinic in-

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stability or large-scale "slope convection." A more recent application of parcel theory to moist symmetric instability occurring in frontal rainbands has been made by Emanuel (1983). This corresponds to the linear conditional symmetric instability described by Bennetts and Hoskins (1979).

This paper will reexamine the use of parcel theory in these cases both to clarify the general method and to assess its usefulness in producing extra dynamical insight. It will be emphasized that the parcel method is not a complete theory, but that its usefulness for gaining insight should not be dismissed.

Before discussing parcel theory, the equations governing the generation of kinetic energy following the motion in a two-dimensional baroclinic flow are derived in section 2. These are necessary background equations as the parcel theory is couched in terms of the change of kinetic energy along a trajectory.

2. Lagrangian kinetic energy equations for a two-dimensional flow

The Boussinesq f-plane equations for frictionless flow independent of the y-direction may be written:

$$\frac{Du}{Dt} - fv + \frac{\partial \phi}{\partial x} = 0$$

$$\frac{Dv}{Dt} + fu = 0$$

$$\frac{Dw}{Dt} + \frac{\partial \phi}{\partial z} - b = 0 \qquad (1)$$

$$\frac{Db}{Dt} = s$$

$$\nabla \cdot \mathbf{u} = 0$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla, \quad \nabla = \left(\frac{\partial}{\partial x}, 0, \frac{\partial}{\partial z}\right) \text{ and } b = g\frac{\theta}{\theta_0}.$$

Consider a basic flow $\bar{\mathbf{u}} = (0, \bar{v}, 0)$ which is fixed in time and in geostrophic balance with a geopotential, and an additional eddy flow $\mathbf{u}' = (u', v', w')$ which is not necessarily of small amplitude. The kinetic energy, K, can be conveniently subdivided in the following way:

$$K = \bar{K} + \tilde{K} + K',$$

where $K = {}^1/_2|\mathbf{u}|^2$ is the total kinetic energy, $\bar{K} = {}^1/_2\bar{v}^2$ is the kinetic energy of the basic flow, $\tilde{K} = \bar{v}v'$ is the term associated with the addition of the eddy field to basic flow, and $K' = {}^1/_2|\mathbf{u}'|^2$ is the kinetic energy of the eddy motion alone. A further subdivision of this "eddy kinetic energy" wil be of use later:

$$K' = K'_L + K'_T,$$

where $K'_L = {}^1/_2 v'^2$ is the kinetic energy of the eddy flow in the longitudinal direction (i.e., that of the mean flow), and $K'_T = \frac{1}{2}(u'^2 + w'^2)$ is the transverse eddy kinetic energy. The transverse eddy kinetic energy is made up of two parts:

$$K'_T = K'_{TX} + K'_{TZ},$$

where $K_{TX} = {}^1/_2 u'^2$ and $K_{TZ} = {}^1/_2 w'^2$.

In Fig. 1 we show the evolution of the various contributions to the kinetic energy following the motion in diagrammatic form. The terms contributing to the rate of change of kinetic energy are associated with either generation, G, or conversion C:

$$\tilde{C} = -u'\frac{\partial \bar{K}}{\partial x} - w'\frac{\partial \bar{K}}{\partial z}$$

$$C'_L = -u'v'\frac{\partial \bar{v}}{\partial x} - v'w'\frac{\partial \bar{v}}{\partial z}$$

$$C'_T = fu'v'$$

$$\tilde{G} = -u'\frac{\partial \bar{\phi}}{\partial x}$$

$$\left.\begin{aligned} G'_{TX} &= -u'\frac{\partial \phi'}{\partial x} \\ G'_{TZ} &= -w'\frac{\partial \phi'}{\partial z} + w'b' \end{aligned}\right\} G'_T = G'_{TX} + G'_{TZ}. \qquad (2)$$

Figure 1 can be converted into equations; for example,

$$\frac{DK'}{Dt} = C'_L - C'_T.$$

It should be noted that $\tilde{K}$, $\tilde{C}$, $\tilde{G}$ are first order in the eddy motion whereas K'_L, K'_T, G'_T, C'_T, C'_L are second order in the eddy motion. Also $\tilde{K}$ is *not* a positive definite quantity. Integrating over a region which has zero eddy motion at its boundaries gives net generation terms:

$$\int (\tilde{G} + \tilde{C})dV = \int \tilde{G}dV = \int w'\bar{b}dV$$

and

$$\int G'_T dV = \int w'b'dV.$$

The pressure force terms in G'_{TX} and G'_{TZ} can only lead to a redistribution of the buoyancy generation term $w'b'$ from K'_{TZ} to K'_{TX}. This provides the only coupling of K'_{TZ} with the other contributions. Indeed, $G'_{TZ} = 0$ if the eddy is hydrostatic and the energy from the buoyancy generation is completely transferred by the action of the pressure force, to K'_{TX}.

It is clear from Fig. 1 that even for a simple basic flow the kinetic energy generation can be rather complicated. The conversion and generation terms are re-

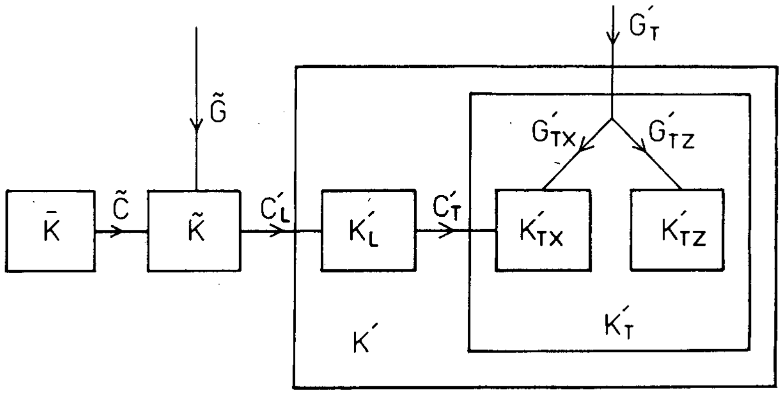

FIG. 1. The Lagrangian kinetic energy conversions, C, and generations, G, in a baroclinic flow in two dimensions. The suffix L indicates longitudinal, T transverse, x the direction perpendicular to the mean flow, and z the vertical direction. The overbar indicates terms associated with the basic flow which is time independent, and the tilde indicates the contribution due to the presence of the eddy flow in the direction of the basic flow. The terms are all defined in Eq. (2).

