XVI CONGRESSO BRASILEIRO DE ENGENHARIA MECÂNICA
16th BRAZILIAN CONGRESS OF MECHANICAL ENGINEERING

ORBIT DETERMINATION USING GPS AND IMPROVED RECURSIVE LEAST SQUARES METHOD

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Abstract. *In this paper, we discuss a method of orbit determination for an artificial satellite based on the signals of the GPS constellation. The Global Positioning System is a powerful and low cost process to compute orbits for some artificial Earth satellites. This work presents a method of orbit determination for satellites with an onboard GPS receiver. Pseudoranges are used in the measurement equations for the orbit estimator. The estimator considered is the recursive least squares method, numerically improved with orthogonal Givens rotations and thus avoiding problems concerning inversion of matrices. Up to high order geopotential perturbations are taken into account. Preliminary results indicate that precision better than 10m is easily obtained using batches of one orbital period for the TOPEX satellite (two hours of orbital period). Standard deviation of about 5m resulted for the residuals of pseudo-range measurements.*

Keywords: *Global Positioning System, orbit determination, pseudo-range, recursive least squares, Givens rotation.*

1. Introduction

The Global Positioning System is a powerful and low cost process to compute orbits for low Earth orbits of artificial satellites. Theoretically four GPS satellites, simultaneously visible, are enough to determine the position parameters of an artificial satellite carrying an onboard GPS receiver.

Orbit determination for artificial satellite is a non-linear problem where the perturbing forces are not easily modelled. The GPS satellites send signals such that accurate measurements of distances are performed based in the comparison between received signals and template signals generated by the receiver (Leick, 1995; Parkinson, 1996). Through a GPS receiver on board of an artificial satellite it is possible to obtain such measurements (pseudo ranges) that can be used to estimate the state vector that characterizes the orbit of the satellite.

Using knowledge about the dynamics of the system and assuming statistics for the measurement errors, the state vector is computed based on a set of observations. The error between the nominal value and these resulting from the estimation process is minimised according to the least squares criterion (Bierman, 1977; Kuga, 1989; Chiaradia, 2000).

The aim of this work is to determine the orbit of an artificial satellite carrying a GPS receiver, using pseudo-range measurements between the satellite and the GPS constellation to furnish the data that will be used in the estimator, the recursive least squares method improved with orthogonal Givens rotations.

2. Dynamical Model

The problem of orbit determination is essentially non linear since the dynamical process, that is, the orbital motion is described in an inertial frame by a system of ordinary differential equations of the following form

$$\dot{\mathbf{r}} = \mathbf{v} \tag{1}$$

$$\dot{\mathbf{v}} = -\mu \frac{\mathbf{r}}{r^3} + \tilde{\mathbf{N}} \tag{2}$$

$$\dot{\mathbf{b}} = \mathbf{0} \tag{3}$$

where $\mathbf{r}:(x,y,z)$ is the position vector, $\mathbf{v}$ is the velocity vector, $\tilde{\mathbf{N}}$ stands for modelled perturbations and $\mathbf{b} = [b_0, b_1, b_2]$ are constants referred to the receiver clock bias. In the equations for the estimator the state vector is defined by

$$\mathbf{x} \equiv \begin{bmatrix} \mathbf{r} \\ \mathbf{v} \\ \mathbf{b} \end{bmatrix} \tag{4}$$

with $r = \sqrt{x^2 + y^2 + z^2}$. The Eqs. (1) e (2) can be written as:

$$\dot{\mathbf{x}}(t) = \mathbf{f}_m(\mathbf{x}, t) + \mathbf{f}_{nm}(\mathbf{x}, t) \tag{5}$$

where $\mathbf{f}_m$ is a vector-valued function expressing the adopted modeling in the estimator and the vector valued function $\mathbf{f}_{nm}$ stands for the unmodeled part.

