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16. Summary/Notes

This chapter analyses the propagation of very low frequency waves in a highly conducting fluid. These waves are commonly referred to as magnetohydrodynamic (MHD) waves. The characteristics of these waves are investigated for propagation parallel, perpendicular, and in an arbitrary direction, with respect to an externally applied magnetic field. The effects of the displacement current, and of viscovity, on the propagation of the MHD waves, are also analysed.

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MAGNETOHYDRODYNAMIC WAVES

1. INTRODUCTION

The most fundamental type of wave motion that propagates in a compressible, non-conducting fluid is that of longitudinal *sound waves*. For these waves the variations in pressure (p) and in density (ρ), associated with the compressions and rarefactions of the fluid during the longitudinal wave motion, obey the adiabatic energy equation commonly used in thermodynamics,

$$p\rho^{-\gamma} = \text{constant} \tag{1.1}$$

where γ denotes the ratio of the specific heats at constant pressure and at constant volume. Differentiating (1.1) gives

$$\nabla p = \left(\frac{\gamma p}{\rho}\right) \nabla \rho$$

$$= V_S^2 \nabla \rho \tag{1.2}$$

where

$$V_S = \left(\frac{\gamma p}{\rho}\right)^{1/2} = \left(\frac{\gamma k T}{m}\right)^{1/2} \tag{1.3}$$

is the velocity of propagation of sound waves, known as the *adiabatic sound velocity.* Fig. 1 illustrates the regions of compression and rarefaction of the fluid, associated with the longitudinal motion of sound waves.

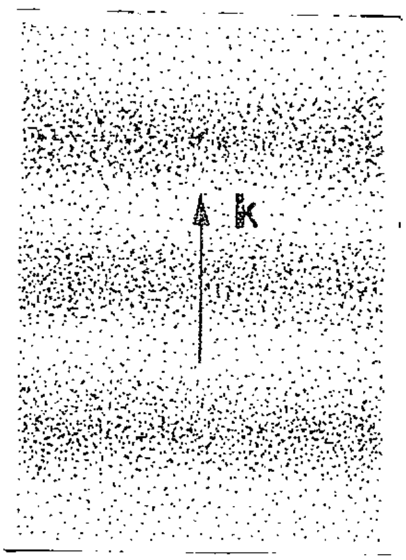

Fig. 1 - Schematic representation of longitudinal *sound waves* that propagate in a compressible, non-conducting fluid, showing the regions of compression and rarefaction associated with the longitudinal wave motion.

1.1 - Alfvén waves

In the case of a compressible, conducting fluid immersed in a magnetic field, other types of wave motion are possible.

We have seen that in a magnetic field of intensity $\underset{\sim}{B}_0$

the magnetic stresses are equivalent to a tension B_o^2/μ_o along the field lines, and an isotropic hydrostatic pressure $B_o^2/2\mu_o$ (see section 5, of Chapter 12). Since the latter can always be incorporated with the fluid pressure, the magnetic field lines behave effectively as elastic cords under a tension B_o^2/μ_o. Further, in a perfectly conducting fluid the plasma particles behave as if they were tied to the magnetic field lines (see section 4, of Chapter 12), so that the lines of force act as if they were mass-loaded strings under tension. Thus, by analogy with the transverse vibrations of elastic strings, we expect that, whenever the conducting fluid is slightly disturbed from the equilibrium conditions, the magnetic field lines will perform transverse vibrations. The velocity of propagation of these transverse oscillations are expected to be given by

$$V_A = \left(\frac{\text{Tension}}{\text{Density}}\right)^{1/2} = \left(\frac{B_o^2}{\mu_o \rho}\right)^{1/2} \tag{1.4}$$

This velocity is known as the *Alfvēn velocity,* since the existence of this type of wave motion was first pointed out by Alfvēn, in 1942. An important property of these waves, as will be shown later, is the absence of any fluctuations in density (ρ) or fluid pressure (p). Fig.2 illustrates the transverse oscillations of the fluid (and of the "frozen in" field lines) for the Alfvēn wave.

1.2 Magnetosonic waves

Longitudinal oscillations are also expected to occur in a compressible, conducting fluid in a magnetic field. For motion of

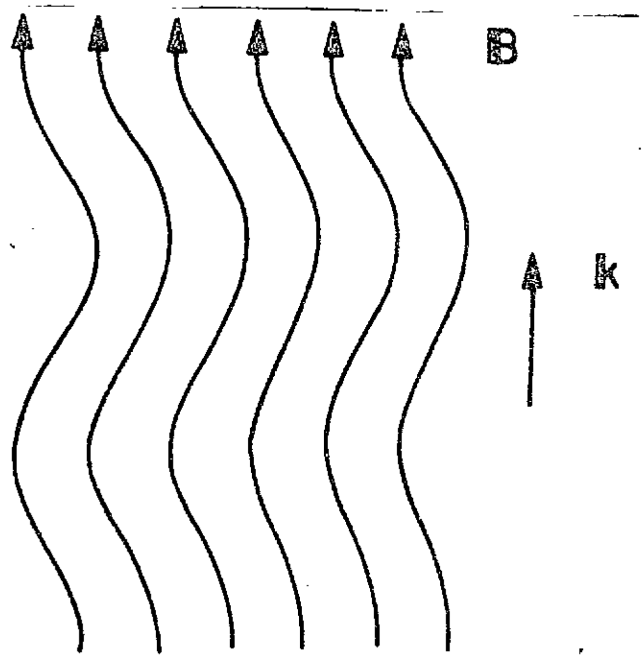

Fig. 2 - Transverse *Alfvén waves* in a compressible, conducting magnetofluid. The velocity of propagation is along the magnetic field lines, and the fluid motion and magnetic field perturbations are perpendicular to the field lines.

the particles, and propagation of the wave, in the direction of the magnetic field there will be no perturbation in the magnetic field, since the particles are free to move in this direction. Thus, in this case, the waves will be ordinary longitudinal sound waves propagating at the velocity V_s along the field lines (Fig. 3).

On the other hand, for motion of the particles, and propagation of the wave, in the direction perpendicular to the magnetic field, a new type of longitudinal wave motion is possible since now, in addition to the fluid pressure p, we must add the

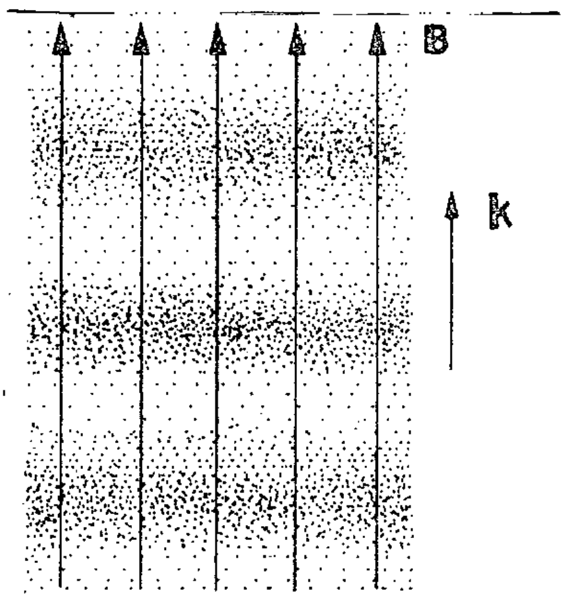

Fig. 3 - Longitudinal *sound waves* propagating along the magnetic field lines in a compressible, conducting magnetofluid.

hydrostatic magnetic pressure $B^2/2\mu_o$ in the plane normal to $\underset{\sim}{B}_o$. Hence, the total pressure is $p + B^2/2\mu_o$ and, consequently, the velocity (V_M) of propagation of these so-called *magnetosonic waves* (Fig.4) satisfies the following relation, analogous to (1.1),

$$\underset{\sim}{\nabla}(p + B^2/\ 2\mu_o) = V_M^2\ \underset{\sim}{\nabla}\rho \tag{1.5}$$

Therefore, we can write

$$V_M^2 = \frac{d}{d\rho}\left(p + \frac{B^2}{2\mu_o}\right)_{\rho=\rho_o} = V_s^2 + \frac{d}{d\rho}\left(\frac{B^2}{2\mu_o}\right)_{\rho=\rho_o} \tag{1.6}$$

where the suffix zero, in ρ, refers to the undisturbed state, and V_s is the adiabatic sound velocity. Since the lines of force are frozen in the conducting fluid, the magnetic flux BdS across an element of surface, $d\underset{\sim}{S}$, whose normal is oriented along the magnetic field, and the mass ρ dS of a unit length of column having $d\underset{\sim}{S}$ as base, are both conserved during the oscillation, in such a way that $(B/\rho) = (B_o/\rho_o)$. Consequently, (1.6) becomes

$$V_M^2 = V_s^2 + \frac{d}{d\rho}\left(\frac{B_o^2\,\rho^2}{2\mu_o\rho_o^2}\right)_{\rho=\rho_o} = V_s^2 + \frac{B_o^2}{\mu_o\rho_o} \tag{1.7}$$

or

$$V_M = (V_s^2 + V_A^2)^{1/2} \tag{1.8}$$

where V_A is the Alfvēn velocity.