ferred to by Eliassen and Kleinschmidt (1957) as the rate of change of an "effective potential energy." It is known from that work that the stability of a flow $\bar{v}(x, z)$ can be investigated by minimizing this effective potential energy for a vertical displacement in the *transverse* direction. This is equivalent in our terminology to maximizing the kinetic energy K'_T. This variational technique can be mathematically complex; see Eliassen and Kleinschmidt (1957) and Shutts and Cullen (1987) for details. Eliassen and Kleinschmidt (1957) make the point that in some cases, such as upright convective instability, the results of a simple parcel theory can lead to the same conclusion as the more elaborate method. However, a note of caution is made there that the gross assumption of the parcel method, namely an undisturbed environment through which to displace the parcel, may lead to errors in general. This assumption is equivalent to the neglect of the perturbation pressure forces.

3. Parcel theory for a baroclinic flow

We now consider the parcel method for various examples of baroclinic instability. The discussion will draw together strands of thinking from a variety of other work. In particular, we reexamine the role of pressure forces in symmetric baroclinic instability and relate the parcel model to the normal mode analysis. Various generalizations of the results obtained by Emanuel (1983) will be developed in section 3d after a brief digression in section 3c to consider nonsymmetric baroclinic instability.

a. *Role of the perturbation pressure forces in symmetric instability*

The basic dynamics of symmetric instability in terms of the restoring forces acting due to a small perturbation away from the basic flow were described by Hoskins (1978). These forces imply a symmetrically unstable flow for displacements in a sector including that between the absolute momentum ($\bar{m} = v + fx$) and neutral buoyancy $\bar{\theta}$ curves, when the $\bar{\theta}$ surface is steeper than the $\bar{m}$ surface. No assumption of the smallness of the perturbation pressure forces are made there and indeed we now show that neglect of these forces can only be made for parcel trajectories close to the neutral buoyancy curve.

Scaling arguments suggest that, in general, the components of the perturbation inertia (Coriolis), buoyancy, and pressure gradient forces in the direction of the slantwise motion, i_s, b_s and p_s, respectively, are all comparable. In the normal direction, i_n is negligible for quasi-hydrostatic motion and there is an approximate balance between b_n and p_n.

Consider the special case of a normal mode in an infinite atmosphere with uniform basic flow gradients. Then p_s is identically zero and the resultant force in the direction of motion is $i_s + b_s$. Therefore, the parcel argument that ignores the perturbation gradient force, $\mathbf{p}$ does give the correct work done on the parcel. This is the reason why the parcel model gives the correct stability criteria and implied growth rate of such normal modes (cf. Eliassen and Kleinschmidt 1957, p. 70, italicized paragraph). However, the parcel displacement would not have occurred without p_n providing the balance in the across-flow direction. Hence, the perturbation pressure force is crucial. For a more general basic state but for motion along gently sloping neutral buoyancy surfaces, $\mathbf{p}$ is zero under the hydrostatic approximation. Thus deductions on the basis of neglecting $\mathbf{p}$ may be expected to be valid only for motion close to such surfaces.

It is illuminating to solve for some simple flows which mimic the displacement of a parcel along a given direction. Consider a symmetrically unstable basic flow in which an external body force is applied locally along an $\bar{m}$ surface. It can be shown that initially the motion is along that $\bar{m}$-surface but with competitive growth of the most unstable normal mode along the neutral buoyancy curve. Eventually the latter mode will dominate. Alternatively we can imagine a local region of warm air which could have arisen from a parcel displacement along an $\bar{m}$-surface. The subsequent motion is essentially vertical, in effect returning the parcel back to its neutral buoyancy curve. During this adjustment the perturbation pressure force is as large as the buoyancy force.

In conclusion, although the subsequent development of parcel theory for symmetric disturbances will be couched in terms of an arbitrary parcel displacement, the parcel model can only be expected to give accurate results for displacements close to the neutral buoyancy curve. Numerical simulations show that for symmetric instability the usual parcel trajectories are indeed close to the neutral buoyancy curve (Bennetts and Hoskins 1979; Innocentini 1986; Thorpe and Rotunno 1989). For other trajectories it is necessary to develop a more complete parcel model in which the perturbation pressure forces are assumed to support the displacement field. This field would be continuous, and much more akin to a normal mode structure. The parcel arguments could then determine the maximum possible kinetic energy that can be realized *if* the pressure forces are able to perform their role in redistributing momentum and kinetic energy. We do not pursue this extension to parcel theory here.

b. The relationship of the parcel theory of symmetric disturbances with the normal mode analysis

To demonstrate the relationship between parcel theory and the linear instability analysis we consider here the idealized case in which the gradients of the basic flow are constants. Referring to Fig. 1 and neglecting the perturbation pressure forces, we can write the following equations for the individual components of the eddy kinetic energy:

$$\frac{D}{Dt}K'_{TX} = fu'v'$$

$$\frac{D}{Dt}K'_{TZ} = w'b'$$

$$\frac{D}{Dt}K'_{L} = -u'v'\bar{\zeta} - v'w'\frac{\partial\bar{v}}{\partial z} \quad (3)$$

where $\bar{\zeta} = f + \partial\bar{v}/\partial x$. The generation and conversion terms on the right-hand side can be considered to be rates of change following a parcel by writing, for example,

$$fu'v' = \frac{D}{Dt}\left[\int_{x_0}^{x_1} fv'dx\right]$$

where (x_0, x_1) are the x-coordinates of the parcel's initial and current positions, respectively, and the prime now refers to the difference between the parcel value and that of the basic flow. The integrand is evaluated along the parcel trajectory. Equation (2) can now be rewritten in the form

$$\Delta K'_{TX} = \int_{x_0}^{x_1} fv'dx$$

$$\Delta K'_{TZ} = \int_{z_0}^{z_1} b'dz$$

$$\Delta K'_{L} = -\int_{x_0}^{x_1} v'\bar{\zeta}dx - \int_{z_0}^{z_1} v'\frac{\partial\bar{v}}{\partial z}dz \quad (4)$$