The state transition matrix which relates the state between t_k and t_{k+1} can be computed by (Kuga, 1989):

$$\hat{\mathbf{Ô}}(t, t_k) = \mathbf{F}(\mathbf{x}, t)\mathbf{Ö}(t, t_k) \tag{6}$$

for $t \in [t_0, t_k]$ and initial condition $\mathbf{Ö}(t_0, t_0) = \mathbf{É}$. The 9×9 matrix $\mathbf{F}(\mathbf{x},t)$ is contains partial derivatives of the accelerations of the satellite with respect to the components of state vector. Thus, the transition matrix can be obtained integrating Eq. (6).

The transition matrix will be numerically integrated together with the orbit, according to the Eqs. (1), (2), (3) and (6).

Let us consider in the recursive least squares method, the dynamical model as perfect, that is, $\mathbf{f}_{nm} = \mathbf{0}$. Thus, setting $\mathbf{f} = \mathbf{f}_m$ for the accelerations of the satellite, the matrix $\mathbf{F}$ can be written as:

$$\mathbf{F} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} \mathbf{0}_{3\times3} & \mathbf{É}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{A}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \end{bmatrix} \tag{7}$$

where $\mathbf{0}_{3\times3}$ is a 3×3 matrix of zeros; $\mathbf{É}_{3\times3}$ is an identity matrix 3×3 and $\mathbf{A}_{3\times3}$ is the gradient matrix 3×3 given by:

$$\mathbf{A}_{3\times3} = \frac{\partial \dot{\mathbf{v}}}{\partial \mathbf{r}} \tag{8}$$

In this work the modelled forces are due to the geopotential taking into account the spherical harmonic coefficients up to 50^{th} order and degree of JGM-2 model, implemented using Pine's recursive formulation (Pines, 1973).

3. Measurement Model

The non-linear equation representing the scalar model of observations is given by:

$$\mathbf{y}_k = \mathbf{h}_k(\mathbf{x}_k, t) + \mathbf{í}_k \tag{9}$$

where: $\mathbf{y}_k$ is the vector of *m* observations; $\mathbf{h}_k(\mathbf{x}_k)$ is the *m*-dimensional non-linear vector function of the state $\mathbf{x}_k$; $\mathbf{í}_k$ is the *m*-dimensional vector of observation errors with statistics given by $\mathbf{í}_k = N(0, R_k)$.

Pseudo-range observations characterize measurements between the GPS satellites and the receiver's antenna, referring to the epochs of emission and reception of the signals. Pseudorange is the type of measurement to be used in an orbit determination process using GPS and can be written as:

$$P_i = \rho_i + c(dt - dT_i) + D_{ion} + D_{trop} + \nu \tag{10}$$

where

P_i is the pseudorange measured by the user with respect to the i_{th} GPS satellite;
ρ_i is the geometric distance;
c is the speed of light;
dt is the user clock offset;
dT_i is the i_{th} GPS satellite clock offset;
D_{ion} are ionospheric corrections (not considered in this work);

D_{trop} are tropospheric corrections (not considered in this work); and
ν denotes measurement random noise.

The user clock offset is modelled as: $cdt = b_0 + b_1\Delta t + b_2\Delta t^2$, where $\mathbf{b} = [b_0, b_1, b_2]$ are parameters to be estimated and their physical meaning are respectively the receiver clock bias, bias rate (drift) and rate of bias rate (drift rate).

The geometric distance is given by:

$$\rho_i = \sqrt{(x - X_i)^2 + (y - Y_i)^2 + (z - Z_i)^2} \tag{11}$$

where $(X,Y,Z)_i$ are the Cartesian coordinates of the i_{th} GPS satellite at the instant of emission of the signal and (x,y,z) are the Cartesian coordinates of the user's satellite at instant of reception of the signal.

The linearized measurement equations can be given as:

$$\Delta\mathbf{y}_k = \mathbf{H}_k \Delta\mathbf{x}_k + \mathbf{í}_k \tag{12}$$

where: $\mathbf{H}_k = \left.\frac{\partial \mathbf{h}_k}{\partial \mathbf{x}}\right|_{x=\bar{x}_k}$ e $\Delta\mathbf{x}_k = (\mathbf{x} - \bar{\mathbf{x}}_k)$

For GPS pseudo-range measurements, the sensitivity matrix $\mathbf{H}_k$ will be given by (Chiaradia et al., 1999; Chiaradia, 2000):

$$\mathbf{H}_k = \begin{bmatrix} \frac{x - X_i}{\rho} & \frac{y - Y_i}{\rho} & \frac{z - Z_i}{\rho} & 0 & 0 & 0 & 1 & \Delta t & \Delta t^2 \end{bmatrix} \tag{13}$$

where Δt is the elapsed time of measurements since the initial epoch.

