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# THE PROBLEM OF TRANSFER ORBITS FROM ONE BODY BACK TO THE SAME BODY

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# THE PROBLEM OF TRANSFER ORBITS FROM ONE BODY BACK TO THE SAME BODY

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The problem of transfer orbits from one body back to the same body (the Moon or a planet) is formulated as a Lambert's problem and solved by Gooding's Lambert routines. We consider elliptic as well as circular orbits for the Moon or a planet and any kind of orbit (elliptic, parabolic or hyperbolic) for the spacecraft. The solutions are plotted in terms of the true anomaly (instead of the eccentric anomaly) for several cases. We show that the use of the true anomaly simplifies the solutions in several ways. We also solved the problem of transfers from this body to the corresponding $L_4$ and $L_5$ points. After that, the same problem is studied in terms of the $\Delta V$ and the time required for the transfer. Among all the possible transfer orbits, a small family with almost zero $\Delta V$ was found. The properties of these orbits are shown in details.

## INTRODUCTION

The problem of transfer orbits from one body back to the same body (the Moon or a planet) is under investigation for a long time. Hénon[1] originally developed a timing condition in the eccentric anomaly for orbits that allows a spacecraft to leave the massless body $M_2$ (the Moon or a planet in our case), goes to an orbit around the other primary $M_1$ (the Earth or the Sun in our case) and meet $M_2$ again, after a certain time. Originally this problem was studied as the problem of consecutive collision orbits in the restricted three body problem. After this paper, several authors worked on improvements of this problem. Hitzl[2] and Hitzl and Hénon[3,4] *studied* stability and critical orbits. Perko[5] derived a proof of existence and a timing condition for what was shown later to be a special case of Hénon's work. Results for the perturbed case $\mu > 0$ (where $M_2$ is assumed to have mass different of zero and cause perturbation on the orbit of $M_3$ around $M_1$) also appeared in the literature. Some examples are the papers published by Gomez and Ollé[6,7] and Bruno[8]. Howell[9,10] extended Hénon's results for the case where the orbit of $M_2$ is elliptic.

In this paper we formulate the problem as an orbit transfer problem, which can be solved with the Gooding's implementation of the Lambert's problem[11]. We gave the solution in terms of the true anomaly, instead of the eccentric anomaly, as has always been done by previous authors. This reveals to be an interesting approach. Both cases, with the target body (Moon or planet) in a circular orbit or in an elliptic orbit are being considered in the present paper. All possible kinds of orbits for the spacecraft $M_3$ are considered: elliptic, parabolic and hyperbolic. At the same time a new problem has been solved: the transfer of the spacecraft from $M_2$ to the corresponding $L_4$ or $L_5$ points, assumed to be on the same circular orbit as $M_2$, either 60 degrees ahead of it or 60 degrees behind of it. The implementation developed is generic with respect to this angle and allows us to study a transfer from $M_2$ to any point in the same circular orbit (not only 60 degrees ahead or behind it). After that, these transfer orbits are studied in terms of the $\Delta V$ and the time required for the transfer. The $\Delta V$s are plotted against the transfer time for several cases and a family of transfer orbits with very small $\Delta V$ (in the order of 0.001 in canonical units) is shown to exist in almost all cases studied. These orbits are study in details. They consist of a family of slight different orbits (when compared to the orbit of $M_2$) that meet all the requirements to provide the transfer desired.

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## FORMULATION OF THE PROBLEM

Let $M_1$ and $M_2$ be the two primaries with masses $(1-\mu)$ and $\mu$ respectively. $M_2$ is in a circular (in the original version of the problem studied by Hénon) or elliptic (in the extension made by Howell[9,10]) orbit around $M_1$. The massless spacecraft $M_3$ leaves $M_2$ from a point P ($t = -\tau$), follows an orbit around $M_1$ and meets again with $M_2$ at a point Q ($t = \tau$). Since only the limiting case $\mu = 0$ is considered, the three-body problem is reduced to the two-body problem and basic equations from Celestial Mechanics do apply. The units are chosen such that the distance between the two primaries and the angular velocity of the system is unit. It is also assumed that all the three bodies involved are points of mass and there are no perturbations from other bodies.

The solution to be found is the coordinate of the point P as a function of the transfer time. The solution is not unique, and a graph including many solutions was published by Hénon[1]. He plotted $\underline{\eta}/\pi$ (where $\underline{\eta}$ is the redefined "eccentric anomaly" of the point P, as defined by Eq. (1)) against $\tau/\pi$ ($\tau$ is half of the transfer time). For the case where the orbit of $M_3$ is hyperbolic, the solutions were plotted in a separate graph[9] with the eccentric anomaly replaced by the analogous hyperbolic eccentric anomaly. Another problem that we consider in the present paper is to calculate the $\Delta V$ and the time required for each of these transfers, in a search for transfer orbits with small $\Delta V$. The solution consists of plots of the $\Delta V$ against the time required for the transfer (both in canonical units). A detailed study of the transfer orbits with small $\Delta V$ is included, with pictures and numeric parameters listed.

Possible applications for this work are: interplanetary research of the Solar System; a basis for a transportation system between Earth ($M_1$) and Moon ($M_2$) that requires no nominal orbit correction; etc.

## HÉNON'S APPROACH

The mathematical formulation used by Hénon to solve this problem is explained in the Appendix A. To express his solutions for this problem, Hénon solved it for a large number of cases and plotted $(\underline{\eta}/\pi)$ against $(\tau/\pi)$, where $\underline{\eta}$ is the redefined "eccentric anomaly", assumed to be:

$$\begin{aligned} \underline{\eta} &= \eta \text{ if } M_3 \text{ pass by the perigee at } t = 0 \\ \underline{\eta} &= \eta - \pi \text{ if } M_3 \text{ pass by the apogee at } t = 0 \end{aligned} \quad (1)$$

Each point in his graph is a Keplerian orbit leaving a point P with eccentric anomaly $-\eta$, making an orbit around $M_1$ and meeting again $M_2$ at a point Q with eccentric anomaly $\eta$, after a time $2\tau$. Hénon did not include hyperbolic orbits for $M_3$ in his famous graph[1], since he was not interested in this kind of orbit, but he had them calculated in a table.

## LAMBERT'S PROBLEM FORMULATION

A different approach used in this paper is to formulate Hénon's problem as a Lambert's problem. The Lambert's problem can be defined as[11]:

"An (unperturbed) orbit, about a given inverse-square-law center of force is to be found connecting two given points, $P_1$ and $P_2$, with a flight time $\Delta t$ ($= t_2 - t_1$) that has been specified. The problem must always have at least one solution and the actual number, which we denote by N, depends on the geometry of the problem - it is assumed, for convenience and with no loss of generality, that $t > 0$."

So, in our formulation, Hénon's problem will be: Find an unperturbed orbit for $M_3$, around $M_1$, that makes it leave the point P at $t = -\tau$ and goes to point Q at $t = \tau$. Since $M_2$ is assumed to have zero mass, we do not need to include it in the equations of motion. It is only use is to relate the time $\tau$ with the eccentric anomaly $\eta$, in such way that $M_3$ has the same position as $M_2$ at P and Q in the times $t = -\tau$ and $t = \tau$, respectively.