where Δ means the change over the parcel path. The interpretation of these quantities as eddy kinetic energies is preferred to the more ambiguous term "available potential energy" as they are really conversions as well as generations of kinetic energy. To determine the eddy quantities appearing on the right-hand side of Eq. (4) we need to notice that from the y-momentum equation, $m = v + fx$ is conserved following the motion. Defining $\bar{m} = \bar{v} + fx$ as the absolute momentum of the basic flow then $m' = v'$. As the parcel conserves its m we can write the parcel m as $\bar{m}_0 = \bar{v}(x_0, z_0) + fx_0$. So $v' = -(\bar{m} - \bar{m}_0)$ and expanding $\bar{m}(x, z) = \bar{m}_0 + (x - x_0)(\partial\bar{m}/\partial x) + (z - z_0)(\partial\bar{m}/\partial z)$ we obtain

$$\delta K'_{TX} = -f\int_0^{x_1}\left(x\frac{\partial\bar{m}}{\partial x} + z\frac{\partial\bar{m}}{\partial z}\right)dx$$

where the initial location of the parcel has been taken as $x_0 = 0$ for convenience. The use of the δ is a reminder that this formula is only true for small displacements in general (or large displacements in a basic flow with exactly constant gradients). The simplest way to evaluate the integral is to use the angles of the parcel trajectory and the mean flow surfaces to the horizontal; see Fig. 2.

Using δs to denote the distance traveled by the parcel, i.e., $\delta s = (x_1^2 + z_1^2)^{1/2}$ we obtain:

$$\delta K'_{TX} = \frac{1}{2}(\delta s)^2 f\frac{\partial\bar{v}}{\partial z}\frac{\cos\alpha}{\cos\alpha_{\bar{m}}}\sin(\alpha - \alpha_{\bar{m}}) \quad (5)$$

$$\delta K'_{TZ} = \frac{1}{2}(\delta s)^2 f\frac{\partial\bar{v}}{\partial z}\frac{\sin\alpha}{\sin\alpha_{\bar{\theta}_{es}}}\sin(\alpha_{\bar{\theta}_{es}} - \alpha). \quad (6)$$

In Eq. (6) use has been made of the relation $b' = zN_w^2 - xf(\partial\bar{v}/\partial z)$ which follows from the conservation of θ_e by the parcel, N_w^2 being defined as in Emanuel (1983). We have also used the good approximation that the horizontal gradient of θ is equal to that for θ_{es}, the saturated equivalent potential temperature.

From Eqs. (5) and (6) we see that $\delta K'_{TX} \geqslant 0$ if $\alpha \geqslant \alpha_{\bar{m}}$ and $\delta K'_{TZ} > 0$ if $\alpha < \alpha_{\bar{\theta}_{es}}$ and the total transverse kinetic energy $\delta K'_T > 0$ if:

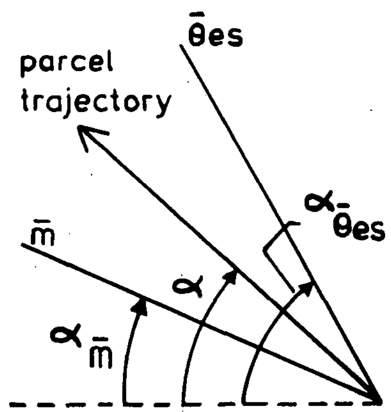

FIG. 2. Orientation of the parcel trajectory and the mean m and Θ_{es} surfaces in the x–z plane, showing the relevant angles to the horizontal. The mean flow gradients are taken to be constant or the parcel displacement to be small.

$$\left[1+\left(1-\frac{\tan\alpha_{\bar{m}}}{\tan\alpha_{\bar{\theta}_{es}}}\right)^{1/2}\right]\tan\alpha_{\bar{\theta}_{es}} \geqslant \tan\alpha$$

$$\geqslant \left[1-\left(1-\frac{\tan\alpha_{\bar{m}}}{\tan\alpha_{\bar{\theta}_{es}}}\right)^{1/2}\right]\tan\alpha_{\bar{\theta}_{es}}$$

which includes the sector $\alpha_{\bar{\theta}_{es}} \geqslant \alpha \geqslant \alpha_{\bar{m}}$. Furthermore, adding the two equations we obtain

$$\delta K'_T = \frac{1}{2}(\delta s)^2\sigma^2 \qquad (6a)$$

where σ is the growth rate of a normal mode as shown in Bennetts and Hoskins (1979). Thus, the transverse kinetic energy is maximized for a parcel displacement along the direction of the most unstable linear mode. Hence, the criteria for linear growth is the same as that for the increase in transverse eddy kinetic energy upon parcel displacement.

Furthermore, the longitudinal eddy kinetic energy can be found directly or from Eq. (4) since $v' = -(\bar{m} - \bar{m}_0)$:

$$\delta K'_L = \int_{\bar{m}_0}^{\bar{m}_1} (\bar{m} - \bar{m}_0) d\bar{m} = \frac{1}{2}(\bar{m}_1 - \bar{m}_0)^2. \qquad (7)$$

This component of the eddy kinetic energy depends only on the differences of the m of the parcel and that of the surrounding air.

Finally, we note that simple expressions can be found by defining the moist Richardson number $R_w = N_w^2/(\partial\bar{v}/\partial z)^2 = N_w^2 z_1^2/(\Delta\hat{v})^2$, and by taking the parcel displacement to be along the neutral buoyancy curve, i.e., $\alpha = \alpha_{\bar{\theta}_{es}}$. We then obtain

$$\frac{\delta K'_T}{\frac{1}{2}(\Delta\hat{v})^2} = R_w\left(1 - R_w\frac{\bar{\zeta}}{f}\right)$$

$$\frac{\delta K'}{\frac{1}{2}(\Delta\hat{v})^2} = \left(1 - R_w\frac{\bar{\zeta}}{f}\right)\left(1 - R_w\left(\frac{\bar{\zeta}}{f} - 1\right)\right). \qquad (8)$$

