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16. Summary/Notes

This is the third chapter, in a series of twenty two, written as an introduction to the fundamentals of plasma physics. It analyses the motion of charged particles in the presence of magnetostatic fields, which have a spatial variation, using the nonrelativistic equations of motion in the first order approximation. This approximation is often referred to as the first order orbit theory. The effects of the divergence, gradient and curvature of the magnetic field lines, on the motion of a charged particle, are studied in some detail. An analysis of the magnetic mirror effect, and of the three adiabatic invariants, is also presented.

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MOTION OF CHARGED PARTICLES IN NONUNIFORM MAGNETOSTATIC FIELDS

1. INTRODUCTION

When the fields are spatially nonuniform, or when they vary with time, the integration of the equation of motion (2.1.1) is a mathematical problem of great difficulty. In this case, since the equation of motion is nonlinear, the theory becomes extremely involved and a rigorous analytic expression for the trajectory of the charged particle cannot, in general, be obtained in closed form. Complicated and tedious numerical methods of integration must be used in order to obtain all the details of the motion.

There is one particularly important case, however, in which it becomes possible to obtain an approximate, but otherwise general solution without recourse to numerical integration, if the details of the particle motion are not of interest. This is the case when the magnetic field is strong and *slowly varying* in both *space* and *time*, and when the electric field is weak. In a wide variety of situations of interest the fields are *approximately* constant and uniform, at least on the distance and time scales seen by the particle during one gyration about the magnetic field. This is the case for many laboratory plasmas, including those of relevance to the problem of controlled thermonuclear reactions, and also for a great number of astrophysical plasmas.

In this chapter we investigate the motion of a charged particle in a static magnetic field *slightly* inhomogeneous in space. The word slightly here means that the spatial variation of the magnetic field inside the particle's orbit is small compared with the magnitude of $\underset{\sim}{B}$. In other words, we shall consider only magnetostatic fields whose spatial change in a distance of the order of the Larmor radius, r_c, is much smaller than the magnitude of the field itself.

To specify more quantitatively this assumption concerning the spatial changes of $\underset{\sim}{B}$, let δB represent the spatial change in the magnitude of $\underset{\sim}{B}$, in a distance of the order of r_c, that is,

$$\delta B \simeq r_c \, | \underset{\sim}{\nabla} B | \tag{1.1}$$

where $\underset{\sim}{\nabla} B$ is the gradient of the magnitude of $\underset{\sim}{B}$. It is assumed therefore that

$$\delta B \ll B \tag{1.2}$$

Consequently, in what follows we limit our discussion to problems where the deviations from uniformity are small and solve for the trajectory only in the first order approximation. The analysis of the motion of charged particles in stationary fields based on this approximation, is often referred to as the *first order orbit theory*. This theory was first used systematically by the Swedish scientist Alfvén, and it is also known as the *Alfvén approximation* or the *guiding center approximation*.

The concept of guiding center is of great utility in the development of this theory. We have seen that in a uniform magnetic field the particle motion can be regarded as a superposition of a circular motion about the direction of $\underset{\sim}{B}$, with a motion of the guiding center along the magnetic field lines. In the case of a nonuniform $\underset{\sim}{B}$ field, satisfying the condition (1.2), the value of $\underset{\sim}{B}$ at the position of the particle differs only slightly from its value at the guiding center. The component of the motion of the particle, in a plane normal to the field line that passes through the instantaneous position of the guiding center, will still be nearly circular (see Fig. 1).

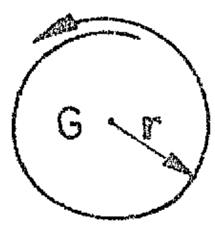

Fig. 1 - The motion of a charged particle in a magnetostatic field slightly inhomogeneous is nearly circular.

However, due to the spatial variation of $\underset{\sim}{B}$, we expect in this case a gradual drift of the guiding center across the magnetic field lines, as well as a gradual change of its velocity along the field lines.

The rapid gyrations of the charged particle about the direction of $\underset{\sim}{B}$ are not usually of great interest and it is convenient to eliminate them from the equations of motion, and focus attention on the motion of the guiding center. In the motion of the guiding center, the small oscillations (of amplitudes small compared with the cyclotron radius) occurring during one gyration period may be averaged out, since they represent the effect of perturbations due to the spatial variation of the magnetic field. The problem is thus reduced to the calculation of the average values over one gyration period (and not the instantaneous values) of the *transverse drift velocity* and the *parallel acceleration* of the guiding center.

2. SPATIAL VARIATION OF THE MAGNETIC FIELD

Any of the three components of the magnetic flux density, $\underset{\sim}{B} = B_x \, \hat{\underset{\sim}{x}} + B_y \, \hat{\underset{\sim}{y}} + B_z \, \hat{\underset{\sim}{z}}$, may vary with respect to the three coordinates x, y, and z. Consequently, nine parameters are needed to completely specify the spatial variation of $\underset{\sim}{B}$. These parameters can be conveniently represented by the dyadic (or tensor) $\underset{\sim}{\nabla} \, \underset{\sim}{B}$, which can be written in matrix form as

$$\underset{\sim}{\nabla}\underset{\sim}{B} = (\underset{\sim}{\bar{x}}\ \underset{\sim}{\hat{y}}\ \underset{\sim}{\hat{z}}) \begin{pmatrix} \partial B_x / \partial x & \partial B_y / \partial x & \partial B_z / \partial x \\ \partial B_x / \partial y & \partial B_y / \partial y & \partial B_z / \partial y \\ \partial B_x / \partial z & \partial B_y / \partial z & \partial B_z / \partial z \end{pmatrix} \begin{pmatrix} \underset{\sim}{\hat{x}} \\ \underset{\sim}{\hat{y}} \\ \underset{\sim}{\bar{z}} \end{pmatrix} \qquad (2.1)$$

Of these nine components only eight are independent, since the following Maxwell equation

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \qquad (2.2)$$

shows that only two of the divergence terms are independent.

If in the region where the particle is moving the condition $\underset{\sim}{J} = 0$ is satisfied, then other restrictions exist in the number of independent components of $\underset{\sim}{\nabla}\underset{\sim}{B}$ since, under these circumstances, the relation $\underset{\sim}{\nabla} \times \underset{\sim}{B} = 0$ holds. This means that in regions where there are no electric currents $\underset{\sim}{B}$ can be written as the gradient of a *scalar magnetic potential,*

$$\underset{\sim}{B} = \underset{\sim}{\nabla} \phi_m \qquad (2.3)$$

where the magnetic potential ϕ_m satisfies Laplace equation

$$\nabla^2 \phi_m = 0 \qquad (2.4)$$

In regions where an electric current density exists, we have $\nabla \times \underset{\sim}{B} = \mu_o \underset{\sim}{J}$ and we cannot define a scalar magnetic potential, ϕ_m, as indicated. The number of independent components of $\underset{\sim}{\nabla}\underset{\sim}{B}$ cannot, in this case, be reduced without knowing the current density $\underset{\sim}{J}$.

Let us consider a Cartesian coordinate system such that at the origin the magnetic field is in the z-direction, that is,

$$\underset{\sim}{B}\,(0,\,0,\,0) = \underset{\sim}{B}_o = B_o\,\hat{\underset{\sim}{z}} \tag{2.5}$$

The nine components of the dyadic $\underset{\sim}{\nabla}\underset{\sim}{B}$ can be conveniently grouped into four categories:

(a) Divergence terms: $\dfrac{\partial B_x}{\partial x}$, $\dfrac{\partial B_y}{\partial y}$, $\dfrac{\partial B_z}{\partial z}$ (2.6a)

(b) Gradient terms: $\dfrac{\partial B_z}{\partial x}$, $\dfrac{\partial B_z}{\partial y}$ (2.6b)

(c) Curvature terms: $\dfrac{\partial B_x}{\partial z}$, $\dfrac{\partial B_y}{\partial z}$ (2.6c)

(d) Shear terms: $\dfrac{\partial B_x}{\partial y}$, $\dfrac{\partial B_y}{\partial x}$ (2.6d)

2.1 - Divergence terms

We shall initially discuss the magnetic field line geometry corresponding to the *divergence* terms of ∇B. The presence of a small variation of the component B_z in the z-direction (i.e., $\partial B_z / \partial z \neq 0$), implies that at least one of the terms $\partial B_x / \partial x$ or $\partial B_y / \partial y$ is also present, as can be seen from (2.2). It is of great utility to make use here of the concept of *magnetic flux lines* which, at any point, are parallel to the B field at that point and whose density at each point is proportional to the local magnitude of B. To determine the differential equation of a line of force, let

$$ds = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z} \qquad (2.7)$$

be an element of arc along the magnetic field line. Then, we must have

$$ds \times B = 0 \qquad (2.8)$$

since ds is parallel to B, which gives by expansion of the cross product,

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z} \qquad (2.9)$$

Since we are focusing attention only on the divergence terms of B, and in the region of interest the field is considered to be mainly in the z-direction, we may expand B_x and B_y in a Taylor series about the origin as follows (see Fig. 2)

$$B_x (x_1, 0, 0) = B_x (0, 0, 0) + \left(\frac{\partial B_x}{\partial x}\right) x_1 = \left(\frac{\partial B_x}{\partial x}\right) x_1 \qquad (2.10)$$

$$B_y (0, y_1, 0) = B_y (0, 0, 0) + \left(\frac{\partial B_y}{\partial y}\right) y_1 = \left(\frac{\partial B_y}{\partial y}\right) y_1 \qquad (2.11)$$

where the second and higher order terms were neglected. Note that at the origin $B_x = B_y = 0$. Therefore, the magnetic field line crossing the $z = 0$ plane at the point $(x_1, y_1, 0)$, when projected in the x-z plane ($y = 0$) and in the y-z plane ($x = 0$), satisfies the following differential equations, respectively,

$$\frac{dx}{dz} = \frac{B_x}{B_z} = \frac{1}{B_z} \left(\frac{\partial B_x}{\partial x}\right) x_1 \qquad (y = 0) \qquad (2.12)$$

$$\frac{dy}{dz} = \frac{B_y}{B_z} = \frac{1}{B_z} \left(\frac{\partial B_y}{\partial y}\right) y_1 \qquad (x = 0) \qquad (2.13)$$

These equations show that the field lines converge or diverge in the x-z plane or in the y-z plane, depending on the sign of the divergence terms of $\underset{\sim}{B}$. Fig.3 illustrates the field line geometry for the case when $\partial B_x/\partial x$ and $\partial B_y/\partial y$ are positive.

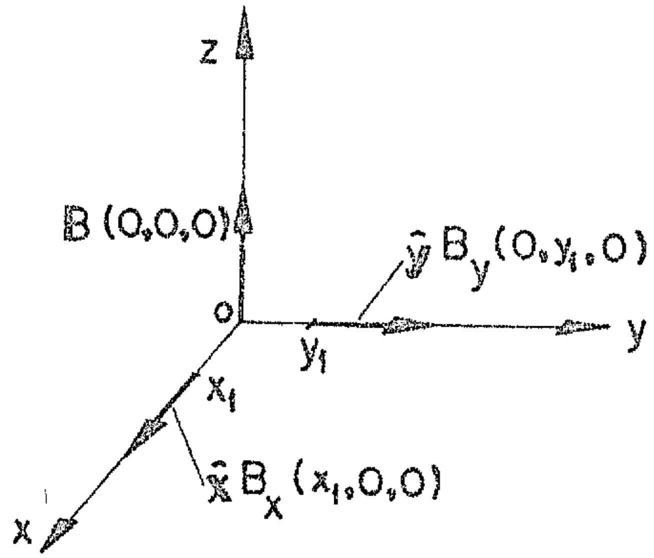

Fig. 2 - The magnetic field components B_x and B_y at the points $(x_1, 0, 0)$ and $(0, y_1, 0)$, near the origin.