## MATHEMATICAL FORMULATION

In terms of mathematical formulation, Hénon's problem formulated as a Lambert's problem can be described as follows. We have the following information available:

1. The position of $M_3$ at $t = -\tau$ (point P). It can be specified by the radius vector $R_1$ and the angle $-\tau$. We can relate $R_1$ with $-\tau$ by using the equation $R_1 = a(1-e^2)/(1+e\cos(-\tau))$ for the orbit of $M_2$, since $M_2$ and $M_3$ occupy the same position at $t = -\tau$;

2. The position of $M_3$ at $t = \tau$ (point Q). It can be specified by the radius vector $R_2$ and the angle $\tau$. We can relate $R_2$ with $\tau$ by using the same equation that we used in the above paragraph;

3. The total time for the transfer, $\Delta t = 2\tau$. Remember that the angular velocity of the system is unit, so we can consider $\tau$ as the time or the angle;

4. The total angle the spacecraft must travel to go from P to Q, that we will call $\phi$. For the case where the orbit of $M_3$ is elliptic this variable has several possible values. First of all, we have to consider two possible choices for the transfer: the one that use the sense of the shortest possible angle between P and Q (that we will call the "short way"), and the one that use the sense of the longest possible angle between these two points (that we will call the "long way"). Which one is the shortest or the longest will depend on the value of $\tau$. After considering these two choices, we also have to consider the possibilities of multi-revolution transfers. In this case, the spacecraft leaves P, makes one or more complete revolutions around $M_1$, and then goes to Q. So, by combining these two factors, the possible values for $\phi$ will be: $2\tau+2m\pi$ and $2(\pi-\tau)+2m\pi$, where m is an integer that represents the number of complete revolutions during the transfer. There is no upper limit for m, and our problem has an infinite number of solutions. For the case where the orbit of $M_3$ is parabolic or hyperbolic $\phi$ has a unique value. The multi-revolution transfer does not exist anymore (the orbit is not closed), and the only sense of transfer that has a solution is the one that makes the spacecraft goes in a retrograde orbit passing by the perigee at $t = 0$.

The information that we need (the solution of the Lambert's problem) is the Keplerian orbit that contains the points P and Q and requires the given transfer time $\Delta t = 2\tau$ for a spacecraft to travel between these two points. This solution can be specified in several ways. The velocity vectors at P or Q are two possible choices, since we have the corresponding position vectors. The Keplerian elements of the transfer orbit is also another possible set of coordinates to express the solution of our problem. In the implementation that we made, all three sets of coordinates are obtained, since all of them are useful later.

To obtain the $\Delta V$s, we followed the steps:

1. Find the radial and transverse velocity components of $M_2$ at P and Q. They are also the velocity components of $M_3$ just before the first impulse and just after the second impulse, respectively, since they match their orbits at these points. They are obtained from the equations[12]:

$$V_r = \frac{e \sin(v)}{\sqrt{a(1-e^2)}} \quad V_t = \frac{1+e\cos(v)}{\sqrt{a(1-e^2)}} \qquad (2)\text{-}(3)$$

where $V_r$ and $V_t$ are the radial and transverse components of the velocity vector, a and e are the semi-major axis and the eccentricity of the transfer orbit and v is the true anomaly of the spacecraft.

2. Find an unperturbed orbit for $M_3$ that allows it to leave the point P at $t = -\tau$ and arrives in point Q at $t = \tau$. We found this orbit by solving the associate Lambert's problem, as better explained in the next section. At this point, we have already the total time for this transfer, $2\tau$.

3. Find the velocity components at these points (P and Q) in the just found transfer orbit. They are the velocity components for $M_3$ just after the first impulse and just before the second impulse. They are provided by the Gooding's Lambert routine[11].

4. With the velocity components just after and just before both impulses we can calculate the magnitude of both impulses ($\Delta V_1$ and $\Delta V_2$) and add them together to get the total impulse required ($\Delta V$) for the transfer.

## GOODING'S IMPLEMENTATION OF THE LAMBERT'S PROBLEM

The solution of the Lambert's problem, as defined in the paragraphs above, is also under investigation for a long time. The approach to solve this problem is to set up a set of non-linear equations (from the two-body problem) and start an iterative process to find an orbit that satisfies all the requirements. There is no closed-form available for the solution of this problem. The major difficulty is to choose the best set of equations and parameter for iterations to guaranty that convergence will occur in all cases. The routine used in this paper is due to Gooding[11]. He chooses $x = \pm\sqrt{1-s/2a}$ as the parameter for convergence, where "a" is the semi-major axis of the transfer orbit and "s" the semi-perimeter of the triangle formed by P, Q and $M_1$. He also made several substitutions of variables, trying to find the best set of equations to guaranty convergence. His implementation allows us to find all possible solutions of the Lambert's problem, including "long way", "short way" and "multi-revolution" transfers. He gave the velocity vectors at P and Q and the Keplerian elements of the transfer orbit in his solution. The basic equations used by Gooding are shown in Appendix B, and more details can be found in reference[11].

Including all phases of this paper, Gooding's routine was called about 3 million times with no failure detected. The average time required to solve one time the Lambert's problem was about 2 milliseconds (in a compatible IBM-PC 486/33 MHz with 256 K cache memory).

## SOLUTION IN TERMS OF THE TRUE ANOMALY

Another new aspect presented in this paper is the modification in the coordinates used to express the position of the point P ($t = -\tau$) in the solution of the problem. The eccentric anomaly (that was used by all other authors, since Hénon's first paper[1]) was replaced by the true anomaly. Fig. 1 shows the solutions in the new coordinate. To keep similarity with the previous authors, we used the redefined true anomaly, that is:

$$\begin{aligned} \underline{\nu} &= \nu \text{ if } \varepsilon'' = +1 \\ \underline{\nu} &= \nu - \pi \text{ if } \varepsilon'' = -1 \end{aligned} \qquad (4)$$

where $\nu$ is the standard true anomaly.

It is easy to see that they are straight lines inclined by 45 degrees, forming squares. In this much simple form, the information required to express the solution can be stored as the inclinations and the extreme points of the segments, instead of the much more complex form given by the use of the eccentric anomaly. Table 1 shows the extreme points for the segments of the straight lines shown in Fig. 1. The solutions found for the case where the orbit of $M_3$ is hyperbolic is not shown here, to save space, but they were also easily obtained. A new advantage of the use of the true anomaly is that the solutions having hyperbolic or elliptic orbits for $M_3$ can be combined in the same graph, since the definition of the true anomaly remains unchanged for all kinds of orbits. In the other side, when the eccentric and hyperbolic eccentric anomalies are used, we have to keep them plotted in separate graph, since they are slight different quantities with different physical meaning and range of values. The parabolic solution is restricted to only one point, at $\tau/\pi = 0.16393$, that separate the elliptic from the hyperbolic orbits.

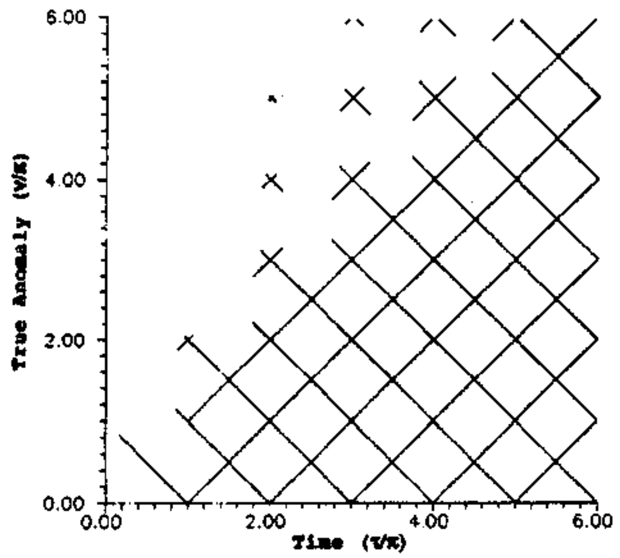


**Fig.1 Elliptic Solutions in Terms of True Anomaly.**

**Table 1**
**THE EXTREME POINTS FOR THE STRAIGHT LINES SEGMENTS SHOWN IN FIG. 1**

| τ/π | ν/π | τ/π | ν/π | τ/π | ν/π | τ/π | ν/π | τ/π | ν/π |
|---|---|---|---|---|---|---|---|---|---|
| 0.172 | 0.828 | 1.801 | 2.801 | 2.062 | 4.155 | 2.908 | 5.908 | 3.817 | 5.817 |
| 0.856 | 1.144 | 1.891 | 3.109 | 2.134 | 3.866 | 3.115 | 5.885 | 4.231 | 5.769 |
| 0.883 | 1.883 | 1.900 | 3.900 | 2.152 | 3.152 | 3.124 | 5.124 | 4.240 | 5.240 |
| 0.964 | 2.036 | 1.945 | 4.055 | 2.755 | 3.245 | 3.196 | 4.804 | 4.708 | 5.708 |
| 1.054 | 2.054 | 1.963 | 4.963 | 2.755 | 3.755 | 3.205 | 4.205 | 4.708 | 5.292 |
| 1.378 | 1.622 | 1.990 | 5.001 | 2.845 | 4.155 | 3.727 | 4.273 | | |
| 1.423 | 1.577 | 2.017 | 5.017 | 2.854 | 4.854 | 3.727 | 4.727 | | |
| 1.792 | 2.208 | 2.044 | 4.956 | 2.899 | 5.101 | 3.808 | 5.192 | | |

## ELLIPTIC CASE

Another improvement made in the original Hénon's work[1], made by Howell[9,10], was to study the case where $M_2$ is in an elliptic orbit around $M_1$. The approach that she used was the same one used by Hénon. Two-body problem equations were written and solved to find the points [(η/π),(τ/π)]. Two different cases were studied: the one where $M_2$ is at perigee at t = 0 and the one where $M_2$ is at apogee at t = 0.

