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THE NEUTRAL PARTICLE DENSITY FIELD AROUND A SATELLITE

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14. Abstract/Notes

The determination of the density field behind a satellite is of importance for calibrating satellite mounted probes. The free molecular theory of rarefied gas dynamics is applied to claculate the neutral particle density field behind a spherical satellite. The characteristics of the Boltzmann equation are obtained and through them the expression for the number density is derived in the form of an integral. This has been evaluated numerically. The density field, while being nearly spherically symmetric at the front, shows considerable angular dependence in the wake.

15. Remarks *This work was presented in the 34th Annual Meeting of SBPC from July 6, 1982 to July 14, 1982 in Campinas, São Paulo*

CONTENTS

LIST OF FIGURES

Page

1 - INTRODUCTION

The perturbations in the ambient flow created by a satellite is of importance in calibrating satellite mounted probes and in determining atmospheric densities. The problem is also of fundamental engineering and scientific interest. Here, the problem of mapping the density field around a spherical satellite moving with a constant velocity is analyzed. In earlier works (Dolph and Weil, 1959; Kiel, 1966), the problem has been solved based on geometrical considerations. In this work, the method of characteristics (Venkataraman, 1980) has been used to solve the collisionless Boltzmann equation. The density field is obtained by integrating the distribution function in the velocity space. An analytical expression for the number density is provided.

2 - THE FORMULATION AND SOLUTION

The geometry considered is shown in Figure 1. A spherical satellite of radius R is moving with a constant velocity U in a rarefied atmosphere. It is assumed that at large distances from the satellite the gas is in equilibrium with a Maxwellian distribution. It is also assumed that the intermolecular collisions can be neglected. The governing equation is the collisionless Boltzmann equation, which can be written in a coordinate system fixed to the satellite as (Kogan, 1969)

$$c \cos \beta \frac{\partial f}{\partial r} + \frac{c}{r} \sin \beta \cos \gamma \frac{\partial f}{\partial \theta} - \frac{c}{r} \sin \beta \frac{\partial f}{\partial \beta}$$

$$- \frac{c}{r} \frac{\sin \beta \sin \gamma}{\tan \theta} \frac{\partial f}{\partial \gamma} = 0, \qquad (1)$$

where f is the velocity distribution function; r, θ and ϕ are the spherical coordinates of the point P; c_r, c_θ and c_ϕ are the components of the velocity $\vec{c}$ along the directions r, θ and ϕ respectively. The speed c, together with the angles β and γ, are the spherical coordinates of the velocity vector $\vec{c}$, with respect to the c_r, c_θ, c_ϕ system. Because of axial symmetry there is no ϕ dependence. Thus we have