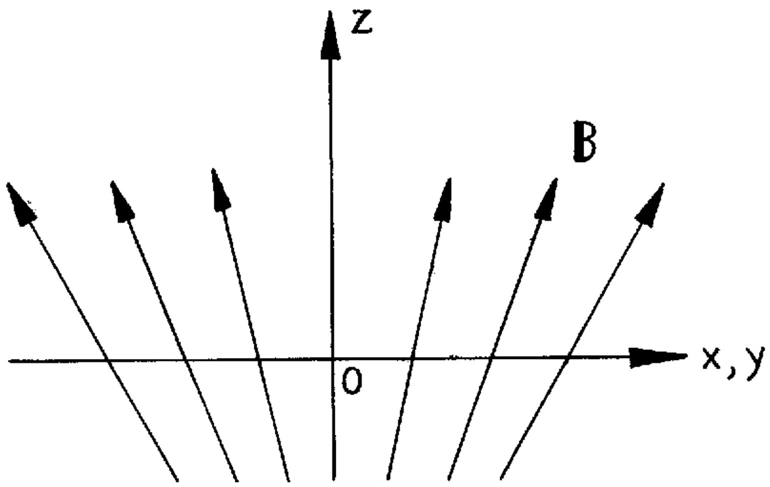

Fig. 3 - Geometry of the magnetic field lines corresponding to the divergence terms $\partial B_x/\partial x$ or $\partial B_y/\partial y$, when they are positive.

2.2 - Gradient and curvature terms

An example of a vector field having a *gradient* in the x - direction is

$$\underset{\sim}{B} = B_z \, \hat{\underset{\sim}{z}} = B_0 \, (1 + \alpha x) \, \hat{\underset{\sim}{z}} \tag{2.14}$$

(see Fig. 4). We must note, however, that in a region where $\underset{\sim}{J} = 0$ the vector field of (2.14) does not satisfy the Maxwell equation $\underset{\sim}{\nabla} \times \underset{\sim}{B} = 0$. In order that $\underset{\sim}{B}$ satisfies this equation, we must add to the $\underset{\sim}{B}$-field of (2.14) a term of *curvature*, given by $B_x \, \hat{\underset{\sim}{x}} = B_0 \, \alpha z \, \hat{\underset{\sim}{x}}$. Therefore, a magnetic field having gradient and curvature, and which satisfies $\underset{\sim}{\nabla} \times \underset{\sim}{B} = 0$, is

$$\underset{\sim}{B} = B_0 \left[\alpha z \, \hat{\underset{\sim}{x}} + (1 + \alpha x) \, \hat{\underset{\sim}{z}} \right] \tag{2.15}$$

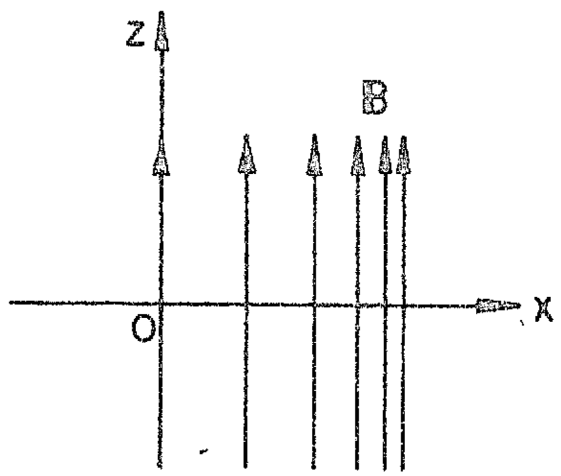

Fig. 4 - Geometry of the magnetic field lines when $\underset{\sim}{B}$ has a gradient in the x-direction, according to Eq. (2.14). Note, however, that this field geometry does not satisfy $\underset{\sim}{\nabla} \times \underset{\sim}{B} = 0$.

The geometry of the magnetic field lines corresponding to this equation is indicated in Fig. 5.

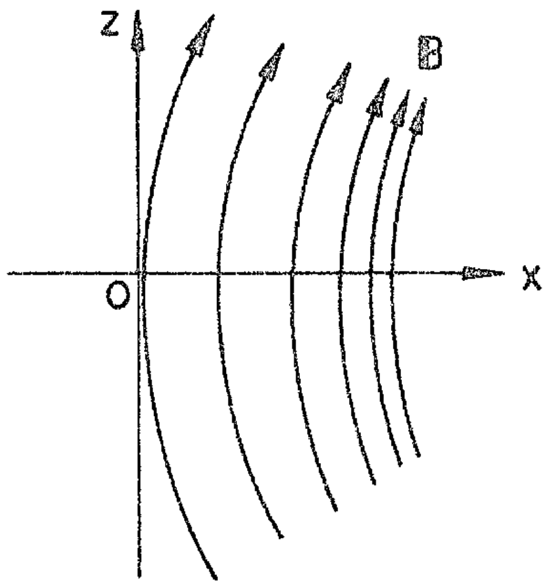

Fig. 5 - Geometry of the magnetic field lines corresponding to Eq. (2.15), having gradient and curvature terms.

Generally, all terms corresponding to divergence, gradient and curvature are simultaneously present. Fig. 6 illustrates a $\underset{\sim}{B}$ field having divergence, gradient and curvature. A good example is provided by Earth's magnetic field. Later in this section we will investigate the effects of each group of terms of (2.6) separately, on the motion of the charged particle. The net effect will be the sum of each effect separately, since in the first order approximation the equations are linear.

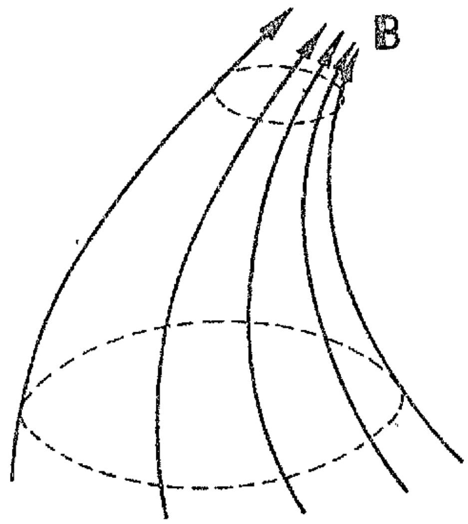

Fig. 6 - Schematic representation of a magnetic field having divergence, gradient and curvature terms.

2.3 - Shear terms

The *shear terms* of (2.6), $\partial B_x / \partial y$ and $\partial B_y / \partial x$, enter into the z-component of $\nabla \times B$, that is, into $B \cdot (\nabla \times B)$, and cause twisting of the magnetic field lines about each other. They do not produce any first order drifts, although the shape of the orbit can be slightly changed. Therefore, they do not give rise to any particularly interesting effect on the motion of charged particles and will not be considered any further.

3. EQUATION OF MOTION IN THE FIRST ORDER APPROXIMATION

We consider that the magnetic field $\underset{\sim}{B}_0$, which exists at the origin in the guiding center coordinate system, is in the z-direction,

$$\underset{\sim}{B}\ (0,\ 0,\ 0) \equiv \underset{\sim}{B}_0 = B_0\ \hat{\underset{\sim}{z}} \tag{3.1}$$

The particle motion in the neighborhood of the origin can be described by considering only a linear approximation to the magnetic field near the origin. Let $\underset{\sim}{r}$ be the momentary position vector of the particle in the guiding center coordinate system (Fig. 1). In the region of interest (near the origin) the magnetic field can be expressed by a Taylor expansion,

$$\underset{\sim}{B}\ (\underset{\sim}{r}) = \underset{\sim}{B}_0 + \underset{\sim}{r} \cdot (\underset{\sim}{\nabla}\underset{\sim}{B}) + \ldots \tag{3.2}$$

where the derivatives of $\underset{\sim}{B}$ are to be calculated at the origin. Note that actually the instantaneous position of the guiding center changes slightly during one period of rotation of the particle, while the origin is kept fixed during this time.

Since we are assuming that the spatial variation of $\underset{\sim}{B}$ in a distance of the order of the Larmor radius is much smaller than the magnitude of $\underset{\sim}{B}$ itself, the higher order terms of (3.2) can be neglected. The condition

$$|\underset{\sim}{r} \cdot (\underset{\sim}{\nabla} \underset{\sim}{B})| \ll |\underset{\sim}{B}_o| \tag{3.3}$$

is clearly met [see Eq. (1.2)] . Thus, the magnetic field at the position of the particle differs only slightly from that existing at the guiding center. The term of first order, $\underset{\sim}{r} \cdot (\underset{\sim}{\nabla} \underset{\sim}{B})$, can be written explicitly as

$$\underset{\sim}{r} \cdot (\underset{\sim}{\nabla} \underset{\sim}{B}) = (\underset{\sim}{r} \cdot \underset{\sim}{\nabla}) \underset{\sim}{B} = (x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}) \underset{\sim}{B} =$$

$$= (x \frac{\partial B_x}{\partial x} + y \frac{\partial B_x}{\partial y} + z \frac{\partial B_x}{\partial z}) \hat{\underset{\sim}{x}} +$$

$$+ (x \frac{\partial B_y}{\partial x} + y \frac{\partial B_y}{\partial y} + z \frac{\partial B_y}{\partial z}) \hat{\underset{\sim}{y}} +$$

$$+ (x \frac{\partial B_z}{\partial x} + y \frac{\partial B_z}{\partial y} + z \frac{\partial B_z}{\partial z}) \hat{\underset{\sim}{z}} \tag{3.4}$$

where the partial derivatives are to be calculated at the origin of the coordinate system.

The substitution of (3.2) into the equation of motion (2.1.5) with $\underset{\sim}{E} = 0$, gives

$$m \frac{d\underset{\sim}{v}}{dt} = q (\underset{\sim}{v} \times \underset{\sim}{B}_o) + q \underset{\sim}{v} \times \left[\underset{\sim}{r} \cdot (\underset{\sim}{\nabla} \underset{\sim}{B})\right] \tag{3.5}$$

The last term in the right-hand side is of first order compared to the first one. The particle velocity can be written as a superposition

$$\underset{\sim}{v} = \underset{\sim}{v}^{(0)} + \underset{\sim}{v}^{(1)} = \frac{d\,\underset{\sim}{r}^{(0)}}{dt} + \frac{d\,\underset{\sim}{r}^{(1)}}{dt} \tag{3.6}$$

where $\underset{\sim}{v}^{(1)}$ is a first-order perturbation term

$$|\underset{\sim}{v}^{(1)}| \ll |\underset{\sim}{v}^{(0)}| \tag{3.7}$$

and $\underset{\sim}{v}^{(0)}$ is the solution of the zero-order equation

$$m \frac{d\,\underset{\sim}{v}^{(0)}}{dt} = q\,(\underset{\sim}{v}^{(0)} \times \underset{\sim}{B}_o) \tag{3.8}$$

which has already been discussed in section 4 of Chapter 2. Neglecting second-order terms we can write, therefore,

$$\underset{\sim}{v} \times [\underset{\sim}{r} \cdot (\underset{\sim}{\nabla}\,\underset{\sim}{B})] = \underset{\sim}{v}^{(0)} \times [\underset{\sim}{r}^{(0)} \cdot (\underset{\sim}{\nabla}\underset{\sim}{B})] \tag{3.9}$$

The equation of motion (3.5) becomes, under these approximations,

$$m \frac{d\underset{\sim}{v}}{dt} = q\,(\underset{\sim}{v} \times \underset{\sim}{B}_o) + q\,\underset{\sim}{v}^{(0)} \times \left[\underset{\sim}{r}^{(0)} \cdot (\underset{\sim}{\nabla}\underset{\sim}{B})\right] \tag{3.10}$$

The equation satisfied by $\underset{\sim}{v}^{(1)}$ is identical to (3.10) and may be obtained by subtracting (3.8) from (3.10),

$$m \frac{d\underset{\sim}{v}^{(1)}}{dt} = q\ (\underset{\sim}{v}^{(1)} \times \underset{\sim}{B}_0) + q\ \underset{\sim}{v}^{(0)} \times \left[\underset{\sim}{r}^{(0)} \cdot (\underset{\sim}{\nabla} \underset{\sim}{B})\right] \qquad (3.11)$$

The second term in the right-hand side constitutes the force term of Eq. (2.6.1). This additional force, however, is not constant but it depends on the *instantaneous* position of the particle. Thus, small oscillations occur during one period of gyration of the particle. Since we are interested in the smoothed motion of the guiding center we shall eliminate these small oscillations by averaging this force term over one gyration period. Therefore, in what follows we will be involved in calculating the average value over one gyration period of the force term $q\ \underset{\sim}{v}^{(0)} \times \left[\underset{\sim}{r}^{(0)} \cdot (\underset{\sim}{\nabla} \underset{\sim}{B})\right]$, which will allow us to compute the parallel acceleration of the guiding center and its transverse drift velocity using Eq. (2.6.2).