In the recursive least squares process the matrix $\mathbf{H}_k$, must be multiplied by the transition matrix, thus

$$\mathbf{H} = \begin{bmatrix} H_0 \\ H_1 \Phi_{1,0} \\ \vdots \\ H_k \Phi_{k,0} \\ \vdots \\ H_f \Phi_{f,0} \end{bmatrix} \tag{14}$$

where $\Phi_{k,0}$ is the transition matrix between the instants t_0 and t_k, according to Eq. (6).

4. Recursive Least Squares Method Through Givens Rotations

An estimator that uses the least squares theory, starting from an initial information, should have its convergence checked. Taking this into account and in order to validate the method for an orbit determination application program, it will be used here a recursive approach, that is, the Recursive Least Squares Method. Thus, using Kalman formulation, the equations for the state estimates are given by (Kuga et al., 2000):

- Kalman's gain:

$$\mathbf{K}_i = \mathbf{P}_{i-1}\mathbf{H}_i^T\left[\mathbf{H}_i\mathbf{P}_{i-1}\mathbf{H}_i^T + \mathbf{R}_i\right]^{-1} \tag{15}$$

- estimated state:

$$\hat{\mathbf{x}}_i = \hat{\mathbf{x}}_{i-1} + \mathbf{K}_i\left(\mathbf{y}_i - \mathbf{H}_i\hat{\mathbf{x}}_{i-1}\right) \tag{16}$$

- state error covariance matrix:

$$\hat{\mathbf{P}}_i = \left(\mathbf{I} - \mathbf{K}_i\mathbf{H}_i\right)\mathbf{P}_{i-1} \tag{17}$$

Extensive analysis of Eqs. (15-17) has shown that such equations may be sensitive to numerical errors (Bierman, 1977). On the other hand, the non-recursive normal equations needs matrix inversions which should be avoided. In order to overcome such problems, there exist in the literature several methods, using orthogonal transformations, to yield best numerical performance concerning problems due to error's propagation or uncertainty in the informations. The main aim to apply orthogonal transformations in matrix and vectors in the least squares method is to substitute the brute force, used in matrix inversion, by a method more robust and less prone to numerical errors. In this respect, Givens rotations are the most adequate transformations to selectively annihilate elements of a matrix, making easy to implement the least squares method in a recursive way. Using Givens rotations a matrix can be made a triangular matrix through a sequence of rotations using classical orthogonal matrices.

Indeed, the solution of the normal equations can be sensitive to small errors in the matrix H that are inevitable when forming the product $(\mathbf{H}^T\mathbf{H})$, with a limited machine accuracy. In order to avoid the normal equations given by $(\mathbf{H}^T\mathbf{H})x = \mathbf{H}^T y$, we can use a process based on the QR factorisation (Golub and Van Loan, 1989) choosing:

$$\mathbf{H}_{m\times n} = \mathbf{Q}_{m\times m}\begin{pmatrix} \mathbf{R}_{n\times n} \\ \mathbf{0}_{(m-n)\times n} \end{pmatrix} \tag{18}$$

in other words, the matrix $\mathbf{H}$ is factored into an orthogonal matrix $\mathbf{Q}$ and an upper triangular matrix $\mathbf{R}$.