In the present paper, these same extensions of Hénon's work were studied by using the Lambert's problem approach. Very few modifications in the implementation developed for the circular case were necessary. Fig. 2 to Fig. 6 show some of the results obtained in both coordinates (True and Eccentric anomaly) for several cases studied, including elliptic, parabolic and hyperbolic orbits for $M_3$.

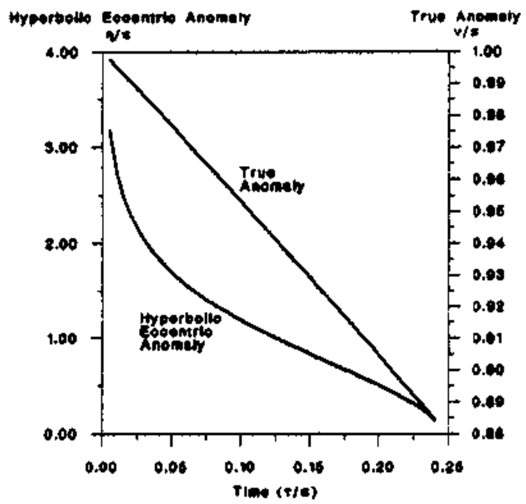


Fig.2 Elliptic Case (e = 0.4, $M_2$ at Apoapsis) in Eccentric and True Anomalies for Hyperbolic Transfer Orbits.

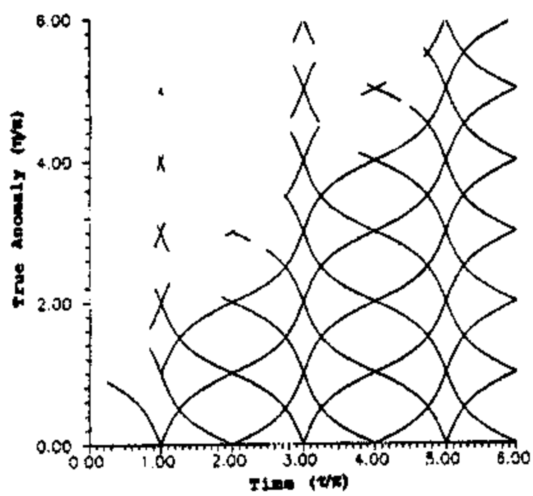


Fig.3 Elliptic Case (e = 0.4, $M_2$ at Apoapsis) in True Anomaly for Elliptic Transfer Orbits.

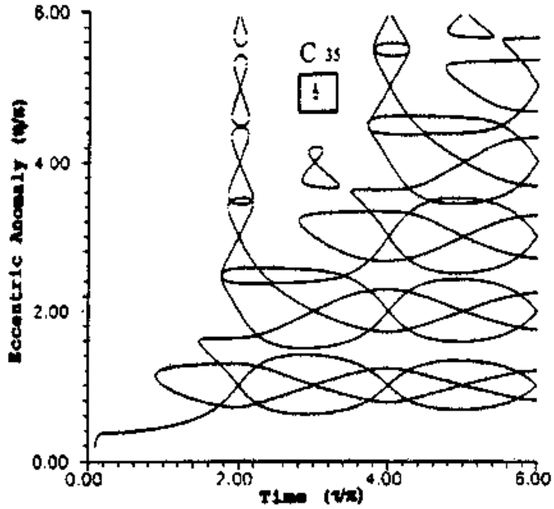


Fig.4a - Elliptic Case (e = 0.4, $M_2$ at Periapsis) in Eccentric Anomaly for Elliptic Transfer Orbits and The New Family $C_{35}$.

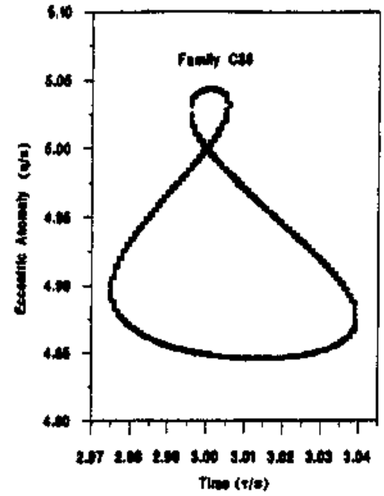


Fig. 4b - The New Family $C_{35}$ in Detail

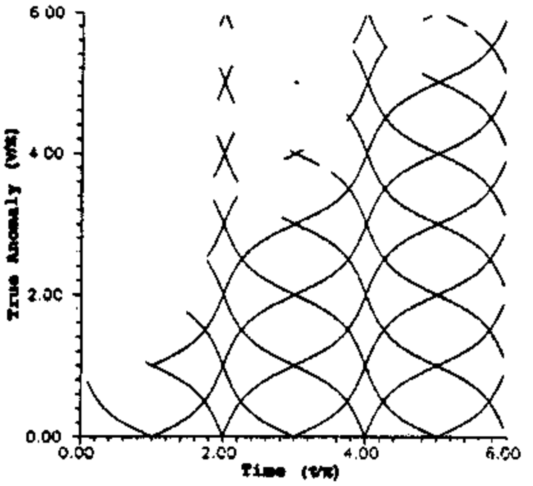


Fig.5 - Elliptic Case (e = 0.4, $M_2$ at Periapsis) in True Anomaly for Elliptic Transfer Orbits.

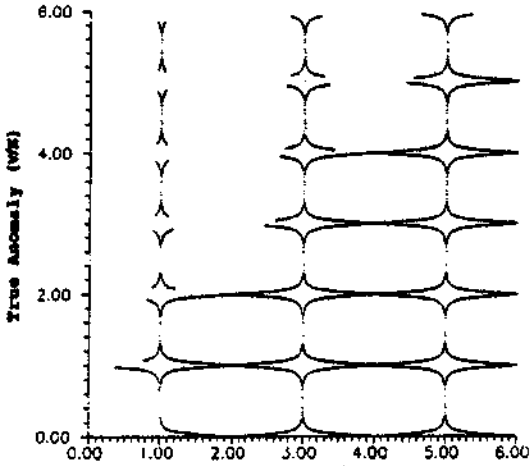


Fig.6 - Elliptic Case (e = 0.97, $M_2$ at Apoapsis) in True Anomaly for Elliptic Transfer Orbits.

It is noted that the family $C_{35}$ (following the standard nomenclature for a family around $\tau/\pi = 3$ and $\eta/\pi = 5$) that appeared in Fig. 4a (and detailed in Fig. 4b) was found when solving the problem by the Lambert's approach, but it is not present in Howell original paper[9]. The use of the true anomaly still has a simpler form, since the solutions are composed of patterns that repeat themselves.

**TRANSFER TO $L_4$ AND $L_5$**

Another improvement made in the present paper was to extend Hénon's problem to the one where the objective is to transfer a spacecraft from $M_2$ to the corresponding Lagrangian equilibrium points $L_4$ or $L_5$. In this version, the spacecraft $M_3$ leaves $M_2$ at P, goes to an orbit around $M_1$ and rendezvous with $L_4$ or $L_5$ (instead of $M_2$) at Q. Fig. 7 to Fig. 9 show the results in the true and eccentric anomaly for elliptic, parabolic and hyperbolic transfer orbits for the transfer to $L_4$. Similar results are available for $L_5$, but they are omitted in this paper to save space. The use of the true anomaly has the advantages of linear graph, that are the original ones (transfer from $M_2$ to $M_2$ again) with a shift of 60 degrees. It is also noted that, in this case, two families of hyperbolic transfer orbits appeared. The usual one and the one that makes the spacecraft goes in the direct sense passing by the perigee in a positive abscissa. This is due to the extra 60 degrees involved in the transfer.