For propagation in a direction inclined with respect to the magnetic field the waves are more complex. This subject will be considered in some detail in section 5.

2. MHD EQUATIONS FOR A COMPRESSIBLE, NONVISCOUS, CONDUCTING FLUID

2.1 Basic equations

To investigate the propagation of waves in a conducting magnetofluid, let us consider a compressible, nonviscous, perfectly conducting fluid immersed in a magnetic field. The appropriate system of equations governing the behavior of this type of fluid, and the assumptions involved, have been summarized in section 1, of Chapter 12. These equations are

$$\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla} \cdot (\rho\, \underset{\sim}{u}) = 0 \tag{2.1}$$

$$\rho \frac{\partial \underset{\sim}{u}}{\partial t} + \rho(\underset{\sim}{u} \cdot \underset{\sim}{\nabla})\, \underset{\sim}{u} = - \underset{\sim}{\nabla} p + \underset{\sim}{J} \times \underset{\sim}{B} \tag{2.2}$$

$$\underset{\sim}{\nabla} p = V_s^2\, \underset{\sim}{\nabla} \rho \tag{2.3}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{B} = \mu_0\, \underset{\sim}{J} \tag{2.4}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{E} = - \partial \underset{\sim}{B}/\partial t \tag{2.5}$$

$$\underset{\sim}{E} + \underset{\sim}{u} \times \underset{\sim}{B} = 0 \tag{2.6}$$

This system of equations can be reduced by combining Eqs. (2.2) to (2.4) in the form

$$\rho \frac{\partial \underset{\sim}{u}}{\partial t} + \rho\, (\underset{\sim}{u} \cdot \underset{\sim}{\nabla})\, \underset{\sim}{u} = - V_s^2\, \underset{\sim}{\nabla} \rho + \frac{1}{\mu_0} (\underset{\sim}{\nabla} \times \underset{\sim}{B}) \times \underset{\sim}{B} \tag{2.7}$$

as well as Eqs. (2.5) and (2.6) in the form

$$\underset{\sim}{\nabla} \times (\underset{\sim}{u} \times \underset{\sim}{B}) = \partial \underset{\sim}{B}/\partial t \tag{2.8}$$

Under equilibrium conditions, the fluid is assumed to be uniform, with constant density ρ_o, the equilibium velocity is zero, and throughout the fluid the magnetic induction $\underset{\sim}{B}_o$ is uniform and constant.

In order to develop a dispersion relation for small-amplitude waves, consider small-amplitude departures from the equilibrium values, so that

$$\underset{\sim}{B}(\underset{\sim}{r}, t) = \underset{\sim}{B}_o + \underset{\sim}{B}_1(\underset{\sim}{r}, t) \tag{2.9}$$

$$\rho(\underset{\sim}{r}, t) = \rho_o + \rho_1(\underset{\sim}{r}, t) \tag{2.10}$$

$$\underset{\sim}{u}(\underset{\sim}{r}, t) = \underset{\sim}{u}_1(\underset{\sim}{r}, t) \tag{2.11}$$

Substituting Eqs. (2.9) to (2.11) into Eqs. (2.1), (2.7) and (2.8), and neglecting second - order terms, we obtain the following linearized equations in the small first - order quantities

$$\frac{\partial \rho_1}{\partial t} + \rho_o (\underset{\sim}{\nabla} \cdot \underset{\sim}{u}_1) = 0 \tag{2.12}$$

$$\rho_o \frac{\partial \underset{\sim}{u}_1}{\partial t} + V_s^2 \underset{\sim}{\nabla} \rho_1 + \frac{1}{\mu_o} \underset{\sim}{B}_o \times (\underset{\sim}{\nabla} \times \underset{\sim}{B}_1) = 0 \tag{2.13}$$

$$\frac{\partial \underset{\sim}{B}_1}{\partial t} - \underset{\sim}{\nabla} \times (\underset{\sim}{u}_1 \times B_o) = 0 \tag{2.14}$$

2.2 Development of an equation for $\underset{\sim}{u}_1$

Eqs. (2.12) to (2.14) can be combined to yield an equation for $\underset{\sim}{u}_1$ alone. For this purpose, we first differentiate (2.13) with respect to time, obtaining

$$\rho_o \frac{\partial^2 \underset{\sim}{u}_1}{\partial t^2} + V_s^2 \underset{\sim}{\nabla}\left(\frac{\partial \rho_1}{\partial t}\right) + \frac{1}{\mu_o} \underset{\sim}{B}_o \times \left[\underset{\sim}{\nabla} \times \left(\frac{\partial \underset{\sim}{B}_1}{\partial t}\right) \right] = 0 \qquad (2.15)$$

Next, using (2.12) and (2.14), we can write (2.15) as

$$\frac{\partial^2 \underset{\sim}{u}_1}{\partial t^2} - V_s^2 \underset{\sim}{\nabla} (\underset{\sim}{\nabla} \cdot \underset{\sim}{u}_1) + \underset{\sim}{V}_A \times \{ \underset{\sim}{\nabla} \times \left[\underset{\sim}{\nabla} \times (\underset{\sim}{u}_1 \times \underset{\sim}{V}_A) \right] \} = 0 \qquad (2.16)$$

where we have introduced the *vector* Alfvēn velocity

$$\underset{\sim}{V}_A = \frac{\underset{\sim}{B}_o}{(\mu_o \rho_o)^{1/2}} \qquad (2.17)$$

Without loss of generality we can consider plane wave solutions of the form

$$\underset{\sim}{u}_1 (\underset{\sim}{r}, t) = \underset{\sim}{u}_1 \exp (i \underset{\sim}{k} \cdot \underset{\sim}{r} - i \omega t) \qquad (2.18)$$

In what follows $\underset{\sim}{u}_1$ can stand for either the amplitude or the entire expression (2.18). Thus, in (2.16) we can replace the operator $\underset{\sim}{\nabla}$ by $i \underset{\sim}{k}$ and the time derivative by $- i\omega$, so that

$$-\omega^2 \underset{\sim}{u}_1 + V_s^2 (\underset{\sim}{k} \cdot \underset{\sim}{u}_1) \underset{\sim}{k} - \underset{\sim}{V}_A \times \{\underset{\sim}{k} \times \left[\underset{\sim}{k} \times (\underset{\sim}{u}_1 \times \underset{\sim}{V}_A) \right] \} = 0 \qquad (2.19)$$

Since, for any three vectors $\underset{\sim}{A}$, $\underset{\sim}{B}$ and $\underset{\sim}{C}$ we have the vector identity

$$\underset{\sim}{A} \times (\underset{\sim}{B} \times \underset{\sim}{C}) = (\underset{\sim}{A} \cdot \underset{\sim}{C})\, \underset{\sim}{B} - (\underset{\sim}{A} \cdot \underset{\sim}{B})\, \underset{\sim}{C} \tag{2.20}$$

we can rearrange (2.19) to read

$$-\omega^2 \underset{\sim}{u}_1 + (V_S^2 + V_A^2)\, (\underset{\sim}{k} \cdot \underset{\sim}{u}_1)\, \underset{\sim}{k} + (\underset{\sim}{k} \cdot \underset{\sim}{V}_A)\big[\, (\underset{\sim}{k} \cdot \underset{\sim}{V}_A)\, \underset{\sim}{u}_1 -$$

$$- (\underset{\sim}{V}_A \cdot \underset{\sim}{u}_1)\, \underset{\sim}{k} - (\underset{\sim}{k} \cdot \underset{\sim}{u}_1)\, \underset{\sim}{V}_A \big] = 0 \tag{2.21}$$

Although this expression appears to be somewhat involved, it leads to remarkably simple solutions for waves propagating in the directions parallel or perpendicular to the magnetic field.