Note that $\Delta\hat{v}$ is the vertical difference in the mean flow over the trajectory. Both the total and the transverse kinetic energy change are positive if $R_w < f/\bar{\zeta}$, which is the criteria for the linear growth of disturbances. However, $\Delta K'_T$ is a maximum for $R_w = f/2\bar{\zeta}$ while $\Delta K'$ is a maximum for $R_w = 0$. These differences are due to the important contribution, most apparent at zero Richardson number, from the eddy kinetic energy of the along-band flow. These calculations suggest that any estimation from soundings of the kinetic energy of moist symmetric motions may be a considerable underestimate if $\delta K'_T$ is used. This leads to the somewhat unfamiliar idea that the stability of the flow is determined by $\delta K'_T$ but the total eddy kinetic energy is $\delta K'$. A more quantitative example of these differences will be given in the next section.

c. Nonsymmetric baroclinic instability

As an aside from our main discussion of symmetric disturbances it is illuminating to consider the same techniques used to describe the nonsymmetric forms of baroclinic instability. In the Introduction, it was noted that Eady (1947) and Green (1960, 1984) interpreted baroclinic instability in terms of slope convection. The parcel model gives an interpretation of the instability which circumvents some of the mathematical details of the linear analysis. In essence, a two-dimensional mode, which is oriented at right angles to the symmetric axis or in the x-direction in our notation, bypasses the angular momentum constraint by introducing longitudinal pressure gradients. Thus the y-momentum equation becomes a statement of quasi-geostrophy $u' = -f^{-1}\partial\phi'/\partial y$ rather than of conservation of angular (or absolute) momentum, m, as in the symmetric mode.

Thus consider, in Fig. 3, a parcel moving "north" in the negative x direction. The tendency of the parcel to turn to the east due to the Coriolis force is balanced exactly by the longitudinal pressure gradient. Using the similar argument for a parcel moving south leads to the necessity of low pressure *between* the two parcels. (The symmetric mode, on the other hand, has zero longitudinal pressure gradient so parcels must conserve m as they move in the transverse direction.) The motion of these two parcels when projected on a transverse-vertical section is then as shown in the left-hand side of Fig. 3. Note that the parcel motion described in Fig. 3 is somewhat different in concept from that described by Green (1984), where the notion of parcel *exchange* is used. Such an exchange is misleading as parcels A and B *must* be at different longitudes as just described and as noted by Green (1984). We prefer to imagine parcels A and B as always being distinct but moving at the same angle α to the mean isentropes.

With the breaking of the angular momentum constraint, Eq. (5) is no longer valid for $\delta K'_{TX}$. However, within the context of the parcel argument, (6) is still valid for $\delta K'_{TZ}$, and in the manner of Green (1984)

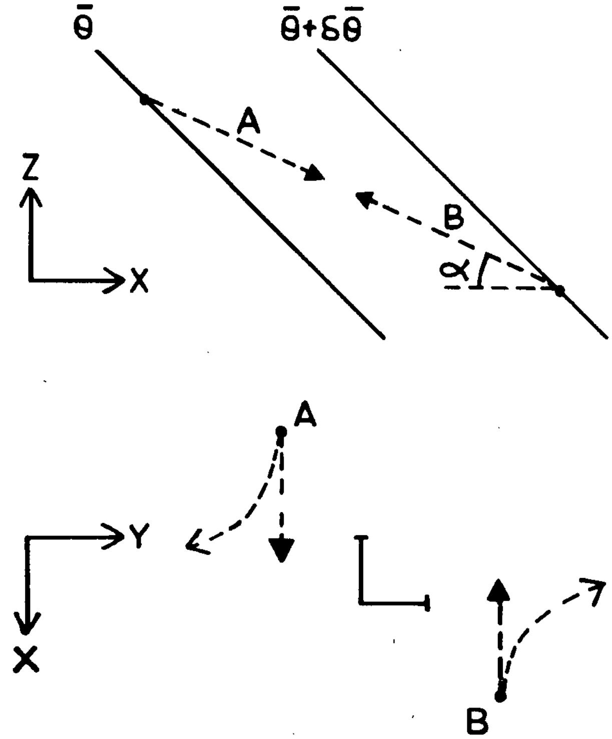

FIG. 3. Parcel trajectories in baroclinic instability. Here A and B are two parcels at different y and θ, starting at different positions in the x–z plane but occupying the same x–z location at some later time: (a) a vertical x–z section showing the basic isentropes and the projections of the trajectories into the section; (b) a plan view on the plane of the trajectories. In the absence of pressure gradients in the y-direction, the trajectories would curve under the action of the Coriolis force as shown by the paths with the open arrowheads. With low pressure (shown by the L) between the trajectories, the motion can be straight and in geostrophic balance.

the trajectory angle α that maximizes this expression can be determined. For unsaturated ascent this gives $\alpha = \alpha_{\bar{\theta}}/2$, i.e., parcel displacements at half the angle of the isentropes, and

$$\delta K'_{TZ}|_{\max} = \frac{1}{4}(\delta s)^2 f \frac{\partial \bar{v}}{\partial z}\tan(\alpha_{\bar{\theta}}/2).$$

The y component of the pressure gradient force has already entered the argument through its role in circumventing the angular momentum constraint. As discussed in section 3a, the fact that the pressure gradient comes into the argument is consistent only with an extention of the notion of individual parcel displacements to that of a continuous field of particle displacements. Again, the role of the (x, z) pressure force, neglected up to this point, is to make the energy generated by the buoyancy realized as $\delta K'_{TX} = \frac{1}{2}(\delta\dot{x})^2$. Further, $\alpha_{\bar{\theta}} \ll 1$ and $\delta s \approx \delta x$, so that

$$\frac{1}{2}(\delta\dot{x})^2_{\max} \approx \frac{1}{4}(\delta x)^2 f \frac{\partial \bar{v}}{\partial z}\frac{\alpha_{\bar{\theta}}}{2} = \frac{1}{2}(\delta x)^2\left[\frac{g}{\theta_0}\frac{\partial \theta}{\partial x}\Big/ 2N\right]^2.$$

Assuming a fixed modal structure, this implies a maximum growth rate:

$$\sigma_{\max} \approx 0.5\left(\frac{g}{\theta_0}\frac{\partial \bar{\theta}}{\partial x}\right)\Big/ N. \quad (9)$$

This estimate of the growth can be compared, for example, to the one from the more complete linear analysis of Eady (1947), which gives the coefficient to be 0.31 rather than 0.5. Lindzen and Farrell (1980) showed that this formula with approximately the same coefficient is remarkably accurate for the Charney model and more general baroclinic instability problems.