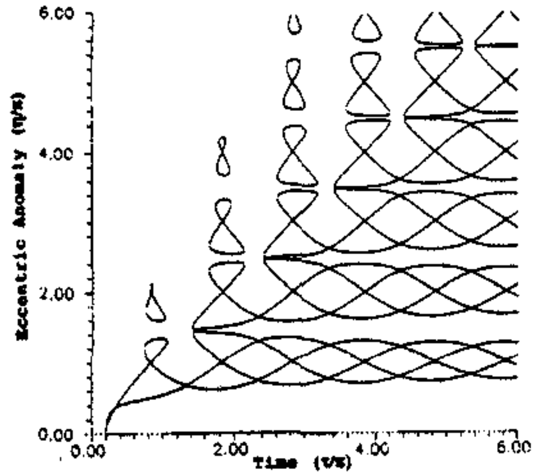


Fig.7 Transfer from $M_2$ to $L_4$ in Eccentric Anomaly for Elliptic Transfer Orbits.

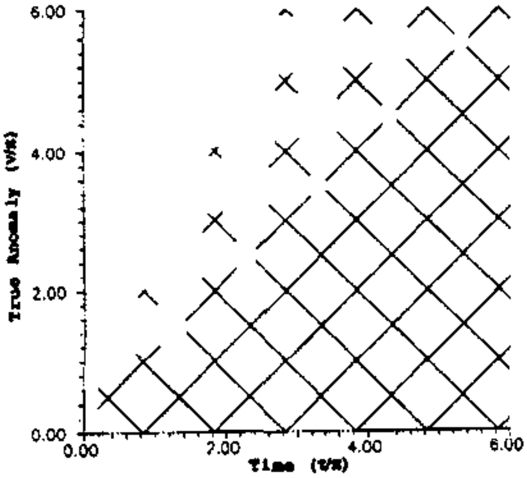


Fig.8 Transfer from $M_2$ to $L_4$ in True Anomaly for Elliptic Transfer Orbits.

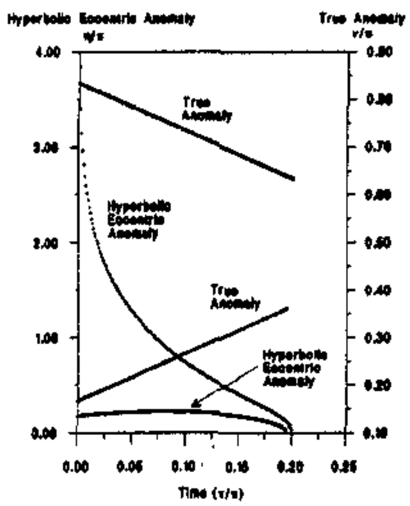


Fig.9 Transfer from $M_2$ to $L_4$ in Eccentric and True Anomalies for Hyperbolic Transfer Orbits.

## THE TRANSFERS WITH MINIMUM $\Delta V$

In the exploratory phase of this study we made plots of $(\Delta V)x(\tau/\pi)$ for thousands of possible transfer orbits. We choose five orbits for $M_2$ around $M_1$:

1-) Circular orbit with semi-major axis equals to one.

2-) Elliptic orbit with eccentricity 0.4 and semi-major axis equals to one, with $M_2$ passing by the perigee at $t = 0$.

3-) Elliptic orbit with eccentricity 0.4 and semi-major axis equals to one, with $M_2$ passing by the apogee at $t = 0$.

4-) Elliptic orbit with eccentricity 0.97 and semi-major axis equals to one, with $M_2$ passing by the perigee at $t = 0$.

5-) Elliptic orbit with eccentricity 0.97 and semi-major axis equals to one, with $M_2$ passing by the apogee at $t = 0$.

The results are shown in Figs. 10 and 11. In the Y-axis we have the total $\Delta V$ in canonical units and in the X-axis we have $\tau/\pi$, where $\tau$ is half of the transfer time. Only elliptic transfer orbits are included in these plots, since the hyperbolic or parabolic transfer orbits are too expensive in terms of $\Delta V$. In these figures, $\tau/\pi$ varies from 0 to 14 and the maximum number of complete revolutions allowed for $M_3$, while in its transfer orbit, is also 14.

Looking in those figures we can see the existence of points (orbits) with very small $\Delta V$. They appear in several locations in the plot and reveal a whole family of small $\Delta V$ transfer orbits. In all cases studied in this paper, this family appeared in the "short transfer time" part of the graph (small $\tau$). A more detailed plot of $(\Delta V)x(\tau/\pi)$ is shown in Fig. 12. It includes only the orbits where $\Delta V < 0.5$ and it is valid for orbit 1 (circular orbit) only. Plots for the other orbits (2 to 5) are very similar and they are omitted in the present paper. We can see that the local minimums increase with time after $\tau/\pi = 6$. An investigation for $\tau/\pi$ varying from zero to 200 and with the maximum number of complete revolution for $M_3$ during the transfer also equals to 200 was done, and no more orbits with $\Delta V < 0.1$ were found.

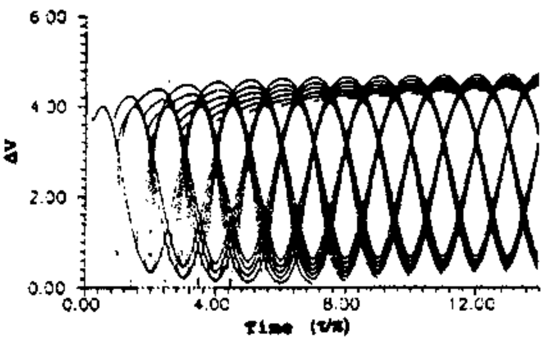


**Fig.10 $(\Delta V)x(\tau/\pi)$ for Orbit 1 for $M_2$.**

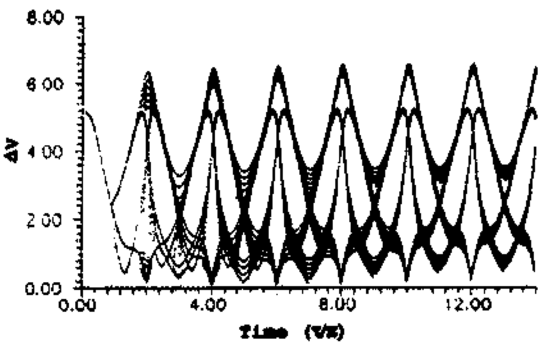


**Fig.11 $(\Delta V)x(\tau/\pi)$ for Orbit 2 for $M_2$.**

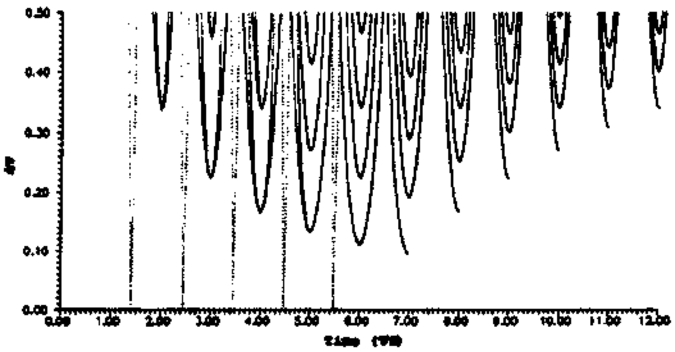

**Fig.12 $(\Delta V)x(\tau/\pi)$ for $\Delta V < 0.5$ (Orbit 1 for $M_2$).**

Table 2 shows the main characteristics of the orbits with $\Delta V < 0.1$ that we found in the circular and elliptic case. It is interesting to see that these orbits appear in pairs (in parts of the Table): one transfer orbit with the perigee in a positive abscissa and one with the perigee in a negative abscissa, very near each other position. In this table the orbit of $M_2$ is assumed to be elliptic with several values for the eccentricity. Both cases, $M_2$ at perigee at t = 0 and $M_2$ at apogee at t = 0 are considered. Fig. 13 shows some of these orbits.