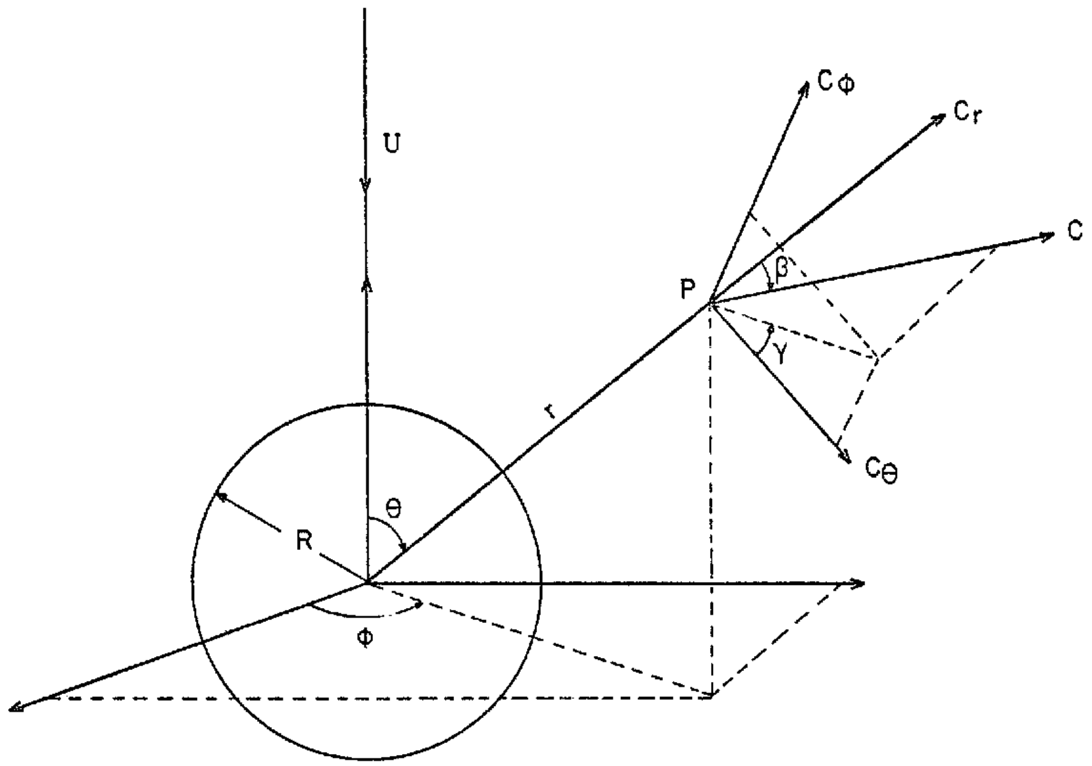

Fig. 1 - The coordinate system.

$$f = f\ (r,\theta,c,\beta,\gamma), \tag{2}$$

$$c_r = c \cos \beta, \tag{3}$$

$$c_\theta = c \sin \beta \cos \gamma, \tag{4}$$

$$c_\phi = c \sin \beta \sin \gamma\ . \tag{5}$$

Far from the satellite, the distribution function in Maxwellian, that is, for large r

$$f = n_o \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left\{ - \frac{m}{2kT} (\vec{c} - \vec{U})^2 \right\}, \tag{6}$$

where n_o is the ambient number density, T is the ambient temperature, m is the mass of the molecule, and k is the Boltzmann constant.

The characteristic equations are (Courant and Hilbert, 1966)

$$\frac{df}{o} = \frac{dr}{c \cos\beta} = \frac{d\theta}{\frac{c}{r}\sin\beta\cos\gamma} = \frac{dc}{o} = \frac{d\beta}{-\frac{c}{r}\sin\beta} = \frac{d\gamma}{-\frac{c}{r}\frac{\sin\beta\sin\gamma}{\tan\theta}}. \quad (7)$$

Integrating the first and fourth terms of Equation 7, we get

f = constant along a dynamic trajectory in phase space

and

$$c = \text{constant}. \quad (8)$$

Integrating the second and fifth expressions of Equation 7, we get

$$r \sin\beta = \text{constant}. \quad (9)$$

Integrating the third and sixth expressions of Equation 7, we get

$$\sin\gamma \sin\theta = \text{constant}. \quad (10)$$

These constants can be evaluated using suitable initial conditions. Using Equations 8 and 9, Equation 10 can be written as

$$m\, c\, r \sin\beta \sin\gamma \sin\theta = \text{constant}. \quad (11)$$

Equation 8 implies energy conservation and Equation 11 implies conservation of angular momentum about the vertical axis (direction of U). This is consistent with the physical situation because, in the absence of intermolecular collision, the molecules are travelling in straight lines.

The characteristic Equation 9 is used to determine admissible trajectories. From Figure 2, it is seen that for a molecule to reach a point at a distance r from the sphere, the condition $\sin^{-1}\frac{R}{r} \leq \beta \leq \pi$ must be satisfied.