4. AVERAGE FORCE OVER ONE GYRATION PERIOD

We consider initially the case when the initial velocity of the particle along $\underset{\sim}{B}$ is zero, so that the particle path differs but little from a circle. In a uniform magnetic field, this would be equivalent to introduce a coordinate system moving with the guiding center with velocity $\underset{\sim}{v}_{\parallel}$. However, when the field lines are bent, a coordinate system gliding along the field line is not an inertial system. The curvature of the field lines give rise to inertial forces and therefore to a *curvature drift* of the particle. This effect will be investigated later in section 7. For the moment we will assume that the field lines are not curved and that the coordinate system moves with velocity $\underset{\sim}{v}_{\parallel}$. The effects of each group of terms of (2.6) can be considered separately since in the region of interest the components of $\underset{\sim}{\nabla}\underset{\sim}{B}$ are small perturbations in the z-component of $\underset{\sim}{B}$, and the first order equation of motion is linear. Thus, we can examine situations where only one inhomogeneity occurs at a time, and the resultant effect will be the sum of the individual effects for each group of terms.

Under the conditions indicated above, the zero-order variables, $\underset{\sim}{v}^{(0)}$ and $\underset{\sim}{r}^{(0)}$, are seen to be situated in the (x, y) plane. The force term

$$\underset{\sim}{F} = q\ \underset{\sim}{v}^{(0)} \times \left[\underset{\sim}{r}^{(0)} \cdot (\underset{\sim}{\nabla}\underset{\sim}{B})\right] \tag{4.1}$$

can be separated in a component along $\underset{\sim}{B}_0$ (z-axis), $\underset{\sim}{F}_{\parallel}$, and a component normal to $\underset{\sim}{B}_0$ (x,y-plane), $\underset{\sim}{F}_{\perp}$. Using a local cylindrical coordinate system (r, Θ, z) with the z-axis pointing along $\underset{\sim}{B}_0$ at the origin (see Fig. 7), we have

$$\underset{\sim}{r}^{(0)} \cdot (\underset{\sim}{\nabla}\underset{\sim}{B}) = r^{(0)}\ (\partial\underset{\sim}{B} / \partial r) \tag{4.2}$$

Of the three components of $\underset{\sim}{B} = B_r\ \hat{\underset{\sim}{r}} + B_\Theta\ \hat{\Theta} + B_z\ \hat{\underset{\sim}{z}}$, $B_\Theta\ \hat{\underset{\sim}{\Theta}}$ is parallel to $\underset{\sim}{v}^{(0)}$ and therefore gives no contribution to $\underset{\sim}{F}$, while

$B_r \hat{\underset{\sim}{r}}$ contributes to $\underset{\sim}{F}_{\parallel}$ and $B_z \hat{\underset{\sim}{z}}$ contributes to $\underset{\sim}{F}_{\perp}$. Hence, from (4.1) and (4.2),

$$\underset{\sim}{F}_{\parallel} = q\ (\underset{\sim}{v}^{(0)} \times \hat{\underset{\sim}{r}})\ r^{(0)} (\frac{\partial B_r}{\partial r}) = |q|\ v^{(0)}\ r^{(0)}\ (\frac{\partial B_r}{\partial r})\ \hat{\underset{\sim}{z}} \qquad (4.3)$$

$$\underset{\sim}{F}_{\perp} = q\ (\underset{\sim}{v}^{(0)} \times \hat{\underset{\sim}{z}})\ r^{(0)} (\frac{\partial B_z}{\partial r}) = -\ |q|\ v^{(0)}\ r^{(0)}\ (\frac{\partial B_z}{\partial r}) \hat{\underset{\sim}{r}} \qquad (4.4)$$

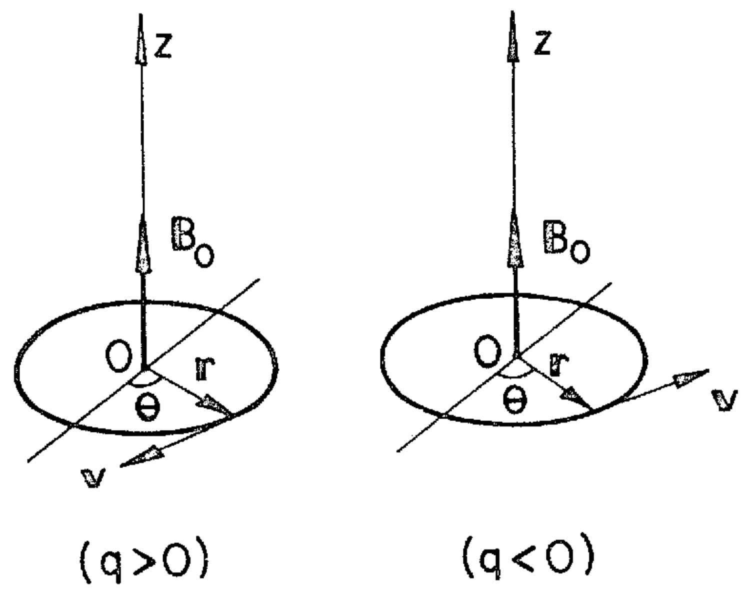

Fig. 7 - Local cylindrical coordinate system with the z-axis pointing in the direction of the field $\underset{\sim}{B}_0$ at the origin.

Note that if $q > 0$ we have $\underset{\sim}{v}^{(0)} \times \hat{\underset{\sim}{r}} = v^{(0)}\hat{\underset{\sim}{z}}$, whereas if $q < 0$ we have $\underset{\sim}{v}^{(0)} \times \hat{\underset{\sim}{r}} = - v^{(0)}\hat{\underset{\sim}{z}}$. Now, $r^{(0)}$ is the cyclotron radius corresponding to $\underset{\sim}{B}_o$,

$$r^{(0)} = \frac{v^{(0)}}{\omega_c} = \frac{v^{(0)}\, m}{|q|\ B_o} \tag{4.5}$$

and using the expression for the magnitude of the magnetic moment $|\underset{\sim}{m}| = m\,(v^{(0)})^2/2B_o$ [Eq. (2.4.34)], we can write (4.3) and (4.4) as

$$\underset{\sim}{F}_{\parallel} = 2\ |\underset{\sim}{m}|\left(\frac{\partial B_r}{\partial r}\right)\hat{\underset{\sim}{z}} \tag{4.6}$$

$$\underset{\sim}{F}_{\perp} = -\ 2\ |\underset{\sim}{m}|\left(\frac{\partial B_z}{\partial r}\right)\hat{\underset{\sim}{r}} \tag{4.7}$$

These results apply to both positively and negatively charged particles.

The average values of $\underset{\sim}{F}_{\parallel}$ and $\underset{\sim}{F}_{\perp}$ over one gyration period are given by

$$< \underset{\sim}{F}_{\parallel} > \ = \ 2\ |\underset{\sim}{m}|\ \hat{\underset{\sim}{z}} \left[\frac{1}{2\pi}\oint \left(\frac{\partial B_r}{\partial r}\right) d\theta\right] = 2\ |\underset{\sim}{m}|\ \hat{\underset{\sim}{z}} < \left(\frac{\partial B_r}{\partial r}\right) > \tag{4.8}$$

$$< \underset{\sim}{F}_{\perp} > \ = -2\ |\underset{\sim}{m}| \left[\frac{1}{2\pi}\oint \left(\frac{\partial B_z}{\partial r}\right)\hat{\underset{\sim}{r}}\ d\theta\right] = -\ 2\ |\underset{\sim}{m}| < \hat{\underset{\sim}{r}}\left(\frac{\partial B_z}{\partial r}\right) > \tag{4.9}$$

The force in (4.8) produces the parallel acceleration of the guiding center, while the force in (4.9) is responsible for the transverse drift velocity of the guiding center. The first one is the result of the *divergence* terms of $\underset{\sim}{B}$, and the second one of the *gradient* terms. We proceed now to evaluate each force term separately.

4.1 - Parallel force

To calculate the *parallel force term* we note that from $\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0$ we have, in cylindrical coordinates,

$$\frac{1}{r}\frac{\partial}{\partial r}(r\, B_r) + \frac{1}{r}\frac{\partial}{\partial \Theta}(B_\Theta) + \frac{\partial}{\partial z}(B_z) = 0 \tag{4.10}$$

The first term can be expanded as

$$\frac{1}{r}\frac{\partial}{\partial r}(r\, B_r) = \frac{\partial B_r}{\partial r} + \frac{B_r}{r} \tag{4.11}$$

Since at $r = 0$ we have $B_r = 0$, and since near the origin B_r changes only very slightly with r, we can take near the origin,

$$\frac{B_r}{r} = \frac{\partial B_r}{\partial r} \tag{4.12}$$

Consequently, in the region near the origin we have from (4.12) and (4.11),

$$\left(\frac{\partial B_r}{\partial r}\right) = -\frac{1}{2}\left[\frac{1}{r}\left(\frac{\partial B_\Theta}{\partial \Theta}\right) + \left(\frac{\partial B_z}{\partial z}\right)\right] \tag{4.13}$$

Hence, taking the average over one gyration period,

$$< \left(\frac{\partial B_r}{\partial r}\right) > = -\frac{1}{2} < \frac{1}{r}\left(\frac{\partial B_\Theta}{\partial \Theta}\right) > - \frac{1}{2} < \left(\frac{\partial B_z}{\partial z}\right) > \tag{4.14}$$

Now, since B is single-valued,

$$< \frac{1}{r}\left(\frac{\partial B_\Theta}{\partial \Theta}\right) > = \frac{1}{2\pi}\oint \frac{1}{r}\left(\frac{\partial B_\Theta}{\partial \Theta}\right) d\Theta = 0 \tag{4.15}$$

Furthermore, since the term $\partial B_z / \partial z$ is a very slowly varying function inside the particle's orbit, it can be taken outside the integral sign, so that we have approximately,

$$< \left(\frac{\partial B_z}{\partial z}\right) > = \frac{1}{2\pi}\oint \left(\frac{\partial B_z}{\partial z}\right) d\Theta = \frac{\partial B_z}{\partial z} = \frac{\partial B}{\partial z} \tag{4.16}$$

It is justifiable to replace B_z by B in (4.16), since all the spatial variations of the magnetic field in the region of interest are very small. Therefore, we have finally from (4.14), (4.15) and (4.16),

$$< \left(\frac{\partial B_r}{\partial r}\right) > = -\frac{1}{2}\left(\frac{\partial B}{\partial z}\right) \tag{4.17}$$

Using this result, the parallel force (4.8) becomes

$$< \underset{\sim}{F}_{\parallel} > = - \,|\underset{\sim}{m}|\, \left(\frac{\partial B}{\partial z}\right) \hat{\underset{\sim}{z}} = - \,|\underset{\sim}{m}|\, (\underset{\sim}{\nabla} B)_{\parallel} \qquad (4.18)$$

or, equivalently,

$$< \underset{\sim}{F}_{\parallel} > = (\underset{\sim}{m} \cdot \underset{\sim}{\nabla})\, B\, \hat{\underset{\sim}{z}} = - \frac{|\underset{\sim}{m}|}{B} \left[(\underset{\sim}{B} \cdot \underset{\sim}{\nabla})\, \underset{\sim}{B} \right]_{\parallel} \qquad (4.19)$$

since $\underset{\sim}{m} = - \,|\underset{\sim}{m}|\, \hat{\underset{\sim}{z}} = - \,|\underset{\sim}{m}|\, \underset{\sim}{B}/B$, and where the derivatives are evaluated at the origin.