**Table 2**

**TRANSFER ORBITS WITH ΔV < 0.1 FOR THE CIRCULAR AND ELLIPTIC CASE**

| | τ/π | a | e | η/π | γ/π | L | P | S | A | ΔV |
|---|---|---|---|---|---|---|---|---|---|---|
| e=0 | 1.400 | 0.993 | 0.0216 | 1.406 | 1.400 | 1 | 1 | 1 | 1 | 0.0417 |
| | 1.410 | 1.003 | 0.0105 | 1.406 | 1.410 | 1 | 0 | 1 | 0 | 0.0204 |
| | 2.440 | 0.997 | 0.0167 | 2.445 | 2.440 | 1 | 0 | 1 | 0 | 0.0331 |
| | 2.450 | 1.002 | 0.0149 | 2.445 | 2.450 | 1 | 1 | 1 | 1 | 0.0295 |
| | 3.460 | 0.999 | 0.0036 | 3.461 | 3.460 | 1 | 1 | 1 | 1 | 0.0072 |
| | 3.470 | 1.003 | 0.0279 | 3.461 | 3.470 | 1 | 0 | 1 | 0 | 0.0555 |
| | 4.460 | 0.997 | 0.0310 | 4.469 | 4.460 | 1 | 0 | 1 | 0 | 0.0618 |
| | 4.470 | 1.000 | 0.0005 | 4.469 | 4.470 | 1 | 1 | 1 | 1 | 0.0010 |
| | 5.470 | 0.998 | 0.0169 | 5.475 | 5.470 | 1 | 1 | 1 | 1 | 0.0336 |
| | 5.480 | 1.001 | 0.0146 | 5.475 | 5.480 | 1 | 0 | 1 | 0 | 0.0292 |
| | 6.990 | 1.108 | 0.9777 | 5.991 | 6.990 | 0 | 0 | 1 | 1 | 0.0955 |
| e2=0.1 S3=-1 | 1.410 | 1.4386 | 1.0025 | 0.1085 | 1.4729 | 1 | 0 | 1 | 0 | 0.0453 |
| | 2.440 | 2.4133 | 0.9979 | 0.1125 | 2.3793 | 1 | 0 | 1 | 0 | 0.0435 |
| | 3.460 | 3.4930 | 0.9995 | 0.0962 | 3.5238 | 0 | 0 | 1 | 0 | 0.0404 |
| | 4.470 | 4.4380 | 1.0002 | 0.0975 | 4.4078 | 1 | 0 | 1 | 0 | 0.0398 |
| | 5.480 | 5.5072 | 1.0011 | 0.1142 | 5.5436 | 0 | 0 | 1 | 0 | 0.0500 |
| | 7.000 | 6.0000 | 1.1082 | 0.1879 | 6.0000 | 0 | 0 | 1 | 1 | 0.0869 |
| e2=0.1 S3=+1 | 1.400 | 1.3747 | 0.9962 | 0.1132 | 1.3420 | 1 | 1 | 1 | 1 | 0.0411 |
| | 2.440 | 2.4772 | 0.9970 | 0.0829 | 2.5036 | 0 | 1 | 1 | 1 | 0.0503 |
| | 3.460 | 3.4293 | 0.9999 | 0.1009 | 3.3982 | 1 | 1 | 1 | 1 | 0.0389 |
| | 4.470 | 4.5018 | 1.0000 | 0.1003 | 4.5337 | 0 | 1 | 1 | 1 | 0.0402 |
| | 5.470 | 5.4435 | 0.9989 | 0.1148 | 5.4078 | 1 | 1 | 1 | 1 | 0.0479 |
| e2=0.2, S3=-1 | 7.000 | 6.0000 | 1.1082 | 0.2782 | 6.0000 | 0 | 0 | 1 | 1 | 0.0793 |
| e2=0.2, S3=1 | 6.000 | 5.0000 | 1.1292 | 0.2916 | 5.0000 | 1 | 1 | 1 | 0 | 0.0917 |
| e2=0.5, S3=-1 | 5.000 | 4.0000 | 1.1604 | 0.5691 | 4.0000 | 1 | 0 | 1 | 1 | 0.0789 |
| e2=0.5 S3=+1 | 4.000 | 3.0000 | 1.2114 | 0.5873 | 3.0000 | 1 | 1 | 1 | 0 | 0.0993 |
| | 4.000 | 5.0000 | 0.8618 | 0.4198 | 5.0000 | 1 | 1 | 1 | 0 | 0.0939 |
| | 6.000 | 5.0000 | 1.1292 | 0.5572 | 5.0000 | 1 | 1 | 1 | 0 | 0.0655 |
| e2=0.6 S3=-1 | 5.000 | 4.0000 | 1.1604 | 0.6553 | 4.0000 | 1 | 0 | 1 | 1 | 0.0685 |
| | 7.000 | 5.0000 | 1.2515 | 0.6804 | 5.0000 | 1 | 0 | 1 | 0 | 0.0992 |
| e2=0.6 S3=+1 | 4.000 | 3.0000 | 1.2114 | 0.6698 | 3.0000 | 1 | 1 | 1 | 0 | 0.0863 |
| | 4.000 | 5.0000 | 0.8618 | 0.5358 | 5.0000 | 1 | 1 | 1 | 0 | 0.0810 |
| | 6.000 | 5.0000 | 1.1292 | 0.6458 | 5.0000 | 1 | 1 | 1 | 0 | 0.0568 |
| e2=0.7 S3=-1 | 3.000 | 2.0000 | 1.3104 | 0.7711 | 2.0000 | 1 | 0 | 1 | 1 | 0.0985 |
| | 3.000 | 4.0000 | 0.8255 | 0.6366 | 4.0000 | 1 | 0 | 1 | 1 | 0.0897 |
| | 5.000 | 4.0000 | 1.1604 | 0.7415 | 4.0000 | 1 | 0 | 1 | 1 | 0.0577 |
| | 7.000 | 5.0000 | 1.2515 | 0.7603 | 5.0000 | 1 | 0 | 1 | 0 | 0.0837 |
| e2=0.7 S3=+1 | 4.000 | 3.0000 | 1.2114 | 0.7524 | 3.0000 | 1 | 1 | 1 | 0 | 0.0728 |
| | 6.000 | 4.0000 | 1.3104 | 0.7711 | 4.0000 | 1 | 1 | 1 | 1 | 0.0985 |
| | 4.000 | 5.0000 | 0.8618 | 0.6519 | 5.0000 | 1 | 1 | 1 | 0 | 0.0679 |
| | 6.000 | 5.0000 | 1.1292 | 0.7343 | 5.0000 | 1 | 1 | 1 | 0 | 0.0478 |

where:

$\tau$ = Half of the transfer time in canonical units

$\underline{v}$ = Redefined true anomaly

$\underline{n}$ = Redefined eccentric anomaly

a = Semi-major axis of the transfer orbit

e = Eccentricity of the transfer orbit

L = 1 for "short way" transfer, 0 for "long way" transfer

P = 1 if perigee is in a positive abscissa, 0 if in a negative abscissa

S = 1 if transfer is direct, 0 if transfer is retrograde

A = 1 if $M_3$ pass by the perigee at t = 0, 0 if it pass by the apogee

$\Delta V$ = Velocity increment in meters/second

Looking in the details of Table 2 and Fig. 13, we can see better the mechanism of the majority of these transfer orbits. They consist of orbits with slight different semi-major axis and eccentricity (compared with the orbit of $M_2$) that have a perigee coincident with the perigee of the orbit of $M_2$. They have mean angular velocity (n) such that $2\tau(1-n) = \pm 2\pi$. So, after $M_3$ makes m complete revolutions in its transfer orbit, $M_2$ makes m+1 or m-1 complete revolutions in its own orbit and they can meet each other at the common perigee, after the time $2\tau$.