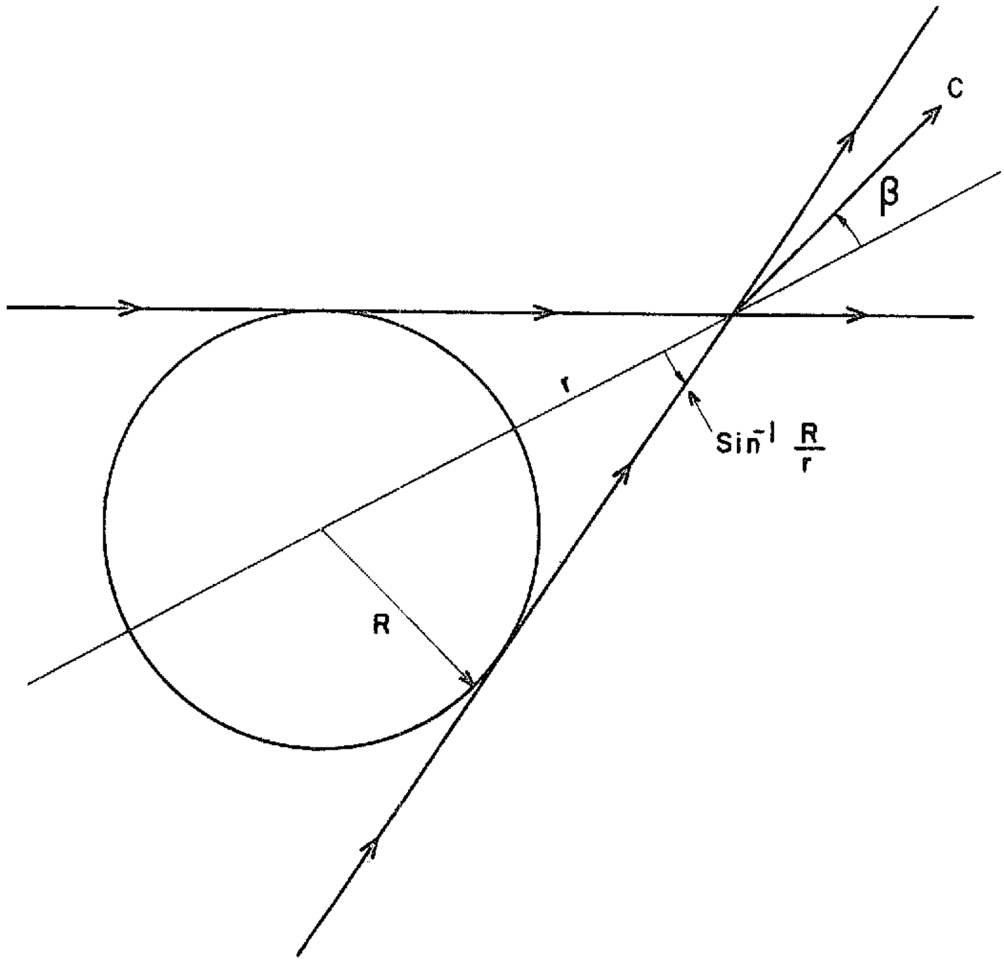

Fig. 2 - The limiting trajectories.

Thus, using the characteristic equations and the boundary condition of Equation 6, the solution can be written as

$$n_o = \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left\{-\frac{m}{2kT}(\vec{c}-\vec{U})^2\right\} H\,(r \sin\beta - R), \tag{12}$$

where H (x) is the Heavyside step function, such that H (x) = 1 if x > 0 and H (x) = 0 if x < 0

3. THE CALCULATION OF NUMBER DENSITY

The number density of particle is given by

$$n = \int f\, d\vec{c}\ , \tag{13}$$

where $d\vec{c}$ is an infinitesimal volume element given by

$$d\vec{c} = c^2 \sin\beta\, dc\, d\beta\, d\gamma\ . \tag{14}$$

From Figure 1, it is seen that

$$\vec{c}\,.\,\vec{U} = c\,U\,(\sin\beta\cos\gamma\sin\Theta - \cos\beta\cos\Theta)\ . \tag{15}$$