4.2 - Perpendicular force

To evaluate the average value of the *perpendicular force component* given in Eq. (4.9), it is convenient to consider a two-dimensional Cartesian coordinate system (x, y) in the perpendicular plane, such that $x = r \cos \Theta$ and $y = r \sin \Theta$ (see Fig. 8). Hence,

$$\hat{\underset{\sim}{r}} = \cos \Theta\, \hat{\underset{\sim}{x}} + \sin \Theta\, \hat{\underset{\sim}{y}} \qquad (4.20)$$

$$\frac{\partial}{\partial r} = \frac{dx}{dr} \frac{\partial}{\partial x} + \frac{dy}{dr} \frac{\partial}{\partial y} = \cos \Theta \frac{\partial}{\partial x} + \sin \Theta \frac{\partial}{\partial y} \qquad (4.21)$$

Therefore, we obtain

$$< \hat{\underset{\sim}{r}} \left(\frac{\partial B_z}{\partial r}\right) > = < (\cos\Theta\ \hat{\underset{\sim}{x}} + \sin\Theta\ \hat{y}) \left[\cos\Theta\left(\frac{\partial B_z}{\partial x}\right) + \sin\Theta\left(\frac{\partial B_z}{\partial y}\right)\right] >$$

$$= < \cos^2\Theta\left(\frac{\partial B_z}{\partial x}\right)\hat{x} > + < \sin\Theta\cos\Theta\left(\frac{\partial B_z}{\partial x}\right)\hat{\underset{\sim}{y}} > +$$

$$+ < \cos\Theta\sin\Theta\left(\frac{\partial B_z}{\partial y}\right)\hat{\underset{\sim}{x}} > + < \sin^2\Theta\left(\frac{\partial B_z}{\partial y}\right)\hat{y} > \quad (4.22)$$

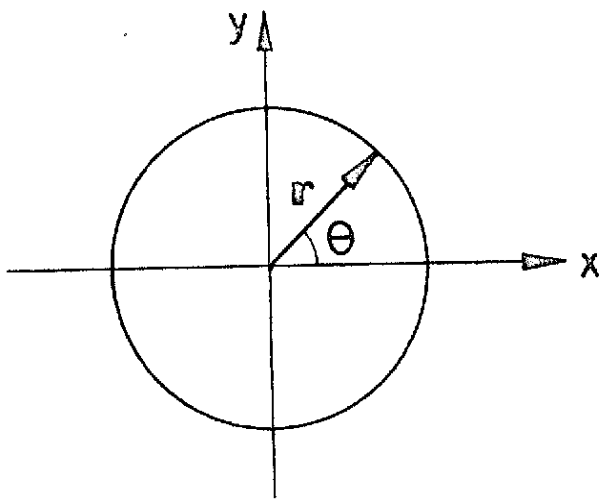

Fig. 8 - Two-dimensional coordinate system in the perpendicular plane, used in the evaluation of $< \underset{\sim}{F}_{\perp} >$.

Next we approximate $(\partial B_z/\partial x)$ by $(\partial B/\partial x)$, and the same for the y-derivative, since these terms are very slowly varying functions inside the particle's orbit, so that we can take them outside the integral sign. Noting that $< \sin\Theta \cos\Theta > = 0$ and $<\cos^2\Theta> = <\sin^2\Theta> = 1/2$, we obtain

$$< \hat{\underset{\sim}{r}} \left(\frac{\partial B_z}{\partial r}\right) > = \frac{1}{2} \left(\frac{\partial B}{\partial x}\right) \hat{\underset{\sim}{x}} + \frac{1}{2} \left(\frac{\partial B}{\partial y}\right) \hat{\underset{\sim}{y}} \tag{4.23}$$

Substituting this result into (4.9), yields

$$< \underset{\sim}{F}_\perp > = - |\underset{\sim}{m}| \left[\left(\frac{\partial B}{\partial x}\right) \hat{\underset{\sim}{x}} + \left(\frac{\partial B}{\partial y}\right) \hat{\underset{\sim}{y}} \right]$$

$$= - |\underset{\sim}{m}| \, (\underset{\sim}{\nabla} B)_\perp \tag{4.24}$$

4.3 - Total average force

We proceed now to write down a general expression for the *total average force* $<\underset{\sim}{F}> = <\underset{\sim}{F}_\parallel> + <\underset{\sim}{F}_\perp>$. From (4.18) and (4.24) we have

$$<\underset{\sim}{F}> = - |\underset{\sim}{m}| \, (\underset{\sim}{\nabla} B)_\parallel - |\underset{\sim}{m}| \, (\underset{\sim}{\nabla} B)_\perp = - |\underset{\sim}{m}| \, \underset{\sim}{\nabla} B \tag{4.25}$$

Alternatively, we can use the vector identity

$$(\underset{\sim}{\nabla} \times \underset{\sim}{B}) \times \underset{\sim}{B} = (\underset{\sim}{B} \cdot \underset{\sim}{\nabla}) \underset{\sim}{B} - \underset{\sim}{\nabla} (B^2/2) \tag{4.26}$$

and write (4.25) in the form

$$< \underset{\sim}{F} > = - \frac{|\underset{\sim}{m}|}{B} \left[(\underset{\sim}{B} \cdot \underset{\sim}{\nabla}) \underset{\sim}{B} - (\underset{\sim}{\nabla} \times \underset{\sim}{B}) \times \underset{\sim}{B} \right] \tag{4.27}$$

Since $\underset{\sim}{m} = - |\underset{\sim}{m}| \, \underset{\sim}{B}/B$, it results

$$<\underset{\sim}{F}> = (\underset{\sim}{m} \cdot \underset{\sim}{\nabla}) \underset{\sim}{B} + \underset{\sim}{m} \times (\underset{\sim}{\nabla} \times \underset{\sim}{B}) \tag{4.28}$$

This is the usual expression for the force acting on a small ring current immersed in a magnetic field with spatial variation. The first term on the right hand side of (4.28) alone gives the force acting on a magnetic dipole.

5. GRADIENT DRIFT

From (2.6.2) and (4.24) we see that the force $< \underset{\sim}{F}_\perp >$, being perpendicular to $\underset{\sim}{B}$, causes the guiding center to drift with the velocity

$$\underset{\sim}{v}_G = \frac{< \underset{\sim}{F}_\perp > \times \underset{\sim}{B}}{q\, B^2} = - \frac{|\underset{\sim}{m}|}{q} \frac{(\underset{\sim}{\nabla} B) \times \underset{\sim}{B}}{B^2} \qquad (5.1)$$

This gradient drift is perpendicular to the $\underset{\sim}{B}$ field and to the field gradient, and its direction depends on the sign of the charge. Thus, positive and negative charges drift in the opposite direction, giving riseto an electric current (Fig. 9).

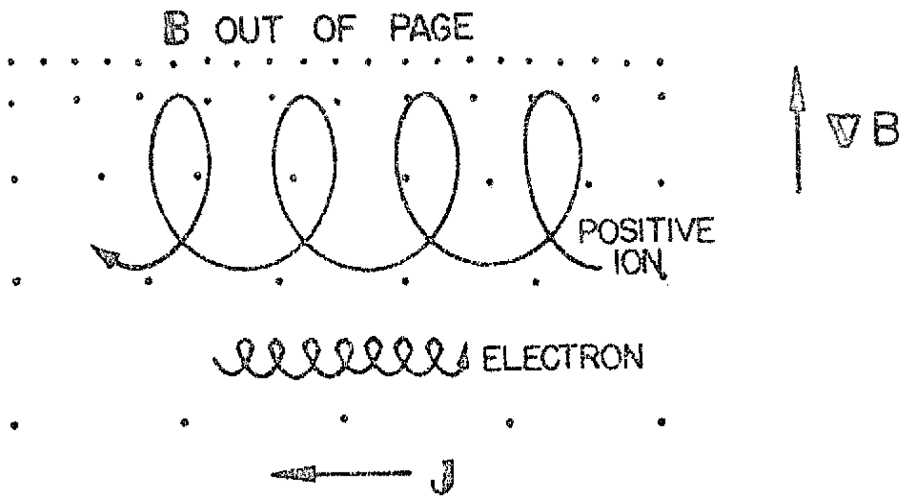

Fig. 9 - Charged particle drifts due to a $\underset{\sim}{B}$ - field gradient perpendicular to $\underset{\sim}{B}$.

The physical reason for this gradient drift can be explained as follows. We have seen [see Eq. (2.4.13)] that the Larmor radius of the particle's orbit decreases as the magnetic field increases, so that the radius of curvature of the orbit is smaller in the regions of stronger $\underset{\sim}{B}$ field. Since the positive ions gyrate in the clockwise direction for $\underset{\sim}{B}$ pointing towards the observer (in Fig. 9), while the electrons gyrate in the counterclockwise direction, the positive ions will drift to the left and the electrons to the right.

In the case of a rarefied collisionless plasma, associated with this gradient drift across the magnetic field lines there is a *magnetization current density* $\underset{\sim}{J}_G$, given by

$$\underset{\sim}{J}_G = \frac{1}{\Delta V} \sum_i q_i \, \underset{\sim}{v}_{Gi} \tag{5.2}$$

where the summation is over all charged particles contained in the element of volume ΔV. Using (5.1) in (5.2), yields

$$\underset{\sim}{J}_G = - \left(\frac{1}{\Delta V} \sum_i \; |\underset{\sim}{m}_i|\right) \frac{(\underset{\sim}{\nabla} B) \times \underset{\sim}{B}}{B^2} \tag{5.3}$$

6. PARALLEL ACCELERATION OF THE GUIDING CENTER

The expression (4.18) for the parallel force $< \underset{\sim}{F}_\parallel >$ shows that, when the magnetic field has a longitudinal variation(i.e., convergence or divergence of the field lines along the z-direction) as shown in Fig. 3, an axial force along z accelerates the particle in the direction of decreasing magnetic field, irrespective of whether the particle is positively or negatively charged. This is illustrated in Fig. 10. There are several important consequences of this respulsion of gyrating charges from a region of converging magnetic field lines, which we proceed to discuss.

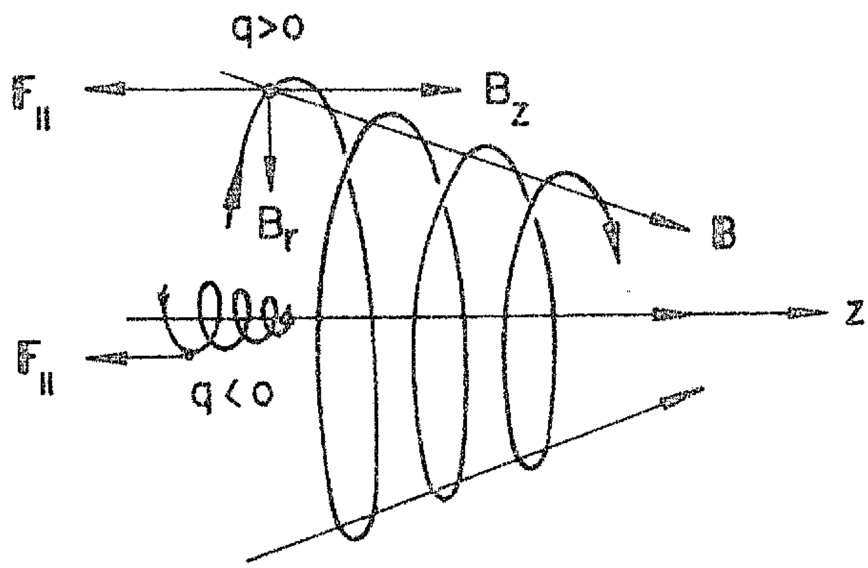

Fig. 10 - Repulsion of gyrating charges from a region of converging magnetic field lines.