A very important aspect of this parcel model is that *no* explicit reference has been made to the boundaries of the motion in the z-direction or to interior potential vorticity gradients. However, the *normal-mode* theory suggests that these ingredients are crucial for the growth to be maintained, but gives maximum growth rates in accord with the parcel theory argument that neglects them.

Normal mode growth is not the only way for baroclinic development to occur. The initial-value calculations described by Farrell (1982) suggest that the "parcel growth rates" as given in Eq. (9) can be sustained for a limited, but not negligible, period even if normal mode growth is not possible. Such nonnormal mode development can be seen in the atmosphere when a finite amplitude upper tropospheric trough or other PV anomaly propagates into a region of tropospheric baroclinity; in that it shows the possibility of eddy energy growth in such a case, the parcel model of baroclinic instability is in some ways more general than the linear normal mode theory. However, it does not prove the existence of such motions in the atmosphere, merely the susceptibility of the atmosphere to such unstable displacements. The linear normal mode theory is important in showing consistent mathematical solutions in which such energy release occurs. Parcel theory can in no way provide a substitute for the potential vorticity view of the development of weather systems (Hoskins et al. 1985) but it is useful in providing an alternative perspective. Indeed, it illustrates the balanced nature which underlies the application of "potential vorticity thinking."

d. Parcel model of conditionally unstable symmetric motion

The method of parcel theory is applied here to a general basic flow without the restriction of uniform gradients. It is important to consider the possibility that the parcel can be lifted part or all of the way from its lifting condensation level to its level of neutral buoyancy. These two levels denote the finite vertical extent of the unstable layer. For symmetric motions these two equilibrium levels become equilibrium points because the parcel not only has zero buoyancy there but also it has zero anomalous m. This configuration

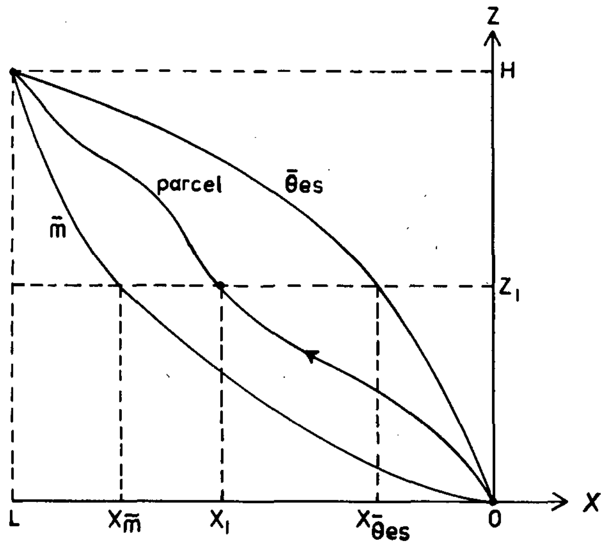

FIG. 4. Parcel trajectory and mean flow surfaces in a confined symmetrically unstable layer. Note that $\bar{\theta}_{es}$ and $\bar{m}$ are taken to increase in the positive x-direction. The horizontal coordinates of the various surfaces at the height, z_1, are defined.

is illustrated schematically in Fig. 4, where a general parcel trajectory is shown lying between the basic flow $\bar{m}$ and $\bar{\theta}_{es}$ surfaces. In this section we follow the methodology of Emanuel (1983) in an attempt to generalize those results and to explicitly evaluate the various contributions to the total eddy kinetic energy.

Following the analysis of section 3a, the expressions for eddy kinetic energy given in Eq. (4) will be used remembering that the neglect of perturbation pressure forces is only valid if the parcel displacement is close to the neutral buoyancy curve, i.e., if $|x_1 - x_{\bar{\theta}_{es}}|/|x_{\bar{\theta}_{es}}| \ll 1$, from Fig. 4. In this section we will first obtain expressions for $\Delta K'_T$ and $\Delta K'$ before finding approximate forms for the individual contributions ($\Delta K'_{TX}$, $\Delta K'_{TZ}$) which can be evaluated from data obtained at a single station. The following expression for v' follows from parcels conservation of m:

$$v' = -fx - (\bar{v} - \bar{v}_0) \tag{10}$$

where $\bar{v}_0 = \bar{v}\,(x = 0, z = 0)$ at the initial position of the parcel. It is convenient to relate the parcel buoyancy to the initial x location of the parcel, $x = 0$, by a first order expansion:

$$b' = \hat{b}' - xf\frac{\partial \bar{v}}{\partial z} \tag{11}$$

where the caret is the value obtained imagining the parcel to rise vertically at $x = 0$; there is no assertion here that the actual parcel follows a vertical path. Note that while b' is small, $\mathbf{b}'$ is typically substantially negative.

From Eq. (4) we find upon substitution of Eqs. (10) and (11) that the *total* eddy kinetic energy at a general point (x_1, z_1) along the trajectory is given by

$$\Delta K' = \int_0^{x_1} fx\frac{\partial \bar{v}}{\partial x}dx + \int_{\bar{v}_0}^{\bar{v}_1}(\bar{v} - \bar{v}_0)d\bar{v} + \text{CAPE}$$

where $\bar{v}_1 = \bar{v}\,(x = x_1, z = z_1)$ is the mean flow *on the parcel trajectory* and CAPE $= \int_0^{z_1}\hat{b}'dz$ as defined from a thermodynamic diagram. Defining $\Delta\bar{v} = \bar{v}_1 - \bar{v}_0$, we can relate this to the vertical change in $\bar{v}$ at $x = 0$ again assuming small horizontal gradients of vorticity:

$$\Delta\hat{\bar{v}} = \Delta\bar{v} - x_1\frac{\partial \bar{v}}{\partial x}.$$

Also $\Delta\hat{\bar{v}} = -\zeta x_{\bar{m}}$ so that Eq. (11), with the assumption of constant $\partial\bar{v}/\partial x$ over the horizontal extent of the trajectory, becomes

$$\Delta K' = \text{CAPE} + \zeta^2\frac{x_{\bar{m}}^2}{2} - \zeta\frac{\partial \bar{v}}{\partial x}\frac{x_1}{2}(2x_{\bar{m}} - x_1). \tag{12}$$

Note that $x_{\bar{m}}$ is known from single station data *and* an estimate of the mean vorticity ζ. A good estimate for x_1 is that found by assuming the parcel trajectory to be close to the neutral buoyancy curve, i.e., $x_1 = x_{\bar{\theta}_{es}}$ where $x_{\bar{\theta}_{es}} = \hat{b}'_1/[f(\partial\bar{v}/\partial z)]$, is evaluated at $z = z_1$.