Thus

$$(\vec{c}.\vec{U})^2 = c^2 + U^2 - 2\,c\,U\,(\sin\beta\cos\gamma\sin\Theta - \cos\beta\cos\Theta)\ . \tag{16}$$

Substituting Equations 12, 14 and 16 in Equation 13, we get

$$\frac{n}{n_o} = \left(\frac{m}{2\pi kT}\right)^{3/2} \int_0^{\infty}\int_0^{2\pi}\int_{\sin^{-1}\frac{R}{r}}^{\pi} \exp\left[-\frac{m}{2kT}\left\{c^2 + U^2 - 2\,c\,U\,(\sin\beta\cos\gamma\sin\Theta - \cos\beta\cos\Theta)\right\}\right] c^2 \sin\beta\, dc\, d\gamma\, d\beta\ . \tag{17}$$

To have generality of results, we introduce the following nondimensional parameters:

Speed ratio, $S = \frac{U}{\sqrt{\frac{2kT}{m}}}$.

Non-dimensional velocity, $V = \frac{c}{\sqrt{\frac{2kT}{m}}}$.

Then Equation 17 becomes

$$\frac{n}{n_o} = \frac{1}{\pi^{3/2}} \int_0^{\infty}\int_0^{2\pi}\int_{\sin^{-1}\frac{R}{r}}^{\pi} v^2 \exp\left[-\left\{ v^2 + S^2 + 2Sv\,(\cos\beta\cos\theta - \sin\beta\cos\gamma\sin\theta)\right\}\right] \sin\beta\, dv\, d\gamma\, d\beta. \quad (18)$$

Now

$$\int_0^{2\pi} \exp\,(-2Sv\sin\beta\sin\theta\sin\gamma)\, d\gamma = 2\pi\, I_0(2Sv\sin\beta\sin\theta) \quad (19)$$

(Abramowitz and stegun, 1964), where $I_0(x)$ is the modified Bessel function of the first kind, order zero of argument x. Thus, the expression for number density becomes

$$\frac{n}{n_o} = \frac{2}{\sqrt{\pi}} \int_0^{\infty}\int_{\sin^{-1}\frac{R}{r}}^{\pi} \exp\,(-v^2 - S^2 - 2vS\cos\beta\cos\theta)\; I_0\,(2Sv\sin\beta\sin\theta)\, v^2 \sin\beta\, dv\, d\beta. \quad (20)$$

Now we transform the variables v, β into new variables x, y as per the following transformation:

$$v \sin\beta = x, \tag{21}$$

$$v \cos\beta = y. \tag{22}$$

Thus, $dx\,dy = v dv d\beta$ (23)

and the limits of integration become

$$o \leqq x \leqq \infty \text{ and } -\infty \leqq y \leqq x \sqrt{\frac{r^2}{R^2} - 1} \tag{24}$$

Using Equations 21 to 24 in Equation 20, we get

$$\frac{n}{n_o} = \frac{2}{\sqrt{\pi}} \int_0^\infty \int_{-\infty}^{x\sqrt{\frac{r^2}{R^2}-1}} \exp\left(-x^2 - y^2 - 2y\,S\cos\theta - S^2\right)$$

$$I_0\,(2Sx \sin\theta)\;x\,dx\,dy. \tag{25}$$

The integration on y can written as

$$\int_{-\infty}^{x\sqrt{\frac{r^2}{R^2}-1}} \exp\left(-y^2 - S^2 - 2yS\cos\theta\right) dy =$$

$$= \exp - (-S^2 \sin^2\theta) \int_{-\infty}^{x\sqrt{\frac{r^2}{R^2}-1}} \exp\left\{-\left(y + S\cos\theta\right)^2\right\} dy =$$

$$= \frac{\sqrt{\pi}}{2} \exp\left(-S^2\sin^2\theta\right) \left\{1 + \text{erf}\left(x\sqrt{\frac{r^2}{R^2} - 1} + S\cos\theta\right)\right\}, \tag{26}$$

where $\mathrm{erf}\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\,dt$ is the error function.