3. PROPAGATION PERPENDICULAR TO THE MAGNETIC FIELD

For the case when the wave vector $\underset{\sim}{k}$ is perpendicular to the magnetic induction $\underset{\sim}{B}_o$, we have $\underset{\sim}{k} \cdot \underset{\sim}{V}_A = 0$, and (2.21) simplifies to

$$-\omega^2 \underset{\sim}{u}_1 + (V_S^2 + V_A^2)\, (\underset{\sim}{k} \cdot \underset{\sim}{u}_1)\, \underset{\sim}{k} = 0 \tag{3.1}$$

from which we obtain

$$\underset{\sim}{u}_1 = (V_S^2 + V_A^2)\, \frac{(\underset{\sim}{k} \cdot \underset{\sim}{u}_1)}{\omega^2}\, \underset{\sim}{k} \tag{3.2}$$

Therefore, $\underset{\sim}{u}_1$ is parallel to $\underset{\sim}{k}$, so that $\underset{\sim}{k} \cdot \underset{\sim}{u}_1 = ku_1$, and the solution for $\underset{\sim}{u}_1$ is a *longitudinal* wave with the *phase velocity*

$$\frac{\omega}{k} = (V_S^2 + V_A^2)^{1/2} \tag{3.3}$$

The magnetic field associated with this longitudinal wave can be obtained from (2.14). Taking

$$\underset{\sim}{B}_1 (\underset{\sim}{r}, t) = \underset{\sim}{B}_1 \exp (i \underset{\sim}{k} . \underset{\sim}{r} - i \omega t) \tag{3.4}$$

we obtain

$$- \omega \underset{\sim}{B}_1 - \underset{\sim}{k} \times (\underset{\sim}{u}_1 \times \underset{\sim}{B}_0) = 0 \tag{3.5}$$

Using the vector identity (2.20), and noting that $\underset{\sim}{k} . \underset{\sim}{B}_0 = 0$, we find

$$\underset{\sim}{B}_1 = \frac{u_1}{(\omega/k)} \underset{\sim}{B}_0 \tag{3.6}$$

The electric field associated with this wave is seen, from (2.6), to be given by

$$\underset{\sim}{E} = - \underset{\sim}{u}_1 \times \underset{\sim}{B}_0 \tag{3.7}$$

This wave is, therefore, similar to an electromagnetic wave, since the time-varying magnetic field is perpendicular to the direction of propagation, but parallel to the magnetostatic field, whereas the time-varying electric field is perpendicular to both the direction of propagation and the magnetostatic field. It is a longitudinal wave, however, since the velocity of mass flow and the fluctuating mass density associated with it are both in the direction of wave propagation. For these reasons, this wave is called the *magnetosonic wave*. The phase velocity of this wave is independent of frequency, so that it is a nondispersive wave. As illustrated in Fig.4, the magnetosonic wave produces compressions and rarefactions in the magnetic field lines without changing their direction. Since the fluid is perfectly conducting, the lines of force and the fluid move together.

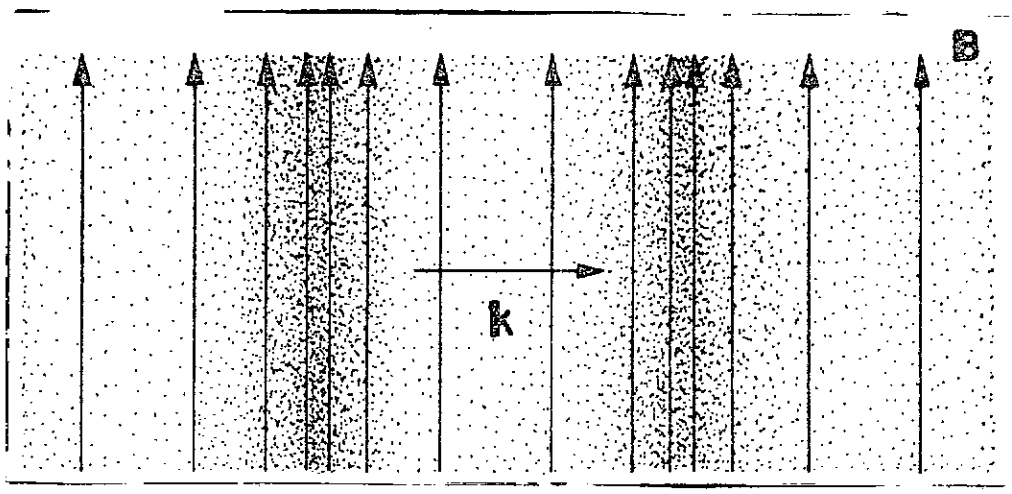

Fig. 4 - The longitudinal *magnetosonic wave* propagates perpendicularly to the magnetic field, causing compressions and rarefactions of both the lines of force and the conducting fluid.

The restoring forces, operating in the magnetosonic wave, are the gradient of the fluid pressure and the gradient of the compressional stresses between the magnetic field lines. If the fluid pressure is much greater than the magnetic pressure, the effect of the magnetic field is negligible, so that $\omega/k \simeq V_S$ and the magnetosonic wave becomes essentially an acoustic wave. On the other hand, if the magnetic field is very strong, so that the magnetic pressure is much larger than the fluid pressure, then the phase velocity of the magnetosonic wave becomes equal to the Alfvén wave velocity V_A.

The magnetosonic wave mode is also known variously as the *compressional Alfvén wave* or *the fast Alfvén wave*.

4. PROPAGATION PARALLEL TO THE MAGNETIC FIELD

For waves propagating along the magnetic field ($\underset{\sim}{k} \| \underset{\sim}{B}_0$), we have $\underset{\sim}{k} \cdot \underset{\sim}{V}_A = k V_A$, and (2.21) simplifies to

$$(k^2 V_A^2 - \omega^2)\, \underset{\sim}{u}_1 + \left(\frac{V_s^2}{V_A^2} - 1\right) k^2\, (\underset{\sim}{u}_1 \cdot \underset{\sim}{V}_A)\, \underset{\sim}{V}_A = 0 \qquad (4.1)$$

In this case there are two types of wave motion possible.

For $\underset{\sim}{u}_1$ parallel to $\underset{\sim}{B}_0$ and $\underset{\sim}{k}$, we find, from (4.1), that a longitudinal mode is possible, with the phase velocity

$$\frac{\omega}{k} = V_s \qquad (4.2)$$

This is an ordinary *longitudinal sound wave*, in which the velocity of mass flow is in the direction of propagation (Fig. 3). There is no electric field, electric current density, and magnetic field associated with this wave.

A transverse wave, with $\underset{\sim}{u}_1$ perpendicular to $\underset{\sim}{B}_0$ and $\underset{\sim}{k}$, is the other possibility. In this case, $\underset{\sim}{u}_1 \cdot \underset{\sim}{V}_A = 0$, and (4.1) gives for the phase velocity of this transverse wave, known as the *Alfvén wave*,

$$\frac{\omega}{k} = V_A \qquad (4.3)$$

Since the phase velocity is independent of frequency, there is no dispersion.

The magnetic field associated with the Alfvén wave is found, from Eqs. (2.14) and (3.5), to be given by

$$\underset{\sim}{B}_1 = - B_o \frac{\underset{\sim}{u}_1}{(\omega/k)} \tag{4.4}$$

Hence, the magnetic field disturbance is normal to the original magnetostatic induction $\underset{\sim}{B}_o$. The small component $\underset{\sim}{B}_1$, when added to $\underset{\sim}{B}_o$, gives the lines of force a sinusoidal ripple, shown in Fig.5. The associated electric field is given by Eq. (3.8).

The Alfvén wave involves no fluctuations in the fluid density or pressure, although both the fluid and the magnetic field lines oscillate back and forth laterally, in the plane normal to $\underset{\sim}{B}_o$. The magnetic energy density of the wave motion, $\underset{\sim}{B}_1^2/2\mu_o$, is equal to the kinetic energy density of the fluid motion, $\rho_o \underset{\sim}{u}_1^2/2$. This equipartition of energy is easily verified from (4.4),

$$\frac{\underset{\sim}{B}_1^2}{2\mu_o} = \frac{B_o^2 \; \underset{\sim}{u}_1^2}{2\mu_o \; (\omega/k)^2} = \frac{B_o^2 \; \underset{\sim}{u}_1^2}{2\mu_o \; V_A^2} = \frac{1}{2} \rho_o \underset{\sim}{u}_1^2$$

where we have used Eqs. (4.3) and (2.17).

The Alfvén wave mode is also known variously as the *shear Alfvén wave* or the *slow Alfvén wave*.