Using Eq. (7), which is true in general as well as for uniform gradients, and the above approximations, we obtain

$$\Delta K'_L = \frac{\zeta^2}{2}(x_1 - x_{\bar{m}})^2 \tag{13}$$

and thus the transverse kinetic energy is obtained by subtracting Eq. (13) for Eq. (12):

$$\Delta K'_T = \text{CAPE} + f\frac{\zeta}{2}x_1(2x_{\bar{m}} - x_1). \tag{14}$$

Notice that over the total trajectory to the point of neutral buoyancy:

$$x_1 = x_{\bar{m}} = x_{\bar{\theta}_{es}} = \text{L}, \quad z_1 = H, \quad \text{and}$$

$$\Delta K' = \Delta K'_T = \text{CAPE} + f\zeta\frac{x_{\bar{m}}^2}{2} = \text{SCAPE}$$

where SCAPE has been defined by Emanuel (1983). The kinetic energy associated with the longitudinal eddy motion peaks around the midlevel of the cloud and is zero at the condensation level and the level of neutral buoyancy. As already noted at the midlevel, $\Delta K'_L$ can be a substantial contribution to $\Delta K'$.

The individual components of $\Delta K'_T$ can be separately evaluated using the method described here. A convenient way to write them is as follows:

$$\Delta K'_{TX} = f\zeta\frac{x_1}{2}(2x_{\bar{m}} - x_1) + \text{CAPE} - f\int_0^{z_1}\frac{\partial \bar{v}}{\partial z}(x_{\bar{\theta}_{es}} - x)dz$$

$$\Delta K'_{TZ} = f\int_0^{z_1}\frac{\partial \bar{v}}{\partial z}(x_{\bar{\theta}_{es}} - x)dz. \tag{15}$$

Note that it is plausible to take $x = x_{\bar{\theta}_{es}}$ for a realistic trajectory making the integral zero. Otherwise, the integral cannot be found in closed form unless $x_1(z)$ is known. Equation (14) is a general form of Eq. (25) of Emanuel (1983), which was derived for a simplified mean atmospheric structure.

Care needs to be taken if the eddy kinetic energy is to be evaluated at a point on the trajectory below cloud top. As shown by Emanuel (1983), the transverse kinetic energy $\Delta K'_T$ is only dependent on the end points of the trajectory and not on the path of the trajectory itself. It is convenient then, if the end points are cloud base and cloud top, to evaluate Eq. (14) with $x_1 = x_{\bar{m}}$. As an example, imagine a cloud layer of depth 4 km, midlayer values of $\hat{b}' = g(-2K)/\theta_0$, $\partial\bar{v}/\partial z = 10^{-2}$ s^{-1}, and $\zeta = 1.5f$. Then $x_{\bar{\theta}_{es}} = -67$ km and $x_{\bar{m}} = -133$ km and from Eq. (14) we find that

$$\Delta K'_T(\bar{\theta}_{es} \text{ trajectory}) = \text{CAPE} + 100 \text{ m}^2 \text{ s}^{-2},$$

$$\Delta K'_T(\bar{m} \text{ trajectory}) = \text{CAPE} + 132 \text{ m}^2 \text{ s}^{-2}.$$

As noted by Emanuel (1983) there is more kinetic energy, for a fixed *vertical* displacement, in the $\bar{m}$ trajectory. However, it is apparent both from linear theory and numerical simulations that parcels ascend close to the neutral buoyancy curve, which indicates the importance of the perturbation pressure forces, neglected in these calculations of $\Delta K'_T$, for $\bar{m}$ surface trajectories. It is also clear, from Eq. (6a), that for a displacement of fixed (slantwise) length that the maximum kinetic energy is produced for the trajectory along the same direction as the slope of the mode of fastest linear growth rate which is very close to a $\bar{\theta}_{es}$ trajectory.

Taking a neutral buoyancy trajectory, the longitudinal kinetic energy in this example can be found from Eq. (13) to be $\Delta K'_L = 50$ m^2 s^{-2}. This is equivalent to a longitudinal eddy flow of $|v'| \sim 10$ m s^{-1}. It seems likely that mesoscale observations of frontal rainbands can most simply be used to assess the validity of the slantwise convection theory by finding v'; stable condensation at fronts or indeed upright convection would have a signature of $v' \approx 0$ in the cloud layer. The above figures for $\Delta K'_L$ imply that an ageostrophic along-front flow of 10 m s^{-1} is typical which is well capable of being detected in field data.

4. Other applications of the parcel method

In an attempt to draw together other applications of the parcel method we now consider its use in small-scale convectively unstable flow and the "extension" to parcel theory described by Green et al. (1966).

a. Parcel model of upright convection

It is informative to return to the more familiar form of convective instability; namely, that which occurs on a small horizontal scale of 50 km or less so that eddy Coriolis forces can be neglected. From the discussion of section 3a it is apparent in this case that parcel trajectories have significant buoyancy so that the neglect of the perturbation pressure forces is *not* a good approximation. However, in kinetic energy terms, the buoyancy term in G'_T feeds directly into the kinetic energy of the vertical motion K'_{TZ}. This is consistent with the amplification of a vertical displacement, requiring no redistribution by the pressure term. Thus, simple parcel arguments may be expected to be more successful in giving information about vertical stability than about horizontal momentum transports. We now illustrate these points by applying the theory developed in section 3.