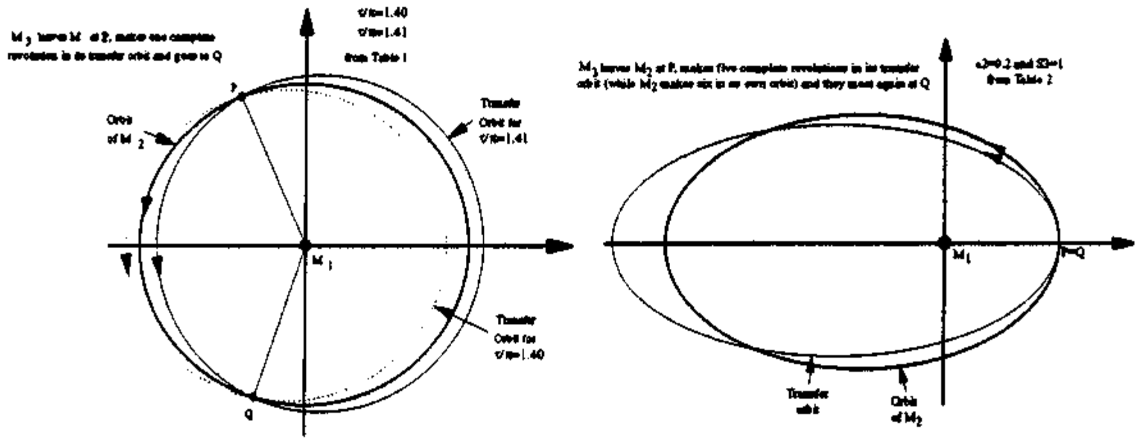


**Fig.13 Some Transfer Orbits with Small $\Delta V$.**

## PRACTICAL APPLICATIONS

To show some of the possible applications for these orbits, we applied them in the case of a transfer from $M_2$ (the Moon or a planet) to the corresponding Lagrangian point $L_4$ or $L_5$. In this case a massless spacecraft $M_3$ has to leave $M_2$ at time $t = -\tau$, goes to an orbit around $M_1$ and meets with $L_4$ or $L_5$ at $t = \tau$. This problem is of great interest in space flight, because the Lagrangian points are good candidates for space stations, since they are equilibrium points and require low fuel consumption for station-keeping. Many trips from the Earth or the Moon to/from the space stations located at the Lagrangian points are expected to happen in the future.

Fig. 14 shows the graph of $(\Delta V)x(\tau/\pi)$ for the transfer from $M_2$ to $L_4$ for elliptic transfer orbits. Results for parabolic and hyperbolic transfer orbits were found, but they are omitted here to save space. We can see the details of these orbits (with $\Delta V < 0.1$) in Table 3. The $\Delta V$ in m/s and the transfer time in days are calculated assuming that the orbital velocity of the Moon around the Earth is V = 1018.31 m/s and that its orbital period is T = 27.322 days. The mechanism used by these transfers is to insert $M_3$ in an elliptic transfer orbit that have an apogee coincident with the apogee of the orbit of $M_2$. These transfer orbits have a mean angular velocity (n) greater than 1, such that $2\tau(n-1) = 1.047$ rad (60 degrees). So, in the same time that $M_3$ makes m revolutions in its transfer orbit, $M_2$ makes m-(1/6) revolutions in its own orbit and $M_3$ can rendezvous with $L_4$ at Q. It is important to remember that these maneuvers are optimal for a two-impulse category of transfer orbits, and it does not mean that a maneuver with more impulses can not be found with a smaller $\Delta V$. It is also important to emphasize that, in this particular example, the spacecraft $M_3$ spends a long time near the body $M_2$, and the mutual influence of these two bodies can be strong. It means that these results have to be better checked with numerical integration of the more realistic case $M_2 \neq 0$. Analogous results for a transfer from $M_2$ to $L_5$ exist, but the figures are not shown in this paper to save space. Table 3 includes some of these transfer orbits (transfer orbits with $\Delta V < 0.1$) to $L_4$ and $L_5$. The same comments made in the transfer to $L_4$ case do apply. The mechanism used by these transfers is to insert $M_3$ in an elliptic transfer orbit that have a perigee coincident with the perigee of the orbit of $M_2$. These transfer orbits have a mean angular velocity (n) smaller than 1, such that $2\pi(1-n) = 1.047$ rad (60 degrees). So, in the same time that $M_3$ makes m revolutions in its transfer orbit, $M_2$ makes m+(1/6) revolutions in its own orbit and $M_3$ can rendezvous with $L_5$ at Q.

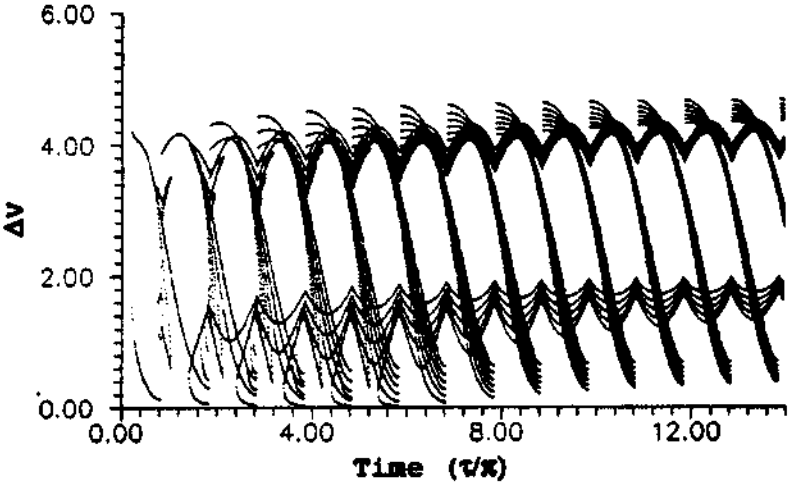


**Fig.14 $(\Delta V)x(\tau/\pi)$ for Transfer to $L_4$ (Elliptic Transfer Orbits).**

**Table 3**

**TRANSFER ORBITS WITH $\Delta V < 0.1$ FOR THE TRANSFER TO $L_4$ AND $L_5$**

| | $\tau/\pi$ | $\eta/\pi$ | a | e | $\nu/\pi$ | L | P | S | A | $\Delta V_C$ | $\Delta T$ | $\Delta V$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| L4 | 1.830 | 2.0000 | 0.9437 | 0.0597 | 2.0000 | 0 | 0 | 1 | 0 | 0.061 | 50.00 | 62.1 |
| | 2.830 | 3.0000 | 0.9626 | 0.0388 | 3.0000 | 0 | 1 | 1 | 1 | 0.039 | 77.32 | 39.7 |
| | 3.830 | 4.0000 | 0.9720 | 0.0288 | 4.0000 | 0 | 0 | 1 | 0 | 0.029 | 104.64 | 29.5 |
| | 4.830 | 5.0000 | 0.9777 | 0.0229 | 5.0000 | 0 | 1 | 1 | 1 | 0.023 | 131.96 | 23.4 |
| | 5.830 | 6.0000 | 0.9814 | 0.0190 | 6.0000 | 0 | 0 | 1 | 0 | 0.019 | 159.29 | 19.3 |
| | 6.830 | 6.0000 | 1.0906 | 0.0830 | 6.0000 | 0 | 0 | 1 | 1 | 0.081 | 186.61 | 82.5 |
| L5 | 1.160 | 1.0000 | 1.1080 | 0.0975 | 1.0000 | 0 | 1 | 1 | 0 | 0.095 | 31.69 | 96.7 |
| | 2.160 | 2.0000 | 1.0548 | 0.0519 | 2.0000 | 0 | 0 | 1 | 1 | 0.051 | 59.01 | 51.9 |
| | 3.160 | 3.0000 | 1.0367 | 0.0354 | 3.0000 | 0 | 1 | 1 | 0 | 0.035 | 86.34 | 35.6 |
| | 4.160 | 4.0000 | 1.0276 | 0.0268 | 4.0000 | 0 | 0 | 1 | 1 | 0.027 | 113.66 | 27.5 |
| | 5.160 | 5.0000 | 1.0221 | 0.0216 | 5.0000 | 0 | 1 | 1 | 0 | 0.022 | 140.98 | 22.4 |
| | 6.160 | 6.0000 | 1.0184 | 0.0181 | 6.0000 | 0 | 0 | 1 | 1 | 0.018 | 168.30 | 18.3 |

where:

$\Delta V_c$ = Velocity increment in canonical units

$\Delta T$ = Total time for the transfer in days

The others symbols are the same defined in Table 2

## CONCLUSIONS

The problem previously called consecutive collision orbits in the three-body problem was formulated as a problem of transfer orbits from one body (the Moon or a planet) back to the same body. Using this approach, the Hénon's problem became a special case of the Lambert's problem.