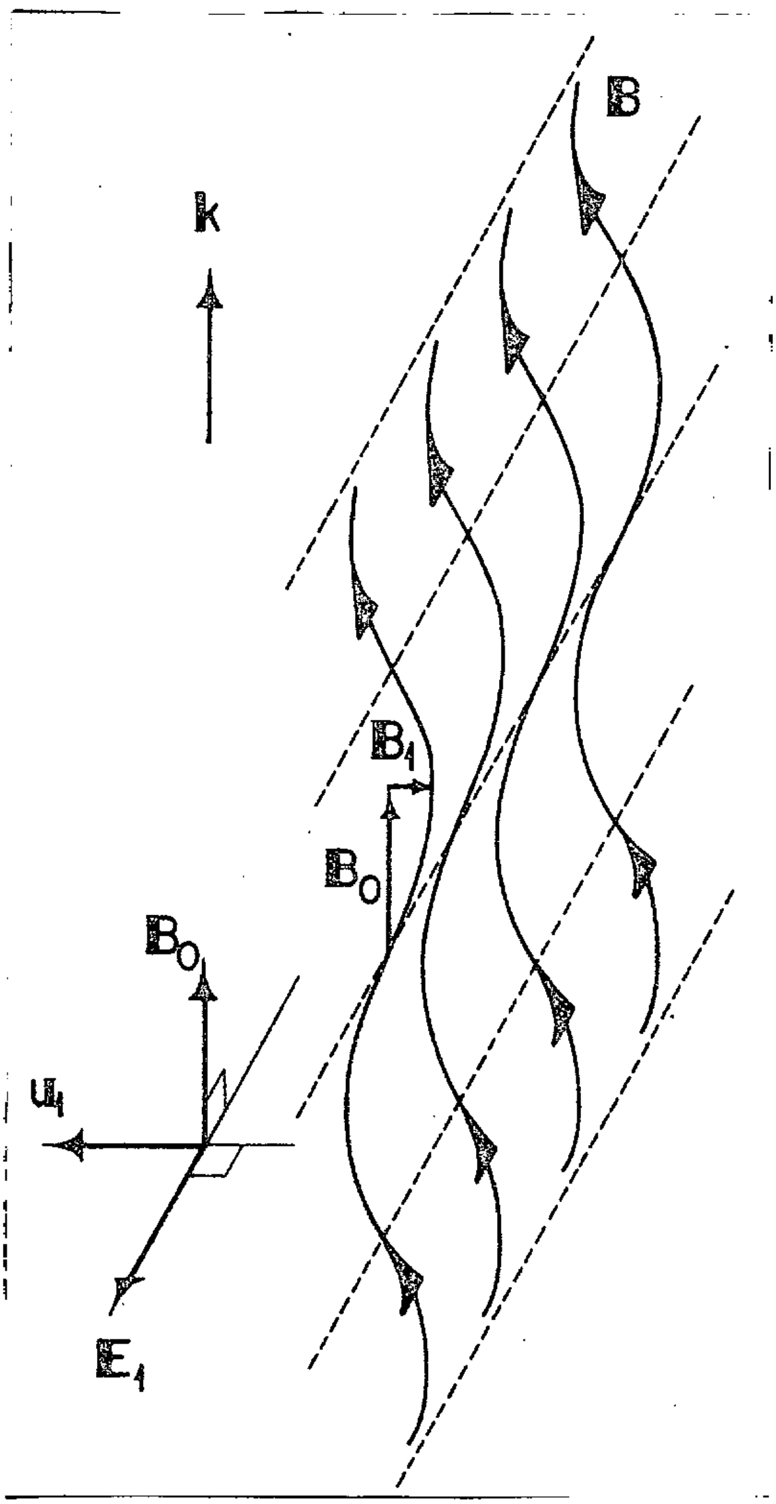

Fig. 5 - Schematic illustration for Alfvén waves propagating along $\underset{\sim}{B}_o$, showing the relations between the oscillating quantities.

5. PROPAGATION IN AN ARBITRARY DIRECTION

Proceeding further, let us now investigate the case of wave propagation in an arbitrary direction with respect to the magnetic induction $\underset{\sim}{B}_o$. With no loss of generality, we introduce a Cartesian coordinate system such that the y-axis is normal to the plane defined by the direction of propagation $\hat{\underset{\sim}{k}}$ and the magnetic induction $\underset{\sim}{B}_o$, and choose $\hat{\underset{\sim}{z}}$ to be along $\underset{\sim}{B}_o$, as shown in Fig.6.

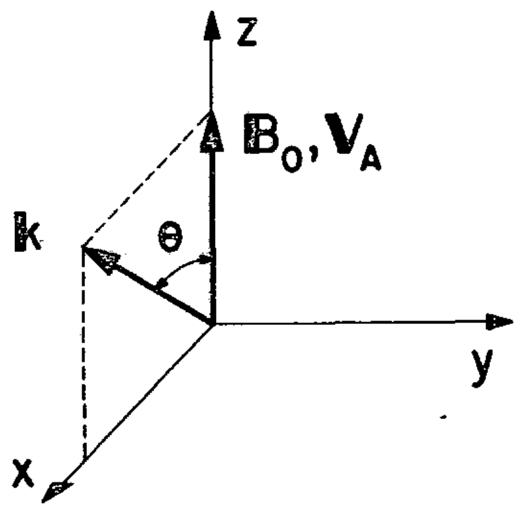

Fig. 6 - Cartesian coordinate system with the relative directions of the vectors $\underset{\sim}{k}$ and $\underset{\sim}{B}_o$.

Denoting by Θ the angle between $\underset{\sim}{k}$ and $\underset{\sim}{B}_o$, we have

$$\underset{\sim}{k} = k\ (\ \hat{\underset{\sim}{x}} \sin\Theta + \hat{\underset{\sim}{z}} \cos\Theta\ ) \tag{5.1}$$

$$\underset{\sim}{V}_A = V_A \hat{\underset{\sim}{z}} \tag{5.2}$$

$$\underset{\sim}{u}_1 = u_{1x} \hat{\underset{\sim}{x}} + u_{1y} \hat{\underset{\sim}{y}} + \underset{\sim}{u}_{1z} \hat{\underset{\sim}{z}} \tag{5.3}$$

$$\underset{\sim}{k} \cdot \underset{\sim}{V}_A = k V_A \cos \Theta \tag{5.4}$$

$$\underset{\sim}{k} \cdot \underset{\sim}{u}_1 = k (u_{1x} \sin\Theta + u_{1z} \cos \Theta) \tag{5.5}$$

$$\underset{\sim}{V}_A \cdot \underset{\sim}{u}_1 = V_A u_{1z} \tag{5.6}$$

Substituting these expressions into equation (2.21) for $\underset{\sim}{u}_1$, performing the required algebra and rearranging the terms, we obtain for the x-component equation,

$$u_{1x} (- \omega^2 + k^2 V_A^2 + k^2 V_S^2 \sin^2 \Theta) + u_{1z} (k^2 V_S^2 \sin \Theta \cos \Theta) = 0 \tag{5.7}$$

for the y - component equation,

$$u_{1y} (- \omega^2 + k^2 V_A^2 \cos^2 \Theta) = 0 \tag{5.8}$$

and for the z - component equation,

$$u_{1x} (k^2 V_S^2 \sin \Theta \cos \Theta) + u_{1z} (- \omega^2 + k^2 V_S^2 \cos^2 \Theta) = 0 \tag{5.9}$$

5.1 Pure Alfvén wave

From (5.8) we see that there is a linearly polarized wave involving oscillations in the direction perpendicular to both $\underset{\sim}{k}$ and $\underset{\sim}{B}_o$ ($u_{1y} \neq 0$), with a phase velocity given by

$$\frac{\omega}{k} = V_A \cos \Theta \tag{5.10}$$

The field components associated with this wave can be seen to be B_{1y}, u_{1y}, E_{1x}, and J_{1x}, so that it is a transverse Alfvén wave. For this reason, this wave is generally referred to as the *pure Alfvén wave*. Note that for propagation along the magnetostatic field ($\Theta = 0$), Eq. (5.10) gives $\omega/k = V_A$, while for propagation across the magnetostatic field ($\Theta = 90°$) this wave disappears, since $\omega/k = 0$. This mode is also known as the *oblique Alfvén wave*.