Methods for performing the QR-factorisation has been proposed by Householder (1958) and Givens (1958), involving orthogonal transformations which subsequently annihilate the sub-diagonal elements of $\mathbf{H}$. The simple method of Householder transformation eliminates all sub-diagonal elements in a given column of $\mathbf{H}$ at a time. When it is necessary to eliminate specific elements (selectively) of a matrix, the *Givens rotations* is the most suitable transformation. Since null elements are introduced in desired positions of the matrix, Givens rotations require more computer time. Nevertheless it is preferred for recursive implementation of the least squares method due to this characteristic of selective annihilation. Thus, the complete transformation can be given by (Montenbruck and Suarez, 1994)**:**

$$\mathbf{Q}^T = (\mathbf{U}_m \mathbf{U}_{m-1} \cdots \mathbf{U}_3 \mathbf{U}_2) \tag{19}$$

where: $\mathbf{U}_i = \mathbf{U}_{i,\max(i-1,n)} \cdots \mathbf{U}_{i,2}\mathbf{U}_{i,1}$ denotes the sequence of rotations required eliminating the sub-diagonal elements in the *i*-th row of $\mathbf{H}$.

A single n-dimensional orthogonal transformation matrix $\mathbf{U}_{ik}(\phi)$ is equal to the identity matrix except for the elements:

$$\begin{pmatrix} U_{ii} & U_{ik} \\ U_{ki} & U_{kk} \end{pmatrix} = \begin{pmatrix} +\cos\phi & +\sin\phi \\ -\sin\phi & +\cos\phi \end{pmatrix} \tag{20}$$

which define a rotation by an angle ϕ in the (ik) plane.

The new elements of $\mathbf{H}' = \mathbf{U}_{ik}\mathbf{H}$ are given by

$$H'_{ij} = +\cos\phi\ H_{ij} + \text{sen}\ \phi\ H_{kj} \qquad (j = i+1,\ldots,n) \tag{21}$$

$$H'_{kj} = -\text{sen}\ \phi\ H_{ij} + \cos\phi\ H_{kj} \tag{22}$$

where $\sin\phi$ and $\cos\phi$ are given by

$$\cos\phi = \frac{1}{\sqrt{H_{ii}^2 + H_{ki}^2}} H_{ii} \tag{23}$$

$$\sin\phi = \frac{1}{\sqrt{H_{ii}^2 + H_{ki}^2}} H_{ki} \tag{24}$$

Orthogonal transformations of matrices have a considerable role in the numerical computations of the least squares problems. In fact, the Euclidean norm of a vector does not change, the same accuracy is yielded with single computer arithmetic that otherwise requires a double precision arithmetic and the problem is solved in a numerically more robust form.

This is the estimation method used in this work. This method is suitable to the recursive character of the Recursive Least Squares Method, and does not require storage of big matrices still providing a good numerical accuracy.

5. Results

Actual Topex/Poseidon satellite (T/P) data is chosen to validate the proposed method. The Topex/Poseidon satellite was launched in August 10th 1992, in a jointly effort conducted by the United States National Aeronautics and Space Administration (NASA) and the French space agency, Centre National d'Etudes Spatiales (CNES). The T/P satellite orbits the Earth at an altitude of 1336 km, inclination 66°, with near zero eccentricity and an orbital period of about 1.87 hours. The T/P satellite has a GPS receiver onboard as experimental equipment to verify several proposed methods for orbit determination, geodetic coordinates translation, ocean level and geopotential models. The receiver can track up to 6 GPS satellite at once on two frequencies if Anti-Spoofing is inactive.

The onboard dual frequency GPS receiver enables to test the ability of Precise Orbit Determination (POD). Rinex format for T/P observations data, GPS group data and navigation messages, can be easily found at Internet.

Here, the following files were used (Chiaradia et al, 2000):

- T/P observation files. Pseudorange codes are transmitted in two frequencies in GPS time step of 10 seconds and are presented by the GPS Data Processing Facility of the Jet Propulsion Laboratory (JPL) in Rinex format;
- Files with the Precise Orbit Ephemeris (POE) generated by the Jet Propulsion Laboratory (JPL) in one minute UTC time steps in Inertial True of date coordinates;
- GPS navigation message files in Rinex format presented by Crustal Dynamics Data Information System (CDDIS) of the Goddard Space Flight Center.

In this work the estimates position and velocity are compared with the T/P Precise Orbit Ephemeris (POE). JPL/POE provides positions estimate with a precision of 15 cm or better.