6.1 - Invariance of the orbital magnetic moment and magnetic flux

Using (4.18), the component of the equation of motion along $\underset{\sim}{B}$ can be written as

$$m \frac{dv_{\parallel}}{dt} \hat{\underset{\sim}{z}} = < \underset{\sim}{F}_{\parallel} >$$

$$= - | \underset{\sim}{m} | \left(\frac{\partial B}{\partial z}\right) \hat{\underset{\sim}{z}} \tag{6.1}$$

If we multiply both sides of this equation by $v_{\parallel} = dz/dt$, we obtain (replacing $|\underset{\sim}{m}|$ by $W_{\perp}/B$)

$$m\, v_{\parallel} \frac{dv_{\parallel}}{dt} \equiv \frac{d}{dt}\left(\frac{1}{2}\, m\, v_{\parallel}^2\right) = - \frac{W_{\perp}}{B} \frac{\partial B}{\partial z} \frac{dz}{dt} \tag{6.2}$$

where $W_{\perp} = m\, v_{\perp}^2/2$ denotes the part of the kinetic energy of the particle associated with its transverse velocity. Since the total kinetic energy of a charged particle in a magnetostatic field is constant,

$$W_{\parallel} + W_{\perp} = \text{constant} \tag{6.3}$$

it follows that

$$\frac{d}{dt}(W_{\perp}) = - \frac{d}{dt}(W_{\parallel}) = - \frac{d}{dt}\left(\frac{1}{2}\, m\, v_{\parallel}^2\right) \tag{6.4}$$

Therefore, from (6.2) and (6.4),

$$\frac{d}{dt}(W_\perp) = \frac{W_\perp}{B}\frac{\partial B}{\partial z}\frac{dz}{dt} = \frac{W_\perp}{B}\frac{dB}{dt} \tag{6.5}$$

where dB/dt represents the rate of change of B as seen by the particle as it moves in the spatially varying magnetic field (i. e., in the frame of reference of the particle). Comparing this result with the following identity

$$\frac{d}{dt}(W_\perp) = \frac{d}{dt}\left(\frac{W_\perp B}{B}\right) = \frac{W_\perp}{B}\frac{dB}{dt} + B\frac{d}{dt}\left(\frac{W_\perp}{B}\right) \tag{6.6}$$

we find that

$$\frac{d}{dt}\left(\frac{W_\perp}{B}\right) = 0 \tag{6.7}$$

or, equivalently,

$$|\underset{\sim}{m}| = \frac{W_\perp}{B} = \text{constant} \tag{6.8}$$

Therefore, as the particle moves into regions of converging or diverging $\underset{\sim}{B}$ its cyclotron radius changes, but the magnetic moment $|\underset{\sim}{m}|$ remains constant. This constancy of the particle's magnetic moment in the guiding center system holds only within the approximation used, that is, when the spatial variation of $\underset{\sim}{B}$ inside the particle's orbit is small compared with the magnitude of $\underset{\sim}{B}$. Consequently, the *orbital magnetic moment* is said to be an *adiabatic invariant*. It is usually referred to as the *first adiabatic invariant*.

The magnetic flux, Φ_m, enclosed by one orbit of the particle is given by

$$\Phi_m = \int \underset{\sim}{B} \cdot d\,\underset{\sim}{S} = \pi\, r_c^2\, B = \pi\, \frac{m^2\, v_\perp^2}{q^2\, B^2}\, B = \frac{2\pi m}{q^2}\left(\frac{W_\perp}{B}\right) \tag{6.9}$$

Therefore,

$$\frac{d}{dt}\,(\Phi_m) = \frac{2\pi m}{q^2}\,\frac{d}{dt}\,|\,\underset{\sim}{m}\,| = 0 \tag{6.10}$$

in view of the invariance of the magnetic moment. Hence, as the charged particle moves in a region of converging $\underset{\sim}{B}$ field the particle will orbit with increasingly smaller radius, so that the *magnetic flux enclosed by the orbit* remains *constant*.

6.2 - Magnetic mirror effect

As a consequence of the *adiabatic invariance* of the orbital magnetic moment of the particle, and of the magnetic flux enclosed by its orbit, as the particle moves into a region of converging magnetic field lines its transverse kinetic energy $W_{\perp}$ increases, while its parallel kinetic energy $W_{\parallel}$ decreases, in order to keep $|\underset{\sim}{m}|$ and the total energy constant. Ultimately, if the $\underset{\sim}{B}$ field becomes strong enough, the particle velocity in the direction of increasing $\underset{\sim}{B}$ field may come to zero and then be reversed. In the reverse direction the opposite happens, i.e., the particle is speeded up in the direction of decreasing field, while its transverse velocity diminishes. Thus, the particle is *reflected* from the region of converging magnetic field lines. This phenomenon is called the *magnetic mirror effect* and it is the basis for one of the primary schemes of plasma confinement.

When two coaxial magnetic mirrors are considered, as illustrated in Fig. 11, the charged particles may be reflected by the magnetic mirrors and travel back and forth in the space between them, being trapped. This trapping region has been called a *magnetic bottle* and it is used in laboratory for the confinement of plasmas. Clearly, it is the parallel force $< \underset{\sim}{F}_{\parallel} >$ which causes the reflection.

The trapping in a magnetic mirror system is not perfect, however. The effectiveness of the coaxial magnetic mirror system in trapping the charged particles, can be measured by the *mirror ratio* B_m/B_o, where B_m is the intensity of the magnetic field at the point of reflection (where the pitch angle of the particle is $\pi/2$) and B_o is the intensity of the magnetic field at the center of the magnetic bottle.

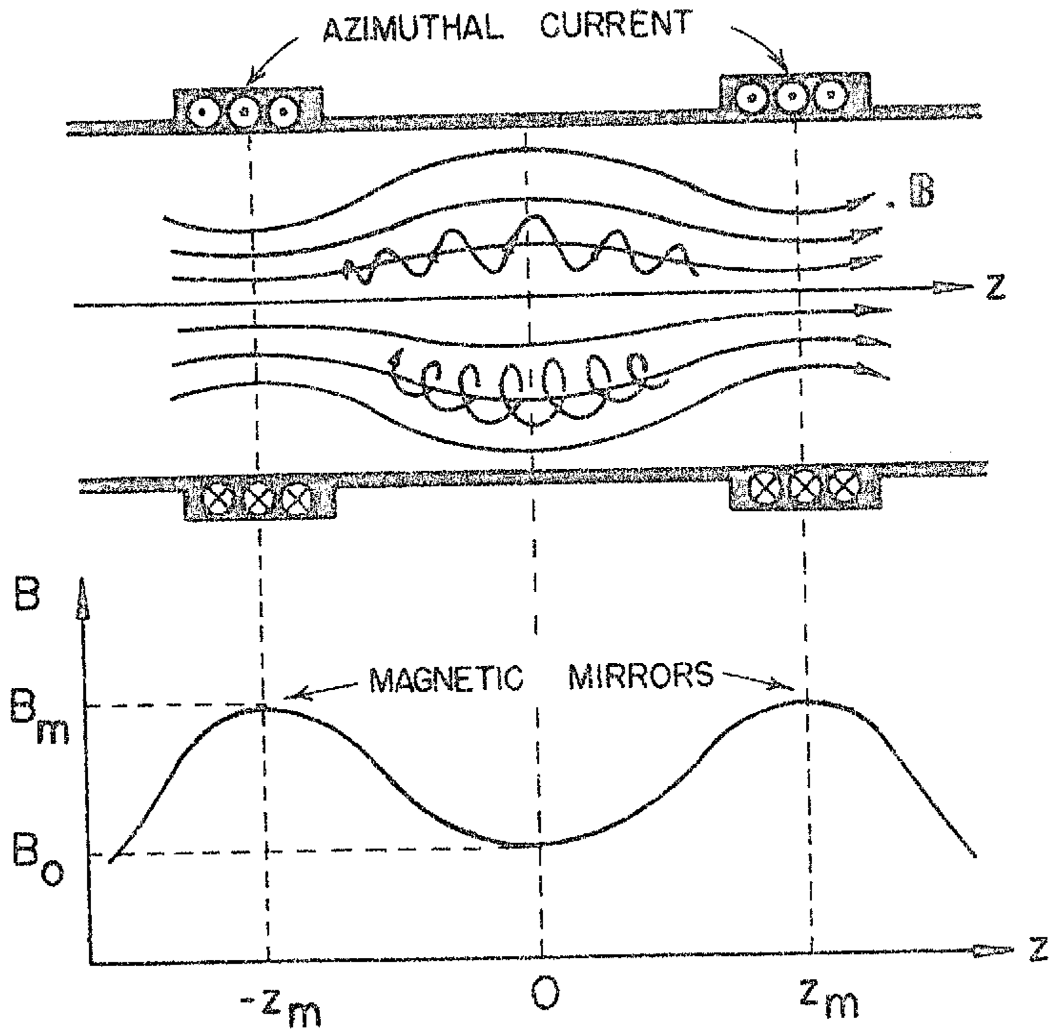

Fig. 11 - Schematic diagram showing the arrangement of coils to produce two coaxial magnetic mirrors facing each other for plasma confinement, and the relative intensity of the magnetic field.

Let us consider a charged particle having a pitch angle α_o at the center of the magnetic bottle. If v is the speed of the particle, which in a static magnetic field remains constant, the constancy of the magnetic moment $|\underset{\sim}{m}| = W_\perp/B$ leads to

$$m\, v^2 \sin^2\alpha/2B \;=\; m\, v^2 \sin^2 \alpha_o/2B_o \tag{6.11}$$

where α is the pitch angle of the particle at a position where the magnetic field intensity is B. Thus, at any point inside the magnetic bottle, for this particle,

$$\frac{\sin^2 \alpha\,(z)}{B\,(z)} = \frac{\sin^2 \alpha_o}{B_o} \tag{6.12}$$

Suppose now that this particle is reflected at the "throat" of the mirror, that is, $\alpha = \pi/2$ for $B(z) = B_m$. Therefore, from (6.12),

$$\frac{\sin^2 \alpha_o}{B_o} = \frac{1}{B_m} \tag{6.13}$$

This means that a particle having a pitch angle α_o, given by,

$$\alpha_o = \sin^{-1} \sqrt{B_o/B_m} = \sin^{-1} (v_\perp/v)_o \tag{6.14}$$

at the center of the bottle, is reflected at a point where the intensity of the field is B_m. Therefore, for a magnetic bottle with a fixed mirror ratio B_m/B_0, the plasma particles having a pitch angle at the center *greater* than α_0, as given by (6.14), will be reflected *before* the ends of the magnetic bottle. On the other hand, if the pitch angle of the particle at the center is less than α_0, its pitch angle will never reach the value $\pi/2$, which implies that at the ends of the bottle the particle has a non-vanishing parallel velocity and, hence, escapes through the ends of the mirror system. There is therefore a *loss cone,* a bi-cone of angle α_0 with its vertex at the center, as shown in Fig. 12, where particles which have velocity vectors with a pitch angle falling inside it are not trapped in the magnetic bottle system. The loss cone is determined by the mirror ratio B_m/B_0 according to (6.14).