5.2 - Fast and slow MHD waves

Eqs. (5.7) and (5.9) constitute a system of two simultaneous equations for the amplitudes of u_{1x} and u_{1z}. To have a solution in which u_{1x} and u_{1z} are nonzero, the determinant of the coefficients of this system of equations must vanish. Therefore, setting

$$\begin{vmatrix} (-\omega^2 + k^2 V_A^2 + k^2 V_S^2 \sin^2\Theta) & (k^2 V_S^2 \sin\Theta \cos\Theta) \\ (k^2 V_S^2 \sin\Theta \cos\Theta) & (-\omega^2 + k^2 V_S^2 \cos^2\Theta) \end{vmatrix} = 0 \tag{5.11}$$

we obtain the following dispersion relation, expressed in terms of the phase velocity ω/k,

$$\frac{\omega^4}{k^4} - (V_s^2 + V_A^2)\frac{\omega^2}{k^2} + V_s^2 V_A^2 \cos^2\Theta = 0 \qquad (5.12)$$

Solving this equation for ω^2/k^2, we obtain two real solutions

$$\frac{\omega^2}{k^2} = \frac{1}{2}(V_s^2 + V_A^2) \pm \frac{1}{2}\left[(V_s^2 + V_A^2)^2 - 4V_s^2 V_A^2 \cos^2\Theta\right]^{1/2} \qquad (5.13)$$

The solutions with the plus-and-minus sign are called, respectively, the *fast* and *slow MHD waves.*

5.3 Phase velocities

All the three MHD waves have constant phase velocities, given by (5.10) and (5.13), for all frequencies, and hence there is no dispersion. Fig.7 displays how the phase velocity varies, for each of these waves, as a funtion of the angle Θ between $\underset{\sim}{k}$ and $\underset{\sim}{B}_o$, for both cases when $V_A > V_s$ and when $V_s > V_A$. The phase velocity of the fast MHD wave increases from V_A (or V_s if $V_s > V_A$) when $\Theta = 0^0$, to $(V_s^2 + V_A^2)^{1/2}$ when $\Theta = 90^0$, while that of the slow MHD wave decreases from V_s (or V_A if $V_s > V_A$) when $\Theta = 0^0$, to zero when $\Theta = 90^0$. Therefore, if $V_A > V_s$, the fast MHD wave becomes the Alfvēn wave for $\Theta = 0^0$, and the magnetosonic wave for $\Theta = 90^0$, while the slow MHD wave becomes the sound wave for $\Theta = 0^0$, and disappears for $\Theta = 90^0$. On the other hand, if $V_s > V_A$, the fast MHD wave becomes the sound wave for $\Theta = 0^0$ and the

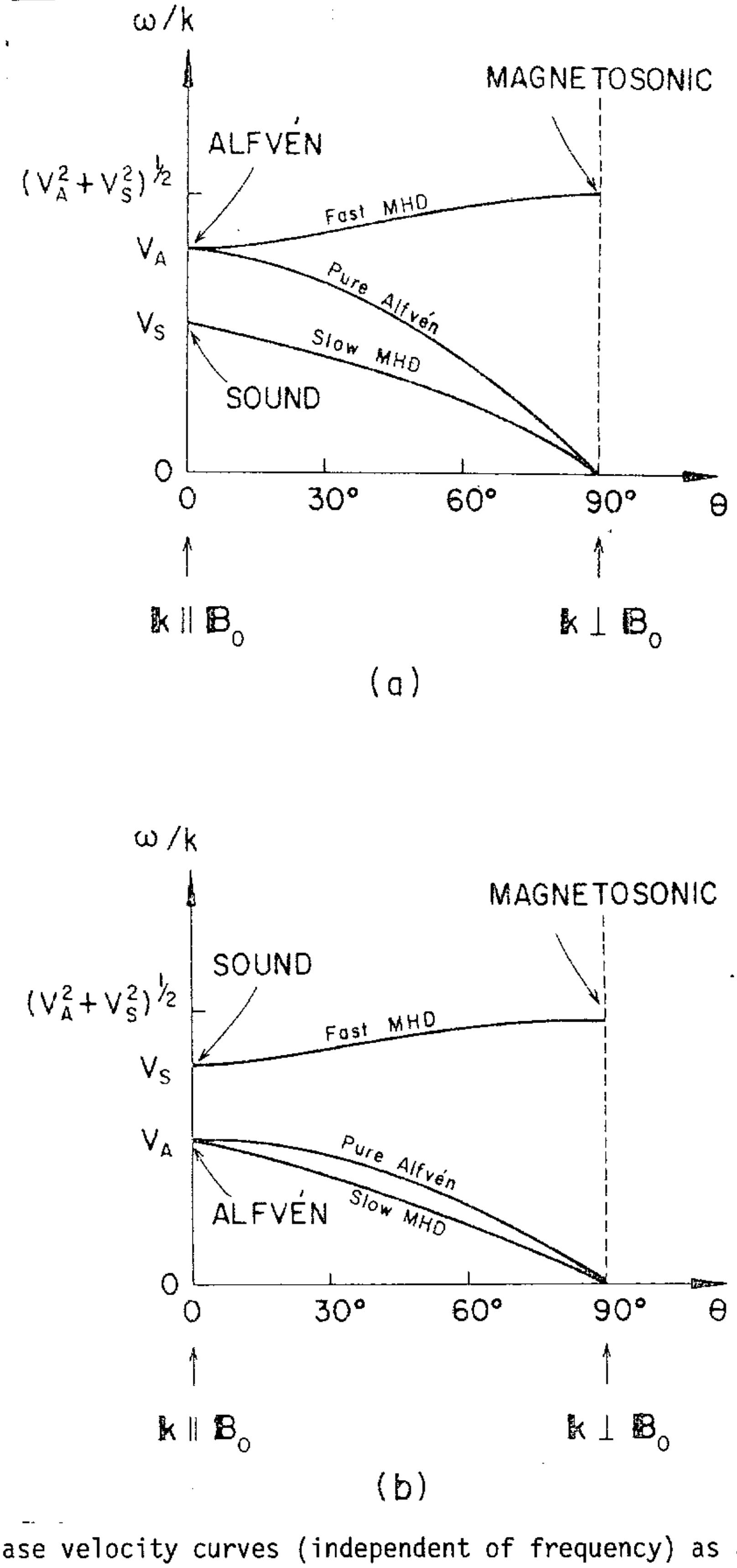

Fig. 7 - Phase velocity curves (independent of frequency) as a function of the angle between $\underset{\sim}{k}$ and $\underset{\sim}{B}_0$, for the pure Alfvén, the fast, and the slow MHD waves, for the cases (a) $V_A > V_s$ and (b) $V_s > V_A$

magnetosonic wave for $\Theta = 90^0$, while the slow MHD wave becomes the Alfvēn wave for $\Theta = 0^0$ and disappears for $\Theta = 90^0$.

5.4 Wave normal surfaces

The propagation of these waves are conveniently represented by means of diagrams called *phase velocity* or *wave normal surfaces,* which give the variations of the magnitude of the phase velocity of plane waves with respect to the magnetic field direction. Fig.8 shows the wave normal diagram for the pure Alfvēn wave, constructed trom Eq. (5.10). The vector drawn from the origin to a point P on the curve represents the phase velocity of a plane wave and the direction of the wave normal with respect to the magnetostatic field. The actual state of affairs, in three dimensions, is obtained by rotating the circles of Fig.8 about the axis oriented along $\underset{\sim}{B}_o$. The three-dimensional surface, thus obtained, is called the *wave normal surface.*

Fig.9 shows the wave normal diagrams for propagation of the pure Alfvēn, the fast and the slow MHD waves, for the two cases $V_A > V_s$ and $V_A < V_s$. The three-dimensional wave normal surfaces are obtained by rotating Fig.9 about the axis oriented along $\underset{\sim}{B}_o$. The wave normal surface corresponding to the fast MHD wave is a smooth, closed surface enclosing the two spheres passing through 0 which correspond to the pure Alfvēn wave. Within each of these spheres, there is another smooth, closed wave normal surface corresponding to the slow MHD wave.

6. EFFECT OF DISPLACEMENT CURRENT

In magnetohydrodynamics, the displacement current ($\varepsilon_o\ \partial \underset{\sim}{E}/\partial t$) term, which appears in Maxwell $\underset{\sim}{\nabla} \times \underset{\sim}{B}$ equation, is usually neglected. This approximation in valid only for fluids of high conductivity at comparatively low frequencies (well below the ion cyclotron frequency), as discussed in section 6, of Chapter 9. The

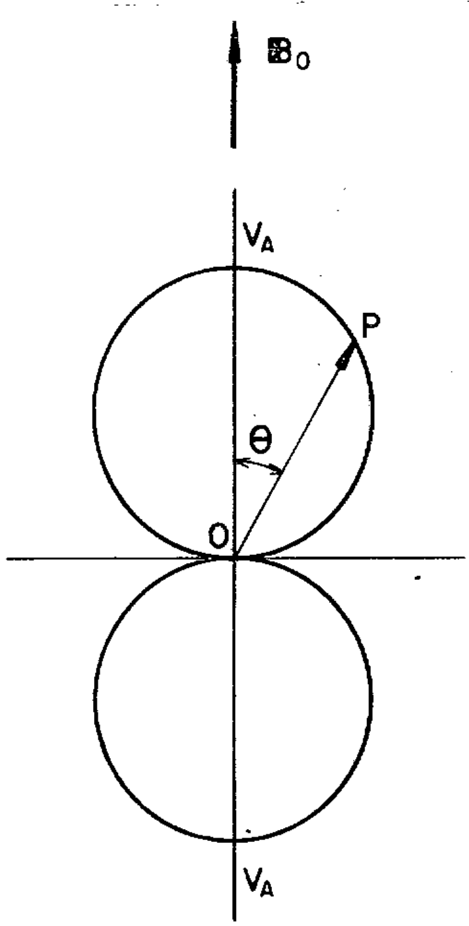

Fig. 8 - Wave normal diagram illustrating the variations of the phase velocity and the direction of the wave normal at any angle θ with respect to $\underset{\sim}{B}_o$, for the pure Alfvén wave.