The test conditions for the proposed problem are the following:

- Actual Topex/Poseidon pseudoranges data set collected by the onboard GPS receiver on November 18th 1993;
- Geopotential perturbations taking into account the spherical harmonic coefficients up to 50th order and degree of JGM-2 model;
- Pseudorange measurements in frequency L_1 (Code);
- Recursive least squares method with Givens rotations;
- 2 hours data set (about 1 orbital period).

The estimates obtained by our method were compared with the POE reference and produced the following statistics:

- position error: 8.22 ± 1.74 (m);
- pseudorange residual: -0.003 ± 4.306 (m).

The Figure 1 shows the position error and Fig. 2 shows the pseudorange residuals versus time, compared with the JPL/POE reference.

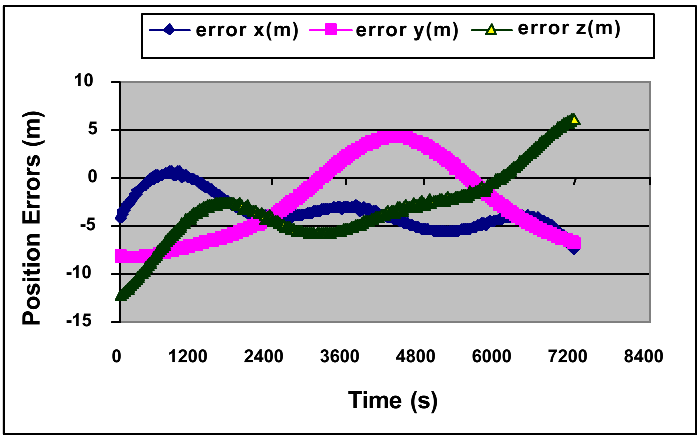

Figure 1 – The position error

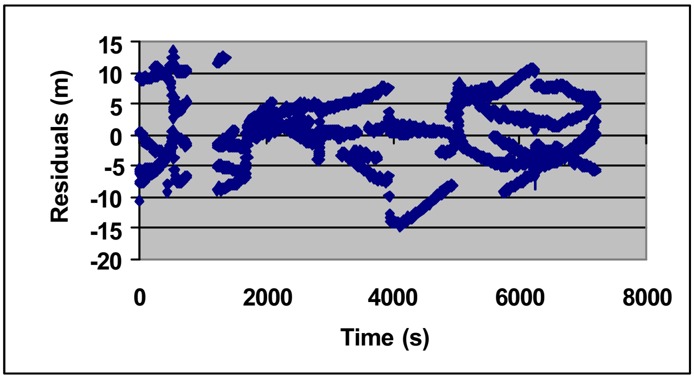

Figure 2 – The pseudorange residuals

6. Conclusions

The aim of this work was to determine the orbit of an artificial satellite using an onboard GPS receiver. Pseudoranges are used in the measurement equations for the orbit estimator. The estimator considered was the recursive least squares method through Givens rotations.

Full 2 hours pseudoranges data set (about one orbital period) collected by the GPS receiver aboard the T/P satellite was used. The modelled forces included the geopotential taking into account the spherical harmonic coefficients up to 50^{th} degree of JGM-2 model. The pseudorange measurements used were those collected in the single frequency L_1 (code).

Results estimated in this work, using T/P data set of November 18^{th}, 1993, and compared against the post-processed GPS ephemeris POE/JPL, has demonstrated accuracy better than 10m. The statistical errors (standard deviation) for pseudorange residuals are about 5 m.

Chiaradia (2000), using Kalman filter technique for orbit determination in real time, also used actual data for the Topex/Poseidon satellite as comparison. The results errors obtained in position and velocity were about 15 to 20m and 0.014 to 0.018m/s, respectively.

In accordance with the obtained results, we can conclude that the recursive least squares with Givens rotations is a reliable and precise method for orbit determination using single frequency GPS measurements.

7. Acknowledgement

The authors wish to express their appreciation for the support provide by FAPESP (Fundação de Amparo à Pesquisa no Estado de São Paulo) – Brazil; under contract #98/14358-2.

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