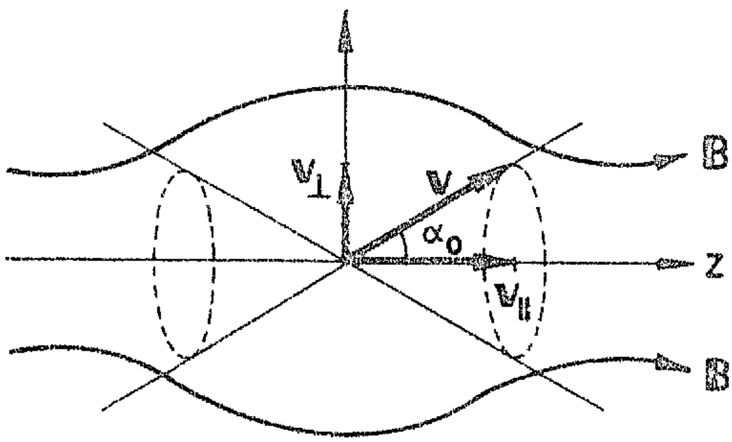

Fig. 12 - The loss cone in a coaxial magnetic mirror system.

Devices that have no ends, with geometries such that the magnetic field lines close on themselves, offer many advantages for plasma confinement. *Toroidal* geometries (see Fig. 13) for example have no ends, but it turns out that confinement of a plasma inside a toroidal magnetic field does not provide a plasma equilibrium situation, because of the radial inhomogeneity of the field. In this case a poloidal magnetic field is normally superposed on the toroidal field, resulting in helical field lines (as in the Tokamak). The major problem in the confinement schemes, however, is that instabilities and small fluctuations from the desired configuration are always present, which lead to a rapid scape of the particles from the magnetic bottle. This instability problem is a fundamental one, and it is likely to occur in any conceivable magnetic confinement scheme.

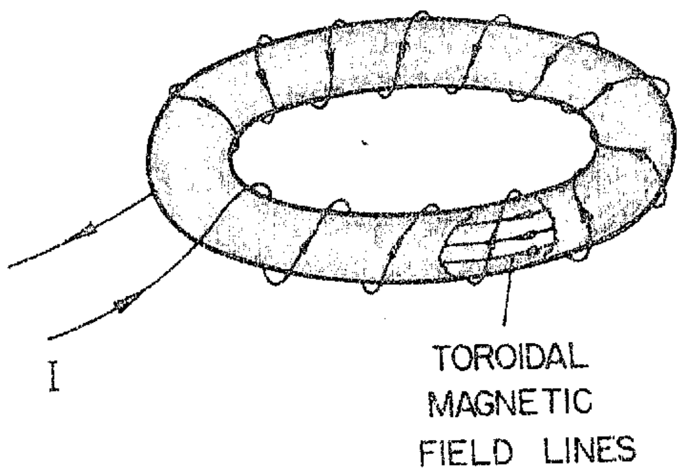

Fig. 13 - Magnetic field with toroidal geometry

A good example of a natural magnetic bottle is the Earth's magnetic field, which traps charged particles of solar and cosmic origin. These charged particles trapped in the Earth's magnetic field constitute the so called *Van Allen radiation belts.* As shown in Fig. 14, the geomagnetic field near the Earth is approximately that of a dipole, with the field lines converging towards the north and south magnetic poles. The electrons and protons spiral in almost helical paths along the field lines towards the magnetic poles, where they are reflected. These particles bounce back and forth between the poles. In addition to this bouncing motion, the charged particle in the Van Allen radiation belts are also subjected to a gradient drift and a curvature drift, to be discussed later in this chapter.

6.3 - The longitudinal adiabatic invariant

Consider a particle trapped between two magnetic mirrors and bouncing between them. Suppose that the separation distance between the two mirrors changes very slowly with time as compared to the bounce period. With the periodic motion of the particle between the two magnetic mirrors (whose separation varies slowly with time) there is associated an adiabatic invariant called the *longitudinal adiabatic invariant,* defined by the integral

$$J = \oint \underset{\sim}{v} \cdot d\underset{\sim}{\ell} = \oint v_{\parallel} \, d\ell \tag{6.15}$$

taken over one period of oscillation of the particle back and forth between the mirror points.

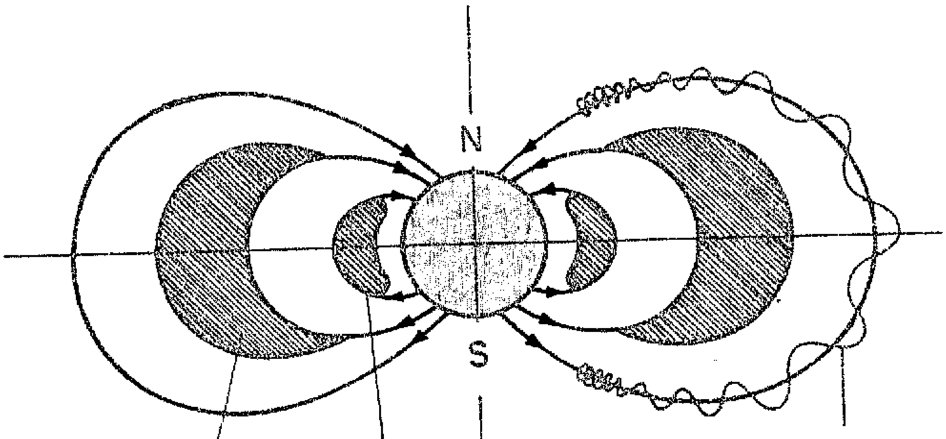

Fig. 14 - Dipole approximation of the Earth's magnetic field. The distance of the Van Allen radiation belts from the center of the Earth, at the equator, is about 1.5 Earth radii for the high-energy protons and about 3 to 4 Earth radii for the high-energy electrons.

For a simple proof of the adiabatic invariance of J, consider the idealized situation illustrated in Fig. 15, where the existing $\underset{\sim}{B}$ field in the z-direction is uniform in space, except near the points M_1 and M_2 where the field increases to form the two mirrors separated by a distance L. The mirror M_1 approaches the other with velocity

$$v_m = - \frac{dL}{dt} \tag{6.16}$$

the negative sign being due to the fact that L decreases with time. It is assumed that this velocity is much smaller than the longitudinal component of the particle velocity, that is, $v_m \ll v_\parallel$. Thus, the distance moved by the mirror M_1 during one period of oscillation of the particle is small compared to the distance L between the mirrors.

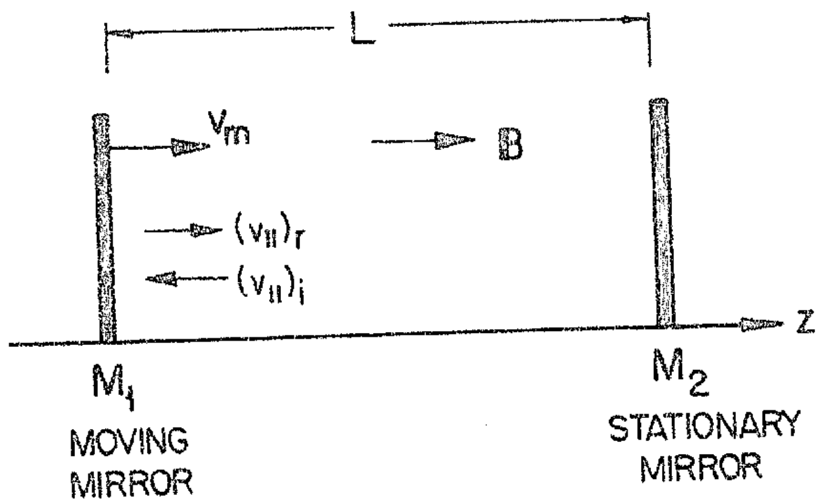

Fig. 15 - Schematic representation of a system of two coaxial magnetic mirrors, approaching each other very slowly.

Further, since B is assumed to be uniform throughout the space between the mirrors (except near the ends), the longitudinal particle speed $v_\parallel$ may be taken to be constant in the space between the mirrors. Neglecting the small end effects at the two mirrors, we can take

$$J = \int_0^{2L} v_\parallel \, d\ell = 2 \, v_\parallel \, L \tag{6.17}$$

The time rate of change of J is

$$\frac{dJ}{dt} = 2 \, v_\parallel \, \frac{dL}{dt} + 2L \, \frac{dv_\parallel}{dt} = - \, 2 \, v_\parallel \, v_m + 2L \, \frac{dv_\parallel}{dt} \tag{6.18}$$

where use was made of (6.16). To calculate $dv_\parallel/dt$, we set

$$\frac{dv_\parallel}{dt} = \frac{\Delta v_\parallel}{\Delta t} = \frac{\Delta v_\parallel}{(2L/v_\parallel)} \tag{6.19}$$

where $\Delta v_\parallel$ denotes the change in the particle speed $v_\parallel$ on reflection from the moving mirror, and $\Delta t = 2L/v_\parallel$ is the period of oscillation between the mirrors. In order to find $\Delta v_\parallel$, it is convenient to transform to a coordinate system moving with the magnetic mirror M_1, at the speed v_m. Let us denote this moving coordinate system by a prime and the incident and reflected particle speeds by subscripts i and r, respectively. Thus,

$$(v_\parallel)_i' = (v_\parallel)_i + v_m \tag{6.20}$$

$$(v_{\parallel})'_r = (v_{\parallel})_r - v_m \tag{6.21}$$

which gives for the change in the particle speed, in one reflection,

$$\Delta v_{\parallel} = (v_{\parallel})_r - (v_{\parallel})_i = 2\, v_m \tag{6.22}$$

since in the moving coordinate system $(v_{\parallel})'_i = (v_{\parallel})'_r$ with only their directions reversed. Therefore, (6.19) becomes

$$\frac{dv_{\parallel}}{dt} = \frac{2\, v_m}{(2L/v_{\parallel})} = \frac{1}{L}\, v_m\, v_{\parallel} \tag{6.23}$$

On substituting this result into (6.18), we find

$$\frac{dJ}{dt} = \frac{d\,(2v_{\parallel}\, L)}{dt} = 0 \tag{6.24}$$

which shows that J is an *adiabatic invariant*. This quantity is also referred to as the *second adiabatic invariant*.

The parallel kinetic energy of a charged particle trapped between the two mirrors is (using $J = 2v_{\parallel}\, L$)

$$W_{\parallel} = \frac{1}{2}\, m\, v_{\parallel}^2 = \frac{m\, J^2}{8\, L^2} \tag{6.25}$$

which increases rapidly as L decreases. The italian physicist Fermi suggested this process as a mechanism for the acceleration of charged particles in order to explain the origin of high energy cosmic rays. Fermi proposed that two stellar clouds moving towards each other, and having a magnetic field greater than in the space between them, may trap and accelerate the cosmic charged particles. There is a limit, however, in the increase in the particle longitudinal speed, since the direction of the particle velocity at the center of the mirror system may eventually enter the loss cone and escape through the ends of the system. It should be noted that a magnetic mirror moving towards a stationary one involves in fact time-varying $\underset{\sim}{B}$ - fields and consequently electric fields, which can lead to a change in the kinetic energy of the particle.

7. CURVATURE DRIFT

So far the effects associated with the curvature of the magnetic field lines have not been considered. As stated previously, a $\underset{\sim}{B}$ field with only curvature terms does not satisfy the equation $\underset{\sim}{\nabla} \times \underset{\sim}{B} = 0$, so that in practice the gradient drift will always be present simultaneously with the curvature drift. In the first order orbit theory the effects corresponding to each of the components of the dyadic $\underset{\sim}{\nabla}\underset{\sim}{B}$ are additive.