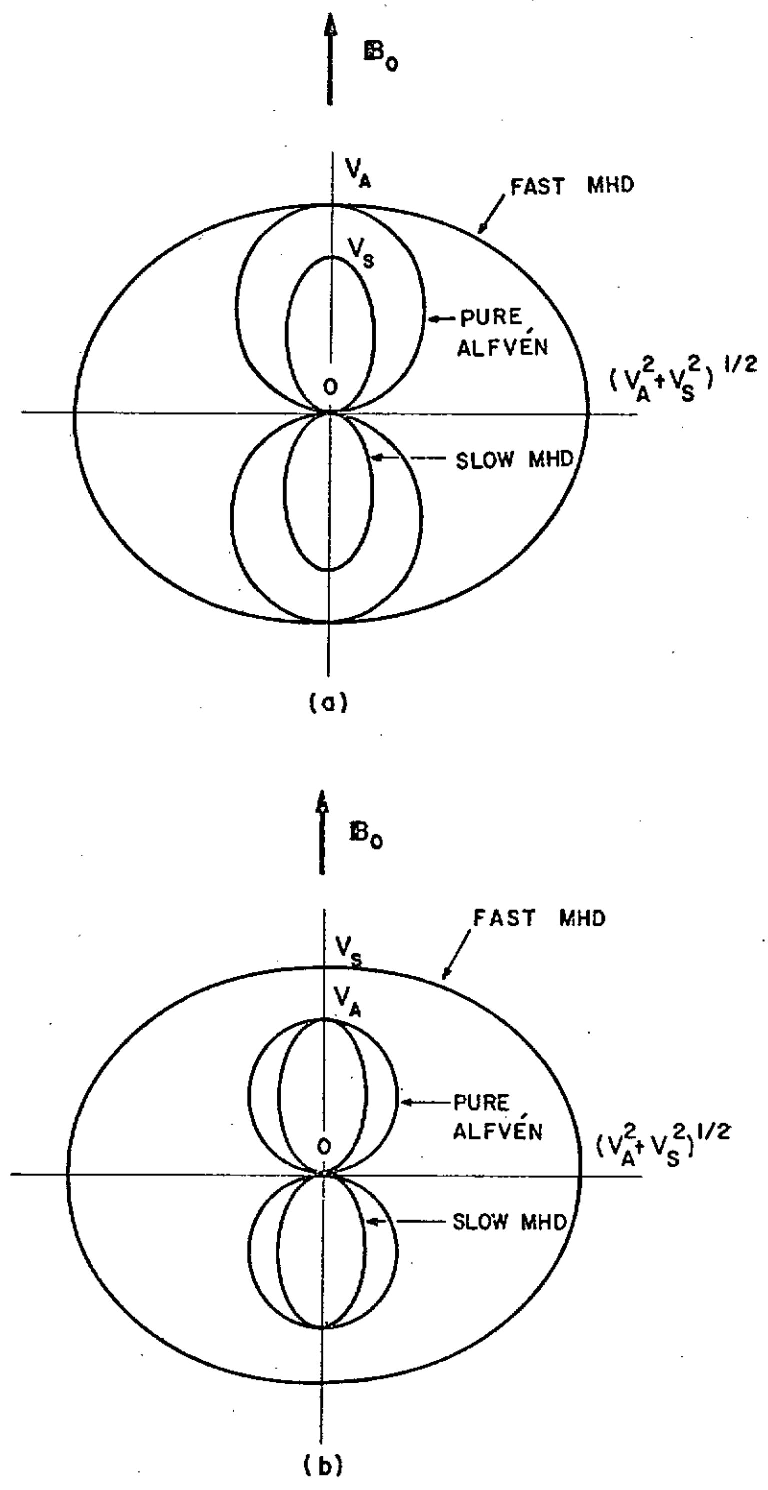

Fig. 9 - Wave normal diagrams for the pure Alfvén, the fast and the slow MHD waves, for (a) $V_A > V_s$ and (b) $V_A < V_s$.

inclusion of the displacement current in the basic equations modify the propagation of the Alfvén and magnetosonic waves. The results obtained, however, are valid only at frequencies where charge separation effects are unimportant.

6.1 Basic equations

To investigate the effect of the displacement current on the propagation of MHD waves in a compressible, nonviscous, perfectly conducting fluid, Eq. (2.4) must be modified to read

$$\underset{\sim}{\nabla} \times \underset{\sim}{B} = \mu_o \underset{\sim}{J} + \frac{1}{c^2} \frac{\partial \underset{\sim}{E}}{\partial t} \tag{6.1}$$

Consequently, the current density to be inserted into the $\underset{\sim}{J} \times \underset{\sim}{B}$ term, in the equation of motion (2.2), is now

$$\underset{\sim}{J} = \frac{1}{\mu_o} \left[\underset{\sim}{\nabla} \times \underset{\sim}{B} + \frac{1}{c^2} \frac{\partial}{\partial t} (\underset{\sim}{u} \times B) \right] \tag{6.2}$$

where use was made of (2.6). Using expressions (2.9) to (2.11) for small-amplitude waves, the set of linearized equations (2.12) to (2.14) for the small quantities ρ_1, $\underset{\sim}{u}_1$ and $\underset{\sim}{B}_1$, become now

$$\frac{\partial \rho_1}{\partial t} + \rho_o \ (\underset{\sim}{\nabla} \cdot \underset{\sim}{u}_1) = 0 \tag{6.3}$$

$$\rho_o \frac{\partial \underset{\sim}{u}_1}{\partial t} + V_s^2 \underset{\sim}{\nabla} \rho_1 + \frac{1}{\mu_o} \underset{\sim}{B} \times (\underset{\sim}{\nabla} \times \underset{\sim}{B}_1 + \frac{1}{c^2} \frac{\partial \underset{\sim}{u}_1}{\partial t} \times \underset{\sim}{B}_o) = 0 \tag{6.4}$$

$$\frac{\partial \underset{\sim}{B}_1}{\partial t} - \underset{\sim}{\nabla} \times (\underset{\sim}{u}_1 \times \underset{\sim}{B}_o) = 0 \qquad (6.5)$$

6.2 Equation for $\underset{\sim}{u}_1$

To obtain an equation for $\underset{\sim}{u}_1$ alone, we take the time derivative of (6.4), and use (6.3) and (6.5), which gives

$$\frac{\partial^2 \underset{\sim}{u}_1}{\partial t^2} - V_S^2 \underset{\sim}{\nabla} (\underset{\sim}{\nabla} \cdot \underset{\sim}{u}_1) + \underset{\sim}{V}_A \times \{\underset{\sim}{\nabla} \times [\underset{\sim}{\nabla} \times (\underset{\sim}{u}_1 \times \underset{\sim}{V}_A)]\} + $$

$$+ \frac{1}{c^2} \underset{\sim}{V}_A \times (\frac{\partial^2 \underset{\sim}{u}_1}{\partial t^2} \times \underset{\sim}{V}_A) = 0 \qquad (6.6)$$

where $\underset{\sim}{V}_A$ is the vector Alfvēn velocity, defined in (2.17). From the vector identity (2.20), we have

$$\underset{\sim}{V}_A \times (\frac{\partial^2 \underset{\sim}{u}_1}{\partial t^2} \times \underset{\sim}{V}_A) = \frac{\partial^2}{\partial t^2} [V_A^2 \underset{\sim}{u}_1 - (\underset{\sim}{V}_A \cdot \underset{\sim}{u}_1) \underset{\sim}{V}_A] \qquad (6.7)$$

so that (6.6) can be rearranged in the form

$$\frac{\partial^2}{\partial t^2} [(1 + \frac{V_A^2}{c^2})\underset{\sim}{u}_1 - (\underset{\sim}{V}_A \cdot \underset{\sim}{u}_1) \frac{\underset{\sim}{V}_A}{c^2}] - V_S^2 \underset{\sim}{\nabla} (\underset{\sim}{\nabla} \cdot \underset{\sim}{u}_1) +$$

$$+ \underset{\sim}{V}_A \times \{\underset{\sim}{\nabla} \times [\underset{\sim}{\nabla} \times (\underset{\sim}{u}_1 \times \underset{\sim}{V}_A)]\} = 0 \qquad (6.8)$$

It is evident that this equation reduces to (2.16), if $V_A^2/c^2 \ll 1$.