Substituting Equation 26 in Equation 25, we have

$$\frac{n}{n_o} = \exp(-S^2 \sin^2\Theta) \int^{\infty} x\, e^{-x^2}\, I_o\,(2x\,S \sin\Theta) \left[1 + \mathrm{erf} \left\{ x \sqrt{\frac{r^2}{R^2} - 1} + S \cos\Theta \right\} \right] dx \qquad (27)$$

But

$$\int_0^{\infty} x \exp(-x^2)\, I_o\,(2x\,S \sin\Theta)\,dx = \frac{\exp(S^2 \sin^2\Theta)}{2}\,, \qquad (28)$$

(Abramowitz and Stegun, 1964)

Substituting Equation 28 in Equation 27, we get

$$\frac{n}{n_o} = \frac{1}{2} + \exp(-S^2 \sin^2\Theta) \int_0^{\infty} x \exp(-x^2)\, I_o\,(2x\,S \sin\Theta)\ \mathrm{erf} \left\{ x \sqrt{\frac{r^2}{R^2} - 1} + S \cos\Theta \right\} dx \qquad (29)$$

4. RESULTS AND DISCUSSIONS

The expression for the number density as given by Equation 29, has been evaluated numerically, for varius speed ratios S, as functions for r and Θ. Constant number density curves are shown for the typical cases of $S = 2$ (Figure 3) and $S = 5$ (Figure 4).

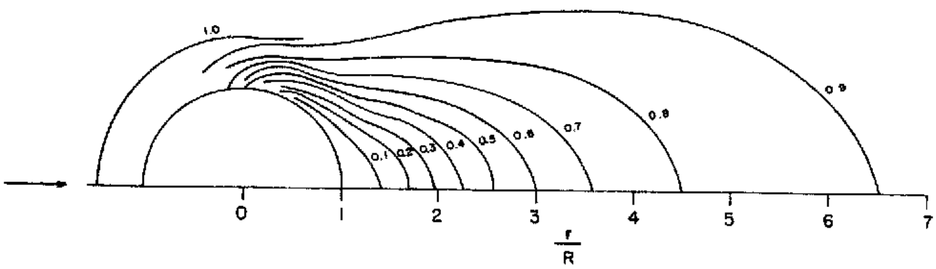

Fig. 3 - Constant number density profiles for S=2

These results show that the density profile is a strong function of θ in the downstream region, but it is practically spherically symmetric in the upstream region. As expected, the downstream wake region increases with S.

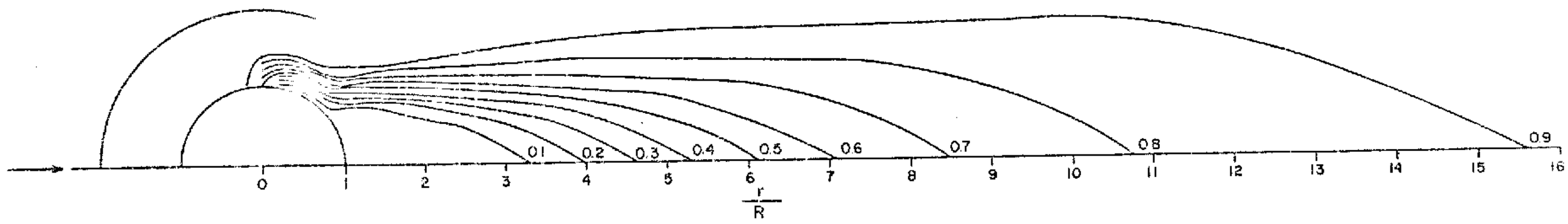

Fig. 4 - Constant number density profiles for S = 5.

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