We investigate now the effect of the curvature terms $\partial B_x/\partial z$ and $\partial B_y/\partial z$ [referred in (2.6c)] on the motion of a charged particle. It will be assumed that these terms are so small that the radius of curvature of the magnetic field lines is very large compared to the cyclotron radius of the particle. Let us introduce a *local* coordinate system gliding along the magnetic field line with the particle's longitudinal velocity $\underset{\sim}{v}_{\parallel}$. Since this is not an inertial system because of the curvature of the field lines, a centrifugal force will be present in this noninertial system. This local coordinate system can be specified by the orthogonal set of unit vectors $\hat{\underset{\sim}{B}}$, $\hat{\underset{\sim}{n}}_1$ and $\hat{\underset{\sim}{n}}_2$, where $\hat{\underset{\sim}{B}}$ is along the field line, $\hat{\underset{\sim}{n}}_1$ is along the principal normal to the field line, and $\hat{\underset{\sim}{n}}_2$ is along the binormal to the curved magnetic field line, as indicated in Fig. 16.

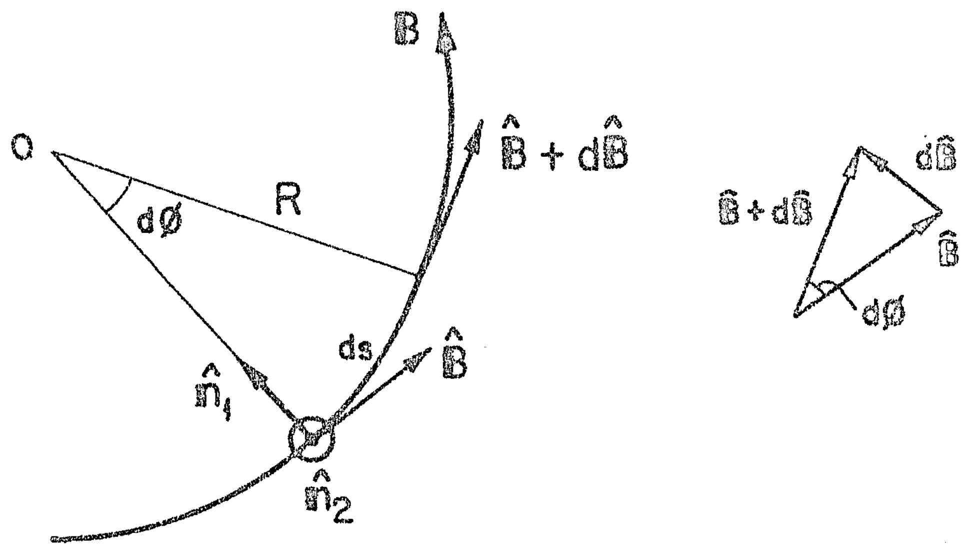

Fig. 16 - Curved magnetic field line showing the unit vector $\hat{\underset{\sim}{B}}$ along the field line, the principal normal $\hat{\underset{\sim}{n}}_1$, and the binormal $\hat{\underset{\sim}{n}}_2$, at an arbitrary point ($\hat{\underset{\sim}{n}}_1 \times \hat{\underset{\sim}{n}}_2 = \hat{\underset{\sim}{B}}$). The local radius of curvature is R.

The centrifugal force, $\underset{\sim}{F}_C$, acting on the particle as seen from this noninertial system, is given by

$$\underset{\sim}{F}_C = - \frac{m\, v_{\parallel}^2}{R}\, \underset{\sim}{\hat{n}}_1 \tag{7.1}$$

where R denotes the local radius of curvature of the magnetic field line and $v_{\parallel}$ is the instantaneous longitudinal speed of the particle of mass m. From (2.6.2), the curvature drift associated with this force is

$$\underset{\sim}{v}_C = \frac{\underset{\sim}{F}_C \times \underset{\sim}{B}}{q\, B^2} = - \frac{m\, v_{\parallel}^2}{R\, q\, B^2}\, (\underset{\sim}{\hat{n}}_1 \times \underset{\sim}{B}) \tag{7.2}$$

To express the unit vector $\underset{\sim}{\hat{n}}_1$ in terms of the unit vector $\underset{\sim}{\hat{B}}$ along the magnetic field line, we let ds represent an element of arc along the field line subtending an angle $d\phi$,

$$ds = R\, d\phi \tag{7.3}$$

If $d\underset{\sim}{\hat{B}}$ denotes the change in $\underset{\sim}{\hat{B}}$ due to the displacement ds (see Fig. 16), then $d\underset{\sim}{\hat{B}}$ is in the direction of $\underset{\sim}{\hat{n}}_1$ and its magnitude is

$$|\, d\underset{\sim}{\hat{B}}\, | = |\, \underset{\sim}{\hat{B}}\, |\, d\phi = d\phi \tag{7.4}$$

Consequently,

$$d\underset{\sim}{\hat{B}} = \underset{\sim}{\hat{n}}_1\, d\phi \tag{7.5}$$

Dividing this equation by (7.3) side by side, gives

$$\frac{d\hat{\underset{\sim}{B}}}{ds} = \frac{\hat{\underset{\sim}{n}}_1}{R} \tag{7.6}$$

The derivative d/ds along $\underset{\sim}{B}$ may be written as $(\hat{\underset{\sim}{B}} \cdot \underset{\sim}{\nabla})$, so that (7.6) becomes

$$\frac{\hat{\underset{\sim}{n}}_1}{R} = (\hat{\underset{\sim}{B}} \cdot \underset{\sim}{\nabla})\, \hat{\underset{\sim}{B}} \tag{7.7}$$

Incorporating this result into equation (7.1), we obtain

$$\underset{\sim}{F}_C = - m\, v_{\parallel}^2\, (\hat{\underset{\sim}{B}} \cdot \underset{\sim}{\nabla})\, \hat{\underset{\sim}{B}} \tag{7.8}$$

This force is obviously perpendicular to the magnetic field $\underset{\sim}{B}$, since it is in the $-\hat{\underset{\sim}{n}}_1$ direction [see (7.7)], and gives rise to a curvature drift whose velocity is

$$\underset{\sim}{v}_C = - \frac{m\, v_{\parallel}^2}{q\, B^2} \left[(\hat{\underset{\sim}{B}} \cdot \underset{\sim}{\nabla})\, \hat{\underset{\sim}{B}}\right] \times \underset{\sim}{B} \tag{7.9}$$

Since $\underset{\sim}{B} = B\hat{\underset{\sim}{B}}$ and writing $W_{\parallel} = m\, v_{\parallel}^2/2$ for the longitudinal kinetic energy of the particle, the expressions (7.8) and (7.9) for the centrifugal force and the curvature drift velocity can be written, respectively, as

$$\underset{\sim}{F}_C = -\frac{2W_\parallel}{B^2}\left[(\underset{\sim}{B}\cdot\underset{\sim}{\nabla})\,\underset{\sim}{B}\right]_\perp \tag{7.10}$$

$$\underset{\sim}{v}_C = -\frac{2W_\parallel}{qB^4}\left[(\underset{\sim}{B}\cdot\underset{\sim}{\nabla})\,\underset{\sim}{B}\right]\times\underset{\sim}{B} \tag{7.11}$$

Thus, at each point, the curvature drift is perpendicular to the osculating plane of the magnetic field line, as shown in Fig. 17. An electric current is associated with the curvature drift, since it is in opposite directions for particles of opposite sign.

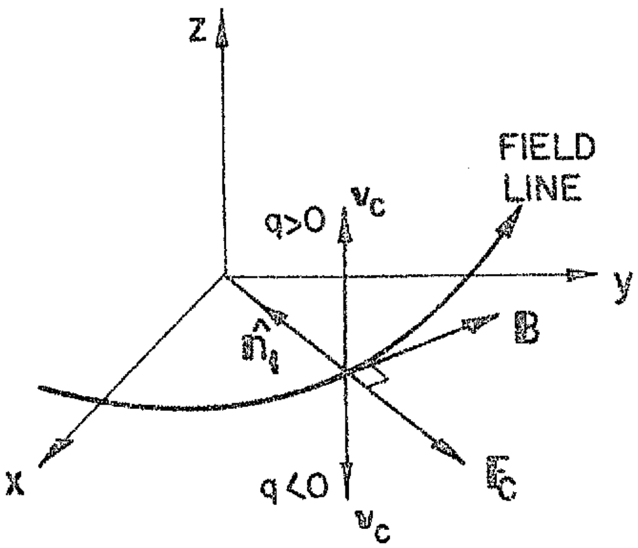

Fig. 17 - Relative direction of the particle drift velocity $\underset{\sim}{v}_c$, due to the curvature of the magnetic field line.

From (7.11) and the definition of the electric current density, we obtain for the curvature drift current density

$$\underset{\sim}{J}_C = -\,2\,\left(\frac{1}{\Delta V}\sum_i W_{i\parallel}\right)\left[(\underset{\sim}{\hat{B}}\cdot\underset{\sim}{\nabla})\,\underset{\sim}{\hat{B}}\right]\times\underset{\sim}{B}\,/\,B^2 \tag{7.12}$$

where the summation extends over all charged particles contained in the small volume element ΔV.

8. COMBINED GRADIENT - CURVATURE DRIFT

The curvature drift and the gradient drift always appear together and both point in the same direction, since the term $\underset{\sim}{\nabla} B$ points in the direction opposite to $\underset{\sim}{F}_C$ (see Fig. 5). These two drifts can therefore be added up to form the combined *gradient - curvature drift*. Thus, from (5.1) and (7.11), and noting that $|\underset{\sim}{m}| = W_\perp/B = m\,v_\perp^2/2B$,

$$\underset{\sim}{v}_{GC} = \underset{\sim}{v}_G + \underset{\sim}{v}_C = -\,\frac{m\,v_\perp^2}{2\,q\,B^3}\,(\underset{\sim}{\nabla} B)\times\underset{\sim}{B} - \frac{m\,v_\parallel^2}{q\,B^4}\left[(\underset{\sim}{B}\cdot\underset{\sim}{\nabla})\,\underset{\sim}{B}\right]\times\underset{\sim}{B} \tag{8.1}$$

When volume currents are not present (in a vacuum field, for example) so that $\underset{\sim}{\nabla}\times\underset{\sim}{B} = 0$, the vector identity (4.26) allows the expression (8.1) to be written in the compact form

$$\underset{\sim}{v}_{GC} = -\,\frac{m}{q\,B^4}\left(v_\parallel^2 + \frac{v_\perp^2}{2}\right)\left(\underset{\sim}{\nabla}\,\frac{B^2}{2}\right)\times\underset{\sim}{B} \tag{8.2}$$

In the Earth's magnetosphere near the equatorial plane both the curvature and the gradient drifts (B decreases with altitude) cause the positively charged particles to slowly drift westward and the negative ones eastward, resulting in an east to west current. This east to west current is known as the *ring current.* Fig. 18 illustrates schematically the motion of a charged particle trapped in the Earth's magnetic field. The particle bounces back and forth along the field line between the mirror points M_1 and M_2, and drifts in longitude as a result of the gradient and curvature of the field lines.

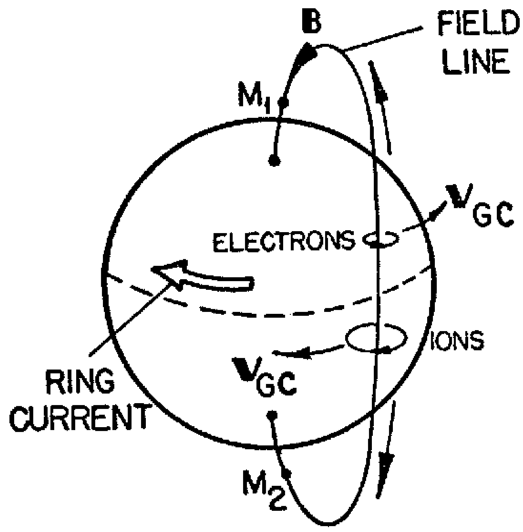

Fig. 18 - Sketch indicating the motion of a charged particle in the Earth's magnetic field. The longitudinal drift velocity $\underset{\sim}{v}_{GC}$, due to the gradient and curvature of $\underset{\sim}{B}$, results in an east to west current called the ring current.