Plane wave solutions of (6.8), in the form (2.18), give

$$- \omega^2 \left[\left(1 + \frac{V_A^2}{c^2}\right) \underset{\sim}{u}_1 - (\underset{\sim}{V}_A \cdot \underset{\sim}{u}_1) \frac{\underset{\sim}{V}_A}{c^2} \right] + (V_S^2 + V_A^2)\, (\underset{\sim}{k} \cdot \underset{\sim}{u}_1)\, \underset{\sim}{k} +$$

$$+ (\underset{\sim}{k} \cdot \underset{\sim}{V}_A) \left[(\underset{\sim}{k} \cdot \underset{\sim}{V}_A) \underset{\sim}{u}_1 - (\underset{\sim}{V}_A \cdot \underset{\sim}{u}_1) \underset{\sim}{k} - (\underset{\sim}{k} \cdot \underset{\sim}{u}_1)\, \underset{\sim}{V}_A \right] = 0 \qquad (6.9)$$

6.3 Propagation across the magnetostatic field

For $\underset{\sim}{k} \perp \underset{\sim}{B}_o$ we have $\underset{\sim}{k} \cdot \underset{\sim}{V}_A = 0$, so that (6.9) gives $(\underset{\sim}{V}_A \cdot \underset{\sim}{u}_1) = 0$ and

$$- \omega^2 \left(1 + \frac{V_A^2}{c^2}\right) \underset{\sim}{u}_1 + (V_S^2 + V_A^2)\, (\underset{\sim}{k} \cdot \underset{\sim}{u}_1)\, \underset{\sim}{k} = 0 \qquad (6.10)$$

This equation is similar to (3.1), except that the square of the frequency is multiplied by the factor $(1 + V_A^2/c^2)$. Thus, the phase velocity of the *longitudinal magnetosonic wave* propagating across $\underset{\sim}{B}_o$ becomes now

$$\frac{\omega}{k} = \frac{(V_S^2 + V_A^2)^{1/2}}{(1 + V_A^2/c^2)^{1/2}} \qquad (6.11)$$

6.4 Propagation along the magnetostatic field

For $\underset{\sim}{k} \parallel \underset{\sim}{B}$, inspection of (6.9) shows that for $\underset{\sim}{u}_1$ parallel to $\underset{\sim}{V}_A$ (i.e., $\underset{\sim}{B}_o$) it becomes identical to (2.21). Thus, for the *longitudinal sound wave* propagating along $\underset{\sim}{B}_o$ there is no change from the results obtained before.

However, for the *transverse Alfvén wave* $(\underset{\sim}{u}_1 \perp \underset{\sim}{k})$ we have $(\underset{\sim}{V}_A \cdot \underset{\sim}{u}_1) = 0$ and (6.9) reduces to

$$-\omega^2 \left(1 + V_A^2/c^2\right) \underset{\sim}{u}_1 + k^2 V_A^2 \underset{\sim}{u}_1 = 0 \tag{6.12}$$

Consequently, the modification introduced in the Alfvēn wave by the displacement current is that the square of the frequency must be multiplied by the factor $(1 + V_A^2/c^2)$. Thus, the phase velocity of the Alfvēn wave becomes

$$\frac{\omega}{k} = \frac{V_A}{(1 + V_A^2/c^2)^{1/2}} \tag{6.13}$$

In the usual limit of $V_A^2/c^2 << 1$, (6.13) reduces to (4.3) and the effect of the displacement current is unimportant. On the other hand, if $V_A^2/c^2 >> 1$, then ω/k becomes equal to the speed of light. In using these results, however, it must be kept in mind that they are valid only at frequencies where charge separation effects are negligible.

7. DAMPING OF MHD WAVES

In this section it is shown that when the fluid is not perfectly conducting, but has a finite conductivity, or if viscous effects are present, the MHD oscillations will be damped. Denoting the kinematic viscosity (viscosity divided by mass density) of the fluid by η_k, and the magnetic viscosity by η_m [see Eq. (6.2.5.)], the linearized set of equations (2.12) to (2.14) are modified to include additional terms, as follows,

$$\frac{\partial \rho_1}{\partial t} + \rho_o \; (\underset{\sim}{\nabla} \cdot \underset{\sim}{u}_1) = 0 \tag{7.1}$$

$$\rho_o \frac{\partial \underset{\sim}{u}_1}{\partial t} + V_s^2 \underset{\sim}{\nabla} \rho_1 + \frac{1}{\mu_o} \underset{\sim}{B}_o \times (\underset{\sim}{\nabla} \times \underset{\sim}{B}_1) - \rho_o \eta_k \; \nabla^2 \underset{\sim}{u}_1 = 0 \tag{7.2}$$

$$\frac{\partial \underset{\sim}{B}_1}{\partial t} - \underset{\sim}{\nabla} \times (\underset{\sim}{u}_1 \times \underset{\sim}{B}_o) - \eta_m \nabla^2 \; \underset{\sim}{B}_1 = 0 \tag{7.3}$$

Although, for a compressible fluid the use of the simple viscous force term $\rho_o \eta_k \nabla^2 \underset{\sim}{u}_1$ is not really allowed, it is, nevertheless, expected to give the correct order of magnitude behavior. The displacement current is not included in the treatment presented in this section.

For plane wave solutions, the differential operators $\partial/\partial t$ and $\underset{\sim}{\nabla}$ are replaced, respectively, by $-i\omega$ and $i\underset{\sim}{k}$, so that the set of differential equations (7.1) to (7.3) are replaced by a correspondent set of algebraic equations. Thus, we have

$$\rho_1 = \rho_o \frac{\underset{\sim}{k} \cdot \underset{\sim}{u}_1}{\omega} \tag{7.4}$$

$$\omega \underset{\sim}{u}_1 = \frac{\rho_1}{\rho_o} V_s^2 \underset{\sim}{k} + \frac{1}{\mu_o \rho_o} \underset{\sim}{B}_o \times (\underset{\sim}{k} \times \underset{\sim}{B}_1) - i \eta_k k^2 \underset{\sim}{u}_1 \tag{7.5}$$

$$\underset{\sim}{B}_1 = - \frac{\underset{\sim}{k} \times (\underset{\sim}{u}_1 \times \underset{\sim}{B}_o)}{(\omega + i \eta_m k^2)} \tag{7.6}$$

Substituting (7.4) and (7.6) into (7.5) , and rearranging, we obtain

$$-\omega^2 \left(1 + i\,\frac{\eta_k k^2}{\omega}\right)\left(1 + i\,\frac{\eta_m k^2}{\omega}\right) \underset{\sim}{u}_1 + \left(1 + i\,\frac{\eta_m k^2}{\omega}\right) V_S^2\, (\underset{\sim}{k} \cdot \underset{\sim}{u}_1)\, \underset{\sim}{k} -$$

$$- \underset{\sim}{V}_A \times \{\underset{\sim}{k} \times [\underset{\sim}{k} \times (\underset{\sim}{u}_1 \times \underset{\sim}{V}_A)]\} = 0 \qquad (7.7)$$

Comparing this equation for $\underset{\sim}{u}_1$ with (2.19), we see that we obtain the same results as before, except that ω^2 must be multiplied by the factor $(1 + i\eta_k\, k^2/\omega)\,(1 + i\eta_m\, k^2/\omega)$, and V_S^2 must be multiplied by the factor $(1 + i\eta_m\, k^2/\omega)$.