Suppose, for simplicity, that the parcel theory is to be applied to a two-dimensional squall line oriented *across* the thermal wind, i.e., along the x-axis. Neglecting Coriolis forces, we can set $u' = 0$ and $\partial/\partial y = 0$ in the kinetic energy equations. A description of the nonlinear dynamics of such squall lines is given in Thorpe et al. (1982). From Eq. (4) the only nonzero terms are

$$\Delta K'_{TZ} = \int_{z_0}^{z_1} b' dz$$

$$\Delta K'_L = -\int_{z_0}^{z_1} v' \frac{\partial \bar{v}}{\partial z} dz$$

where it has been assumed that the basic flow has zero relative vorticity. The y-momentum equation becomes a statement of the conservation of v (since we are using the parcel hypothesis of the neglect of the pressure forces). Consequently, for a parcel displacement from (y_0, z_0) to (y_1, z_1) we have $v' = -(\bar{v} - \bar{v}_0)$ where $\bar{v}_0 = \bar{v}(y_0, z_0)$. Consequently, the expression for $\Delta K'_L$ can be integrated exactly giving

$$\Delta K'_{TZ} = \text{CAPE} \quad \text{and} \quad \Delta K'_L = \frac{1}{2}(\bar{v}_1 - \bar{v}_0)^2$$

where $\bar{v}_1 = \bar{v}(y_1, z_1)$. Note that as in the previous example of slantwise convection the stability is determined by $\Delta K'_T$ alone; consequently, in this case unstable motion occurs if CAPE > 0 or there is positive area on a thermodynamic diagram. For a typical midlatitude sounding $\bar{v}_1 - \bar{v}_0 \sim 30$ m s^{-1} so, given the neglect of pressure forces, it is obvious that the horizontal outflow at anvil level will represent a 30 m s^{-1} deficit compared to the mean flow at that level. Horizontal momentum deficits are features of severe storms but are not consistent with conservation of momentum as envisaged in this simple parcel model. It is well known that horizontal momentum is not conserved in a severe storm updraught as the horizontal pressure forces are crucial.

For upright convection then, although the estimate for the vertical eddy kinetic energy is useful in providing an upper limit to the intensity of convection, the parcel theory is extremely poor for estimating the horizontal component. The steady energetics model of Moncrieff and Green (1972) allows a proper inclusion of the

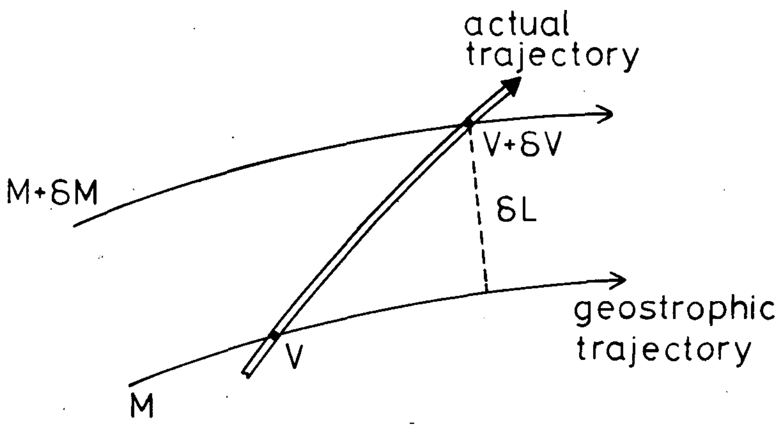

FIG. 5. Isentropic trajectory of an air parcel in synoptic-scale flow (open arrow). Contours of the Montgomery streamfunction, M, are shown by solid lines. The perpendicular distance of the parcel from its geostrophic trajectory position is shown by a dashed line, δl.

pressure forces but, as will be described, abandons parcel theory in favor of the steady flow hypothesis. This hypothesis can also be called into question for a severe storm, although for certain long-lived squall lines the model is reasonably realistic; see Thorpe et al. (1982).

b. "Extended" parcel theory

A totally different concept underlies the "extended parcel theory" which Moncrieff and Green (1972), for example, have applied to cumulonimbus convection and Green et al. (1966) to synoptic scale flow. In the latter case the atmosphere is taken to be continuous and the flow is assumed to be along steady, known trajectories. There is no split into parcel and environment. A conserved energy, akin to the Bernoulli function in fluid mechanics, is used to determine flow speed along the trajectories. The pressure field can be taken into account if the hydrostatic approximation is applicable. There is really only a superficial relationship between this "extended" theory and ordinary parcel theories which arise due to their common use of the kinetic energy equation. The former case is perhaps better described, following Moncrieff (1978), as the steady theory of cumulonimbus convection. This theory uses the, albeit steady, equations of motion to determine a complete flow structure in two-dimensional situations. Detailed observations and/or numerical model data from initial value simulations are needed to say whether any of the supposed flow regimes examined with the steady theory can arise in nature. We consequently make the point that in either case the term "extended" parcel theory is a misnomer.

To illustrate some of the difficulties of applying the steady, energetics model we return to the analysis of Green et al. (1966) and Betts and McIlveen (1969). In these papers the important point was made that in a midlatitude cyclone the air flow is often part of two anticyclone circulations. In the eastern one, warm air rises moist adiabatically from the subtropics, moving northward and eastward while in the other anticyclone, dry, cold air descends, moving towards the south and west. These two circulations meet in a frontal zone, which appears as a narrow cyclonic vortex sheet. This description arose from trajectory analysis. They further use the conserved Bernoulli function in steady motion to estimate the increase in kinetic energy of the air as it progresses northward. A practical difficulty of applying the technique is apparent if we take the kinetic energy equation in isentropic coordinates for (dry) adiabatic frictionless motion:

$$\frac{D}{Dt}\left(\frac{1}{2}|\mathbf{u}|^2 + M\right) = \frac{\partial M}{\partial t}$$

where $M = c_pT + \phi$ is the Montgomery function. In steady motion $|\mathbf{u}|^2/2 + M$ is constant along trajectories. Consider the *almost* geostrophic motion illustrated in Fig. 5. The actual trajectory crosses the M surfaces which themselves would be trajectories if the flow were geostrophic. Using the conserved quantity along the actual trajectory,