The Gooding's implementation of the Lambert's problem[11] was used to solve this problem with great success. It was called about 3 million times with no failure detected and solved the problem one time in about 2 milliseconds.

A new coordinate to express the solution of this problem (the true anomaly) was used and the solutions showed to be of a much simpler form in this variable, when compared to the solutions given by the previous authors in terms of the eccentric anomaly. This coordinate also allows us to plot hyperbolic, parabolic and elliptic orbits in a single graph, in the same variable.

Extension to the elliptic case (when the orbit of $M_2$ is elliptic) and to transfers from $M_2$ to the corresponding Lagrangian points $L_4$ and $L_5$ were made easily in this new approach. They also showed to have a much simpler form for the solutions when expressed in terms of the true anomaly. In the transfer to $L_4$ and $L_5$, a new family of hyperbolic transfer orbits appeared.

A new family of solutions for one of the elliptic cases ($C_{35}$ in Fig. 4) was found by this approach.

The $\Delta$Vs and the transfer time required for these transfers were calculated. Among a large number of transfer orbits, a small family was found, such that the $\Delta V$ required for the transfer is very small. These orbits and its properties were shown in details.

A practical application for these orbits was studied in details: a transfer from the Moon to the corresponding Lagrangian points $L_4$ and $L_5$.

**ACKNOWLEDGMENTS**

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**APPENDIX A - HÉNON'S MATHEMATICAL FORMULATION**

To solve this problem Hénon used standard equations for circular and elliptic two-body problem. For mass $M_2$ in circular orbit (remember that $\omega = 1$):

$$\begin{aligned} x &= \cos(t) \\ y &= \sin(t) \end{aligned} \qquad \text{(A-1)}$$

where x,y are Cartesian coordinates of $M_2$ and t is the time. For $M_3$, in an elliptic orbit with perigee at $x > 0$, moving in the direct sense (same sense as $M_2$) and passing by the perigee at $t = 0$:

$$\begin{aligned} x &= a(\cos(E)-e) \\ y &= a\sin(E)\sqrt{1-e^2} \\ t &= a^{3/2}(E - e\sin(E)) \end{aligned} \qquad \text{(A-2)}$$

where x,y are Cartesian coordinates of $M_3$; "a" is the semi-major axis of the orbit of $M_3$ around $M_1$; "e" is the eccentricity of this orbit; E is the eccentric anomaly; "t" is the time.

To take into account the different possibilities of the orbits of $M_3$ around $M_1$ (perigee at $x < 0$, retrograde orbits and passage by the apogee at $t = 0$) Hénon defined the quantities:

$$\varepsilon = \begin{cases} +1 \\ -1 \end{cases} \text{if the perigee is in a abscissa } \begin{cases} \text{positive} \\ \text{negative} \end{cases}$$

$$\varepsilon' = \begin{cases} +1 \\ -1 \end{cases} \text{if the sense of the orbit is } \begin{cases} \text{direct} \\ \text{retrograde} \end{cases}$$

$$\varepsilon'' = \begin{cases} +1 \\ -1 \end{cases} \text{if the passage at } \tau=0 \text{ (S) is at } \begin{cases} \text{perigee} \\ \text{apogee} \end{cases}$$

Using these quantities the generic equations for the variables x,y,t for $M_3$ is:

$$\begin{aligned} x &= \varepsilon\, a\, (\varepsilon' \cos(E)-e) \\ y &= \varepsilon\, \varepsilon'\, a\sqrt{1-e^2}\, \varepsilon'' \sin(E) \\ t &= a^{3/2}(E-\varepsilon''\, e \sin(E)) \end{aligned} \qquad \text{(A-3)}$$

So, to solve this problem we can write Eqs. (A-1) and (A-3) for $t = \tau$ and $E = \eta$ and obtain:

$$\cos(\tau) = \varepsilon\, a(\varepsilon' \cos(\eta) - e)$$
$$\sin(\tau) = \varepsilon\, \varepsilon'\, a\sqrt{1-e^2}\, \varepsilon'' \sin(\eta) \qquad \text{(A-4)}$$
$$\tau = a^{3/2}(\eta - \varepsilon''\, e \sin(\eta))$$

If $t = -\tau$ and $E = -\eta$ are used, the equations are the same and there is no extra information available. Eq. (A-4) constitute a set of 3 equations in 4 unknowns $(\tau, a, e, \eta)$ and it is used to generate the solutions $(\eta/\pi)x(\tau/\pi)$.

## APPENDIX B - THE GOODING'S ROUTINE TO SOLVE THE LAMBERT'S PROBLEM

The routine developed by Gooding[11] is largely based on the equations developed by Lancaster et al[13,14] To summarize the process, let us show the equations used, step by step.

### The Transformation of Variables

The original variables required as an input to solve the Lambert's problem are:

$R_1$ = the distance from the initial point of the transfer to the center of attraction;
$R_2$ = the distance from the final point of the transfer to the center of attraction;
$\theta$ = the angle between the initial and final points of the transfer;
$\Delta t$ = the time for the transfer;
$\mu$ = the gravitational parameter of the main body

This data is transformed in another set of variables, that are more suitable for the iteration process developed. This new set is constituted by the variables:

$c = \sqrt{R_1^2 + R_2^2 - 2R_1R_2\cos(q)}$ = the distance between the two points involved in the transfer;

$s = \dfrac{R_1 + R_2 + c}{2}$ = the semi-perimeter of the triangle formed by the center of the force and the two points involved in the transfer;

$\theta_r$ = the angle $\theta$, restricted to the interval $[0,2\pi]$;
m = the number of complete revolutions involved in the transfer $(\theta = \theta_r + 2\pi m)$;

$q = \dfrac{\sqrt{R_1R_2}}{2}\cos\left(\dfrac{\theta_r}{2}\right)$ = the parameter used by Lancaster et al[13] in the iteration process;

$1 - q^2 = \dfrac{c}{s}$ , that is calculated from this equation, and not directly from q, to eliminate numerical errors during the computation;

$T = \sqrt{\dfrac{8\mu}{s^3}}\Delta t$ = a variable that replaces the time for the transfer ($\Delta t$) in the iteration process;

$x^2 = 1 - \dfrac{s}{2a}$, that will be the parameter for iteration, instead of the more popular semi-major axis, eccentricity or semi-lactus rectum. This choice was made because x is a Lambert's invariant parameter[11]. The double possibility for the sign of the square root involved give us the two solutions of the Lambert's problem. It is also possible to identify the type of the transfer orbit by looking at x, because $|x| < 1$ for elliptic orbits, $x = 1$ for parabolic orbits and $x > 1$ for hyperbolic orbits. The possibility $x \leq -1$ does not exist, because the time for the transfer would be negative.

**The Parameter T(q,m,x)**

This section describes how to calculate the parameter T, as well as its first three derivatives with respect to x (T', T'', T'''), as a function of x and the parameters q and m. The equations come from Gooding[11] and Lancaster et al[13,14]. The basic algorithm works when m > 0, or x < 0 or |u| > 0.4, where

$$u = 1 - x^2 \qquad \text{(B-1)}$$

We calculate:

$$y = \sqrt{|u|} \qquad z = \sqrt{1-q^2+q^2x^2} \qquad \alpha = z - qx \qquad \text{(B-2), (B-3), (B-4)}$$

$$A = z + qx \qquad \beta = qz - x \qquad B = qz + x \qquad \text{(B-5), (B-6), (B-7)}$$

$$f = \alpha y \qquad g = xz + qu \qquad \text{(B-8), (B-9)}$$

Also, to avoid rounding errors, we compute z directly from the input $1-q^2$ and we use the additional relations:

$$\alpha A = 1-q^2 \qquad \beta B = (1-q^2)(q^2u-x^2) \qquad g = \frac{x^2-q^2u}{xz-qu} \qquad \text{(B-10), (B-11), (B-12)}$$

instead of using the other equations provided (B-4 to B-7 and B-9), in 50% of the time.