7.1 Alfvēn Waves

For the case of the transverse Alfvēn waves propagating along $\underset{\sim}{B}_o$, the relation (4.3) between ω and k becomes

$$k^2 V_A^2 = \omega^2 \left(1 + i\,\frac{\eta_k k^2}{\omega}\right)\left(1 + i\,\frac{\eta_m k^2}{\omega}\right)$$

$$= \omega^2 \left[1 + i\,\frac{k^2}{\omega}(\eta_k + \eta_m) - \frac{\eta_k\,\eta_m\,k^4}{\omega^2}\right] \qquad (7.8)$$

In order to simplify this result, we shall assume that the correction terms corresponding to the kinematic and magnetic viscosity are small, so that the term in the right-hand side of (7.8) can be neglected. Thus,

$$k^2 V_A^2 \simeq \omega^2 \left[1 + i \frac{k^2}{\omega} (\eta_m + \eta_k)\right]$$

$$\simeq \omega^2 \left[1 + i \frac{\omega}{V_A^2} (\eta_m + \eta_k)\right] \quad (7.9)$$

where we have replaced ω/k, in the right-hand side, by the first order result (V_A). Eq. (7.9) can be further simplified to the form (using the binominal expansion $(1 + x)^{1/2} \simeq 1 + x/2$, $x << 1$)

$$k \simeq \frac{\omega}{V_A} + i \frac{\omega^2}{2V_A^3} (\eta_m + \eta_k) \quad (7.10)$$

The positive imaginary part in this expression for $k(\omega)$ implies in damping of the waves. This is easily seen by writing $k = k_r + i\, k_i$, with k_r and k_i real numbers, and noting that

$$e^{ikz} = e^{-k_i z} \cdot e^{ik_r z} \quad (7.11)$$

which represents a wave propagating along the z - axis with wave number k_r, but with an exponentially decreasing amplitude, the amplitude falling to 1/e of its original intensity in a distance of $1/k_i$.

Expression (7.10) shows that the attenuation of Alfvén waves increases rapidly with frequency (or wave number), but decreases rapidly with increasing magnetic field intensity. Also, the attenuation increases with the fluid viscosity and the magnetic viscosity. The latter increases as the fluid conductivity decreases.

7.2 Sound waves

For the longitudinal sound waves propagating along $\underset{\sim}{B}_o$, Eq. (4.2) is modified to read

$$k^2 V_s^2 = \omega^2 \left(1 + i \frac{\eta_k k^2}{\omega}\right) \quad (7.12)$$

Considering that the resistive and viscous correction terms are small, we find

$$k \simeq \frac{\omega}{V_s} + i \frac{\omega^2}{2V_s^3} \eta_k \quad (7.13)$$

This shows that attenuation of sound waves also increases rapidly with frequency, but decreases with increasing sound velocity. It also increases with increasing fluid viscosity, as expected.

7.3 Magnetosonic waves

For longitudinal magnetosonic waves propagating across $\underset{\sim}{B}_o$, the dispersion relation becomes [see Eq. (3.3)]

$$k^2 V_s^2 \left(1 + i\frac{\eta_m k^2}{\omega}\right) + k^2 V_A^2 = \omega^2 \left(1 + i\frac{\eta_k k^2}{\omega}\right) \left(1 + i\frac{\eta_m k^2}{\omega}\right) \quad (7.14)$$

To simplify this expression we consider that the kinematic and magnetic viscosity are small, and neglect the term involving the product $\eta_m \eta_k k^4/\omega^2$. Hence, (7.14) becomes, after some rearrangement,

$$k^2 (V_s^2 + V_A^2) \simeq \omega^2 \{1 + i \frac{k^2}{\omega} [\eta_k + \eta_m (1 - \frac{k^2 V_s^2}{\omega^2})]\} \qquad (7.15)$$

In the terms in the right-hand side of (7.15) we can replace ω^2/k^2 by the approximate result $(V_s^2 + V_A^2)$, so that (7.15) can be further simplified to give the following dispersion relation

$$k = \frac{\omega}{(V_s^2 + V_A^2)^{1/2}} + i \frac{\omega^2}{2 (V_s^2 + V_A^2)^{3/2}} [\eta_k + \frac{\eta_m}{(1 + V_s^2/V_A^2)}] \qquad (7.16)$$

Thus, the attenuation of magnetosonic waves also increases with frequency, and with kinematic and magnetic viscosity, but decreases with increasing magnetic field strength.

PROBLEMS

15.1 - Calculate the speed of an Alfvén wave for the following cases:

(a) In the Earth's ionosphere, considering that $n_e = 10^5\ cm^{-3}$, $B = 0.5$ Gauss and that the positive charge carriers are atomic oxygen ions;

(b) In the solar corona, assuming $n_e = 10^6\ cm^{-3}$, $B = 10$ Gauss and that the positive charge carriers are protons;

(c) In the interstellar space, considering $n = 10^7\ m^{-3}$ and $B = 10^{-7}$ Weber/m^2, the positive charge carriers being protons.

15.2 - Show that Alfvén waves, propagating along the magnetic field, are circularly polarized.

Hint: for this problem it is appropriate to derive first the dispersion relation for transverse electromagnetic waves propagating along $\underset{\sim}{B}_o$, in a two-fluid (electrons and one type of ions) plasma, and then take the limit for very low frequencies.

15.3 - For the pure Alfvén wave, propagating at an angle θ with respect to the magnetostatic field $\underset{\sim}{B}_o$, with phase velocity given by

Eq. (5.10), determine the associated field components B_{1y}, u_{1y}, E_{1x}, and J_{1x}.

15.4 - Include the effect of finite conductivity in the derivation of the equations for the plane Alfvén wave propagating along the magnetic field. Show that the linearized equations are satisfied by solutions of the form $\exp(\alpha z - i\omega t)$ and determine the coefficient α.

15.5 - A plane electromagnetic wave is incident normally on the surface of a conducting fluid of large but finite conductivity (σ), immersed in a uniform magnetic field $\underset{\sim}{B}_o$ such that $\underset{\sim}{k} \perp \underset{\sim}{B}_o$. Assume that the magnetic field ($\underset{\sim}{B}$) of the incoming wave is parallel to $\underset{\sim}{B}_o$. Show that there are two wave modes which penetrate the fluid: an unattenuated magnetosonic wave, and another mode which has an effective skin depth $\delta = (V_S/V_M)\,\delta_{rc}$, where V_S and V_M are the sound and magnetosonic velocities, respectively, and δ_{rc} is the skin depth in a rigid conductor.

15.6 - For the fast and slow MHD waves, let u_ℓ and u_t be the components of the velocity of mass flow which are longitudinal and transverse, respectively, to the direction of propagation. Show that u_ℓ and u_t are in phase for the fast wave and 180^o

out of phase for the slow wave. Also, show that the perturbations of the kinetic and magnetic pressures are in phase for the fast wave and 180^0 out of phase for the slow wave.

15.7 - Consider the following closed set of MHD equations in the so-called Chew, Goldberger and Low approximation:

$$\frac{\partial}{\partial t}\rho + \nabla\cdot(\rho \mathbf{u}) = 0$$

$$\rho \frac{D}{Dt}\mathbf{u} = \rho_c \mathbf{E} - \nabla\left(p_\perp + \frac{B^2}{2\mu_o}\right) + (\mathbf{B}\cdot\nabla)\left\{\left[\frac{1}{\mu_0} - \frac{(p_{\parallel} - p_\perp)}{B^2}\right]\mathbf{B}\right\}$$

$$\frac{D}{Dt}\left(\frac{p_{\parallel} B^2}{\rho^3}\right) = 0$$

$$\frac{D}{Dt}\left(\frac{p_\perp}{\rho B}\right) = 0$$

$$\nabla\times\mathbf{B} = \mu_o \mathbf{J}$$

$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla\cdot\mathbf{E} = \rho_c/\varepsilon_o$$

$$\underset{\sim}{E} + \underset{\sim}{u} \times \underset{\sim}{B} = 0$$

In the equations of this set, involving the pressure tensor $\underset{\approx}{p}$, it is considered that

$$\underset{\approx}{p} = \begin{pmatrix} p_\perp & 0 & 0 \\ 0 & p_\perp & 0 \\ 0 & 0 & p_\parallel \end{pmatrix}$$

(a) Taking the *equilibrium* mean velocity equal to zero, show that the dispersion relation for the magnetohydrodynamic waves is given by

$$\rho_o \omega^2 + k \cos\Theta \left(p_\parallel - p_\perp - \frac{B_o^2}{\mu_o} \right) - k^2 \sin^2\Theta \left(2 p_\perp + \frac{B_o^2}{\mu_o} \right) =$$

$$= \frac{p_\perp^2 \, k^4 \sin^2\Theta \, \cos^2\Theta}{\rho_o \omega^2 - 3 p_\parallel k^2 \cos^2\Theta}$$

where Θ is the angle between $\underset{\sim}{k}$ and $\underset{\sim}{B}_o$, and ρ_o, $p_\parallel$, $p_\perp$ and B_o stand for the unperturbed quantities.

(b) Show that these waves are unstable for all values of Θ less than a critical angle Θ_c, which satisfies the equation

$$\frac{B_o^2}{\mu_o} + p_\perp \,(1 + \sin^2 \Theta_c) = \frac{p_\perp^2}{3p_\parallel} \sin^2 \Theta_c + 2\, p_\parallel \cos^2 \Theta_c$$