$$\frac{1}{2}(V + \delta v)^2 + M + \delta M = \frac{1}{2}V^2 + M$$

which leads, to first order, to the equality

$$V\delta v + \delta M = 0.$$

By definition, for almost geostrophic motion, $fv \approx |\nabla M|$. Also, $\delta M = -|\nabla M|\delta l$ where δl is the distance perpendicular to the M surfaces between the actual and geostrophic trajectories. Therefore, we find that

$$\delta l = +\delta v/f.$$

As is directly apparent from the equations of motion, the ageostrophic displacement of the trajectory is related to the increase in flow speed. Green et al. (1966) use the moist equivalent of this analysis, specifying the trajectory and obtaining the implied flow speed. However, we see from the above that an error of only 100 km in the trajectory is equivalent to an error in flow speed of about 10 m s^{-1}. The ageostrophic displacement is, in general, much more difficult to find from data than the wind speed. Synoptic data with resolution of order 300 km seems too coarse to use this technique with any accuracy. In fact, Danielsen (1961) used a similar kinetic energy equation to *correct* the supposed trajectories rather than to predict the final flow speed.

5. Discussion

The calculation of the generation of eddy kinetic energy in a simple two-dimensional baroclinic flow is one method with which to examine the susceptibility and structure of instabilities developing in the flow. The aim of this paper has been to draw together several studies addressing this approach in synoptic, mesoscale, and convective dynamics. Figure 1 indicates, without approximation, the various generation and conversions occurring in the presence of a developing eddy of, possibly, finite amplitude. The complexity of this energy diagram suggests that, in general, models of eddy development based on energetics will rarely provide a simpler or clarifying view compared to models based on the equations of motion. However, the criteria for the susceptibility of general flows to instabilities uses a variational technique based on the kinetic energy equation [see Fjortoft (1946) and Eliassen and Kleinschmidt (1957)]. Here we have explored a simpler and approximate form of that approach known as parcel theory in the usual form of which the perturbation pressure forces are neglected. We now summarize the points made in this paper:

- Simple parcel theory relies on an abandonment of the continuum hypothesis to replace the fluid by *two* entities—the parcel and the undisturbed environment through which it moves. A good example of the radical nature of this hypothesis is in the case of symmetric instability. The parcel model assumes the parcel to ascend, and reach its neutral buoyancy point where it will perform inertial "circles" until the excess kinetic energy is dissipated by smaller scale mixing; see Fig. 6 for a schematic of the motion of the parcel relative to the mean flow. In contrast, a linear normal mode will predict that as some parcels rise, there is compensating descent, producing the characteristic roll circulation (Bennetts and Hoskins 1979). In upright moist convection the differences in the two approaches may not be so serious as the air ascending to cloud top can flow out freely horizontally; any compensating descent taking place on the scale of the Rossby radius of deformation.
- In general the notion of single parcel displacements being enhanced is inconsistent, as this is possible only if a pressure field exists. Parcel displacements must then be considered as a continuous field, and estimates obtained from the kinetic energy equation provide ideas of unstable structures and their growth rates subject to the possibility of the existence of the necessary pressure field.
- The "extended" parcel theory discussed by Green et al. (1966) and used by Moncrieff and Green (1972) is very different from even the extension of parcel theory described here. Its use on the synoptic scale appears limited. If further dynamical constraints are used it can describe the steady aspects of cumulonimbus convection.
- A consideration in assessing the realism of the parcel method is that it can only predict cloud top height if a parcel traverses the cloud layer in a short time compared with changes in the parcel environment. For moist symmetric motion the time for the parcel

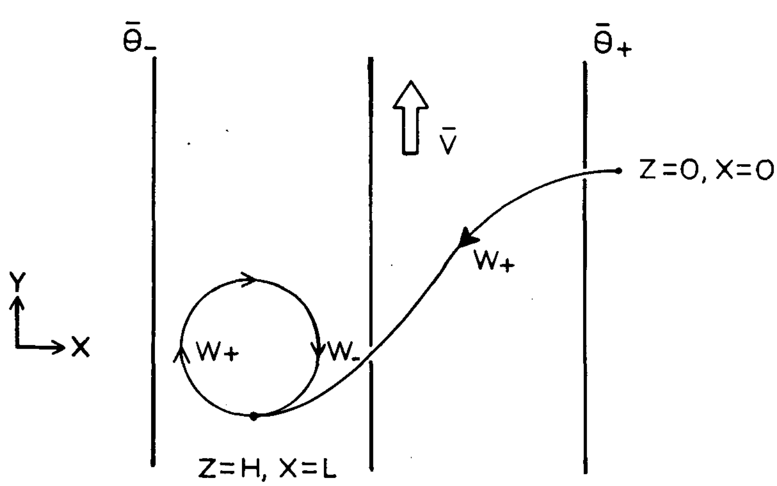

FIG. 6. Plan view of the relative trajectory of a parcel ascending in a moist symmetric cloud. The height of the trajectory and the sense of the vertical velocity are indicated by labels z and w, respectively. Isentropes and the thermal wind are also shown. The radius of the inertial "circle" is $O(u'/f \approx 100$ km) and it is elongated in the y-direction by the factor $(\zeta/f)^{1/2}$.

ascent might be a few hours while for upright convection this time might be 20 min. It is clear that *this* assumption of parcel theory is more justified for upright convection than for symmetric instability.

• Parcel theory gives a description of the growth by baroclinic processes which applies to a wide class of situations. It is however unable to provide a complete dynamical picture.

• For moist symmetric instability the parcel theory can be used only because fluid moves approximately along neutral buoyancy curves; thus, the perturbation pressure forces are small. The kinetic energy generation equations include not only the generation of kinetic energy of the transverse motion but also of the along-band or longitudinal motion. The stability of the flow to moist symmetric disturbances can be estimated by lifting parcels in the transverse direction and determining whether the eddy transverse kinetic energy increases. However, the total eddy kinetic energy is, in general, substantially greater than the transverse kinetic energy. The ageostrophic along-band motion may be the clearest signature of moist symmetric instability in mesoscale observations of frontal rainbands as it is known that the frontal-scale flow is closely in geostrophic balance in this direction. In contrast, the transverse ageostrophic motion of the frontal rainbands may be exceedingly difficult to disentangle from the ageostrophic cross-frontal circulation.

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