Now, we can proceed by calculating:

$$d = m\pi + \arg(g,f) \qquad \text{(B-13)}$$

where arg(g,f) is the angle, in the right quadrant, that the point (g,f) makes with the horizontal axis, or:

$$d = \tanh^{-1}(f/g), \quad \text{for the hyperbolic case.} \qquad \text{(B-14)}$$

Finally, T and its derivatives are given by:

$$T = \frac{2\left(\frac{d}{y}+\beta\right)}{u} \qquad T' = \frac{3xT + \frac{4q^3x}{z} - 4}{u} \qquad \text{(B-15), (B-16)}$$

$$T'' = \frac{3T+5xT'+4\left(\frac{q}{z}\right)^3\left(1-q^2\right)}{u} \qquad T''' = \frac{8T'+7xT''-12x\left(\frac{q}{z}\right)^5\left(1-q^2\right)}{u} \qquad \text{(B-17), (B-18)}$$

Now, it is time to explain the variant of the algorithm for the case m = 0, x > 0 and |u| ≤ 0.4. It is based on series expansion. The basic expressions are:

$$T = \phi(u) - q^3\phi(q^2u) \qquad \text{(B-19)}$$

where

$$\phi(u) = \sum_{n=0}^{\infty} A_n u^n \qquad \text{(B-20)}$$

with

$$A_n = \frac{(2n)!}{2^{2n-2}(n!)^2(2n+3)} \qquad \text{(B-21)}$$

This expression is inaccurate for q near 1, and in this case we replace Eq. (B-19) by:

$$T = \sum_{n=0}^{\infty} B_n u^n \qquad \text{(B-22)}$$

where

$$B_n = A_n b_n \qquad \text{(B-23)}$$

with

$$b_n = 1 - q^{2n+3} \qquad \text{(B-24)}$$

And, for the derivatives, we can use:

$$T' = -2xT' \qquad T'' = -2T' + 4x^2T'' \qquad T''' = 12xT'' - 8x^3T''' \qquad \text{(B-25), (B-26), (B-27)}$$

where $T' = \frac{dT}{du}$

Some more details are available in reference[11]

**The Starting Value of x ($x_0$)**

One of the most important characteristics introduced in this implementation of the Lambert's problem is the method of finding a good initial value for x ($x_0$) to start the iteration process. The method is divided in two parts, depending on the value of m.

a) m = 0 (single revolution transfer)

The first step is to calculate $T_0$, that is the function T(q,m,x) for x = 0. We do it, by using the equations developed in the last section. Then, we divide our study in two cases again:

1. If $T \le T_0$, then $x_0 = \frac{T_0(T_0 - T)}{4T}$ **(B-28)**

2. If $T > T_0$, then we calculate:

$$x_{01} = \frac{T_0 - T}{T - T_0 + 4} \qquad x_{02} = -\sqrt{\frac{T - T_0}{T + \frac{T_0}{2}}} \qquad W = x_{01} + 1.7\sqrt{\left(2 - \frac{\theta}{\pi}\right)} \qquad \text{(B-29)(B-30)(B-31)}$$

and we make:

$$x_0 = x_{01} \qquad \text{if } W \ge 0 \qquad \text{(B-32)}$$

$$x_0 = x_{01} + (-W)^{1/16}(x_{02} - x_{01}) \qquad \text{if } W < 0 \qquad \text{(B-33)}$$

As an additional refinement, to avoid numerical problems, we study separately the case when x is near -1. In this case we multiply the $x_0$ (as given in the previous equations) by λ, that is given by the expression:

$$\lambda = 1 + c_1 x_0 (1 + x_{01}) - c_2 x_0^2 \sqrt{(1 + x_{01})} \qquad \text{(B-34)}$$

where $c_1 = 0.5$ and $c_2 = 0.03$

b) m > 0 (one or more revolutions during the transfer)

The first step is to calculate the quantity $x_M$, that is given by:

$$x_M = x_{M,\pi} = \frac{4}{3\pi(2m+1)} \qquad \text{if } \theta_r = \pi \qquad \text{(B-35)}$$

$$x_M = x_{M,\pi}\left(\frac{\theta_r}{\pi}\right)^{1/8} \qquad \text{if } \theta_r < \pi \qquad \text{(B-36)}$$

$$x_M = x_{M,\pi}\left[2 - \left(2 - \frac{\theta_r}{\pi}\right)^{1/8}\right] \qquad \text{if } \theta_r > \pi \qquad \text{(B-37)}$$

Then, we use the parameter T(q,m,x) to evaluate T for $x_M$. We call it $T_M = T(q,m,x_M)$.

At this point, we have three possibilities:

$T_M > T \Rightarrow$ There is no solution for this Lambert's problem

$T_M = T \Rightarrow$ There is a unique solution ($x = x_M$) for this Lambert's problem

$T_M < T \Rightarrow$ There are two solutions for this Lambert's problem. They are given by:

$$x_0 = x_M + \sqrt{\frac{T - T_M}{\frac{T_M''}{2} + \frac{T - T_M}{(1 - x_M)^2}}} \qquad x_0 = x_M - \sqrt{\frac{T - T_M}{\frac{T_M''}{2} - (T - T_M)\left[\frac{T_M''}{2(T_0 - T_M)} - \frac{1}{x_M^2}\right]}} \qquad \text{(B-38),(B-39)}$$

**The Iteration**

Now that we have a good value for $x_0$, we can make the iterations desired, in the following way:

1. Calculate: $x = x_0$; $T_{in} = \sqrt{\frac{8\mu}{s^3}}\Delta t$;
2. Calculate: $T = T(q,m,x)$ with the equations given in one of the previous sections;
3. Calculate: $DT = T_{in} - T$ and $x = x + \frac{DT * T'}{T'^2 + \frac{DT * T''}{2}}$;
4. Go back to step 2, until three iterations are performed.

The decision of performing three iterations, instead of including a check of convergence for x, is based in several simulations, that showed that three iterations are always enough for convergence.

**The Solution**

After x is found, we have to transform it back to velocities. Remember that the initial and final positions for the transfer are given and, if we can find the velocity vectors at those two points, we have a complete specification of the transfer orbit desired. The equations for the velocities are:

$$V_{R,1} = \frac{\gamma[(qz - x) - \rho(qz + x)]}{R_1} \qquad \text{(B-40)}$$

where $\gamma = \sqrt{\frac{\mu s}{2}}$, $z = +\sqrt{1 - q^2 + q^2x^2}$, $\rho = \frac{R_1 - R_2}{c}$

$$V_{R,2} = -\frac{\gamma[(qz - x) + \rho(qz + x)]}{R_2} \qquad V_{T,1} = \frac{\gamma\sigma(z + qx)}{R_1} \qquad \text{(B-41), (B-42)}$$

where $\sigma = \sqrt{1 - \rho^2} = 2\sqrt{\frac{R_1R_2}{c^2}}\sin\left(\frac{\theta_t}{2}\right)$

$$V_{T,2} = \frac{\gamma\sigma(z + qx)}{R_2} \qquad \text{(B-43)}$$

where the subscripts in the velocity are: T for tangential component, R for radial component, 1 for the initial point and 2 for the final point in the transfer.

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**TÍTULO**
The Problem of Transfer Orbits From One Body Back to the Same Body

**AUTOR**
Antonio Fernando Bertachini de Almeida Prado e Roger Broucke

**TRADUTOR**

**EDITOR**

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[x] EXTERNA [ ] INTERNA [ ] RESERVADA [ ] LISTA DE DISTRIBUIÇÃO ANEXA

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