The trajectory described by the particle is therefore contained in a tire-shaped shell encircling the Earth (Fig. 19).

This tire-shaped shell encircling the Earth defines a surface on which the particle guiding center drifts slowly around the Earth. Connected with the periodic motion of the particle on this drift surface there is an adiabatic invariant, called the *third adiabatic invariant,* which is the total magnetic flux Φ_m enclosed by the drift surface. Clearly, in a static situation this flux is obviously constant. The significant fact is that the total magnetic

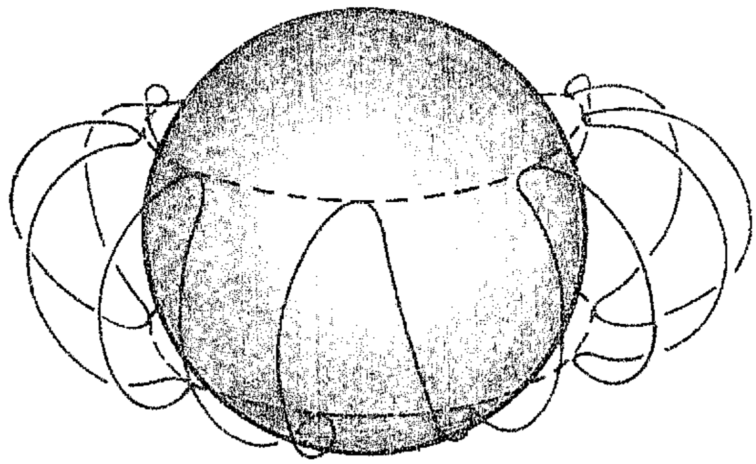

Fig. 19 - Schematic representation of the longitude drift of charged particles around the Earth.

flux Φ_m, enclosed by the drift surface, remains invariant when the field varies slowly in time, that is, when the period of motion of the particle on the drift surface is small compared with the time scale for the magnetic field to change significantly. This invariant has few applications because most fluctuations of $\underset{\sim}{B}$ occur on a time scale short compared with the drift period.

PROBLEMS

3.1 - Describe, in a semiquantitative way, the motion of an electron in the region near the origin under the presence of a constant electric field in the x-direction,

$$\underset{\sim}{E} = E_0 \hat{\underset{\sim}{x}}$$

and a space varying magnetic field given by

$$\underset{\sim}{B} = B_0 \{ \alpha(x+z) \hat{\underset{\sim}{x}} + [1 + \alpha(x-z)] \hat{\underset{\sim}{z}} \}$$

where E_0, B_0 and α are positive constants, $|\alpha x| \ll 1$ and $|\alpha z| \ll 1$. Assume that initially the electron moves with constant velocity in the z-direction, that is, $\underset{\sim}{v}(t=0) = v_0 \hat{\underset{\sim}{z}}$. Verify if this magnetic field satisfies Maxwell equations.

3.2 - Verify if there is any drift velocity for a charged particle in a magnetic field given by

$$\underset{\sim}{B} = B_y(x) \hat{\underset{\sim}{y}} + B_0 \hat{\underset{\sim}{z}}$$

where $B_y(x)$ and $\partial B_y/\partial x$ are very small quantities. Does this field satisfy Maxwell equations?

3.3 - Consider a system of two coaxial magnetic mirrors whose axis coincides with the z-axis, being symmetrical about the plane $z = 0$ as shown schematically in Fig. P3.1. Describe, in a semiquantitative manner, the motion of a charged particle in this magnetic mirror system considering that at $z = 0$ the particle has $v_{\parallel} = v_{\parallel}^{0}$ and $v_{\perp} = v_{\perp}^{0}$. What relation must exist between $\underset{\sim}{B}_0 = \hat{\underset{\sim}{z}}\, B(z=0)$, $\underset{\sim}{B}_m = \hat{\underset{\sim}{z}}\, B(z = \pm z_m)$ and α_0 (particle's pitch angle at $z = 0$) so that the particle be reflected at z_m?

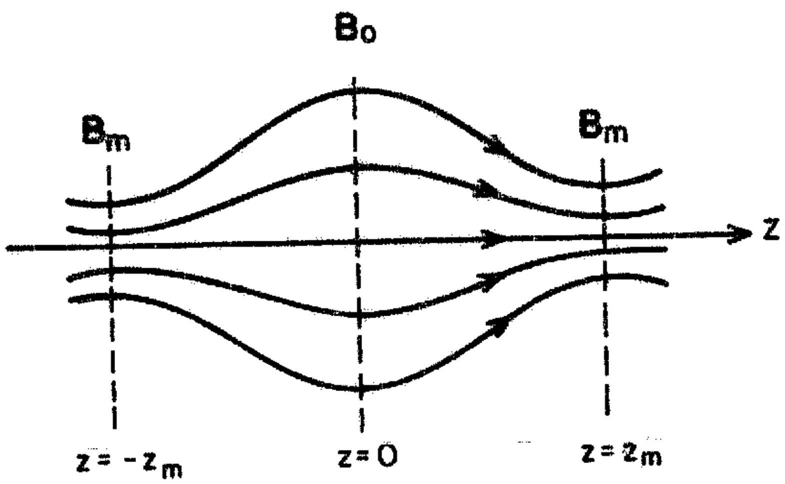

Fig. P3.1

3.4 - For the magnetic mirror system of Problem 3.3 suppose that the axial magnetic field changes in time, that is, $\underset{\sim}{B}_{axial} = \hat{\underset{\sim}{z}}\, B(z,t)$. Considering that the magnetic moment $|\underset{\sim}{m}| = m\, v_{\perp}^2(z,t)/2\, B(z,t)$ is an adiabatic invariant (note that its value is the same at $z = 0$ and $z = \pm z_m$, and that $v^2 = v_{\parallel}^2 + v_{\perp}^2$), show that the *longitudinal adiabatic invariant* can be written in the form

$$\int_{-z_m}^{z_m} \left[\, B(z_m,t) - B(z,t) \,\right]^{1/2} dz = \text{constant}$$

3.5 - Consider the magnetic mirror system shown in Fig. P3.1. Suppose that the axial magnetic field is given by

$$B(z) = B_0 \left[\, 1 + (z/a_0)^2 \,\right]$$

where B_0 and a_0 are positive constants, and that the mirroring planes are given by $z = -z_m$ and $z = z_m$.

(a) For a charged particle trapped in this mirror system, show that the component of the particle velocity along the z-axis is given by

$$v_{\parallel} = \left\{ \frac{2\,|\underset{\sim}{m}|\, B_0}{m} \left[\left(\frac{z_m}{a_0}\right)^2 - \left(\frac{z}{a_0}\right)^2 \right] \right\}^{1/2}$$

(b) The average force acting on the particle guiding center, along the z-axis, is given by

$$\langle F_{\parallel} \rangle = - |\underset{\sim}{m}| \frac{\partial B}{\partial z}$$

Show that the particle performs a simple harmonic motion between the mirroring planes, with a period given by

$$T = 2\pi a_0 \left(\frac{m}{2 |\underset{\sim}{m}| B_0} \right)^{1/2}$$

(c) If the motion of the particle is to be limited to the region $|z| < z_m$, what restriction must be imposed on the total energy and the magnetic moment?

3.6 - Consider a toroidal magnetic field, as shown in Fig. P3.2.

(a) Show that the magnetic flux density along the axis of the torus is given by

$$\underset{\sim}{B} = \underset{\sim}{\hat{\phi}} B_a \frac{a}{r}$$

where B_a denotes the magnitude of $\underset{\sim}{B}$ at the radial distance $r = a$.

(b) In what direction is the gradient drift associated whith the variation of B_ϕ in the radial direction? Examine qualitatively the type of charge separation that occurs. Neglect the effect of the curvature of the magnetic field lines.

(c) If $\underset{\sim}{E}$ denotes the induced electric field due to the charge separation, in what direction is the $\underset{\sim}{E}\times\underset{\sim}{B}$ drift?

(d) Show that it is not possible to confine a plasma in a purely toroidal magnetic field, because of the gradient drift and the $\underset{\sim}{E}\times\underset{\sim}{B}$ drift.

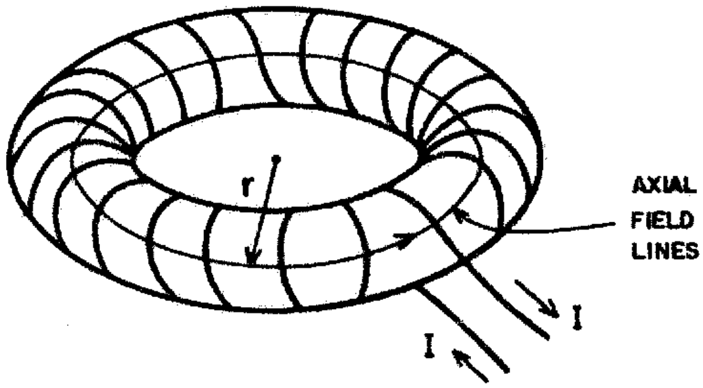

Fig. P3.2

3.7 - Consider a spatially nonuniform magnetostatic field expressed in terms of a Cartesian coordinate system by

$$\underset{\sim}{B}(x,z) = B_o \left[\alpha z \hat{\underset{\sim}{x}} + (1 + \alpha x) \hat{\underset{\sim}{z}} \right]$$

where B_o and α are positive constants, $|\alpha x| << 1$ and $|\alpha z| << 1$.

(a) Show that this magnetic field is consistent with Maxwell equations, so that both gradient and curvature terms are present. Determine the equation of a magnetic flux line.

(b) Write down the Cartesian components of the equation of motion for an electron moving in the region near the origin under the action of this magnetic field.

(c) Consider the following initial conditions for the electron:

$$\underset{\sim}{r}(0) = \hat{\underset{\sim}{x}}(x_o + v_{\perp o}/\omega_c)$$

$$\dot{\underset{\sim}{r}}(0) \equiv \underset{\sim}{v}(0) = \hat{\underset{\sim}{y}}\, v_{\perp o} + \hat{\underset{\sim}{z}}\, v_{zo}$$

Solve the equation of motion using a perturbation technique, retaining only terms up to the first order in the small parameter α. Show that the leading terms in the velocity components, after eliminating the time-periodic parts, are given by

$$\underset{\sim}{v}_x = \alpha v_{zo}^2 \ t \ \hat{\underset{\sim}{x}}$$

$$\underset{\sim}{v}_y = -\left(\frac{\alpha}{\omega_c}\right)\left(\frac{v_{\perp o}^2}{2} + v_{zo}^2\right) \hat{\underset{\sim}{y}}$$

$$\underset{\sim}{v}_z = v_{zo} \ \hat{\underset{\sim}{z}}$$

(d) Show that the average position of the electron in the x-z plane follows the magnetic flux line that passes through its initial position.

(e) Show that the gradient and curvature drift velocities are given, respectively, by

$$\underset{\sim}{v}_G = -\hat{\underset{\sim}{y}} \left(\frac{\alpha}{\omega_c}\right) \frac{v_{\perp o}^2}{2}$$

$$\underset{\sim}{v}_C = -\hat{\underset{\sim}{y}} \left(\frac{\alpha}{\omega_c}\right) v_{zo}^2$$

so that the total drift velocity is precisely the nonperiodic part of $\underset{\sim}{v}_y$.

3.8 - The Earth's magnetic field can be represented, in a first approximation, by a magnetic dipole placed in the Earth's center, at least up to distances of a few Earth radii (R_E).

(a) Using the fact that, at one of the magnetic poles, the field has a magnitude of approximately 0.5 Gauss, calculate the dipole magnetic moment.

(b) Consider the motion of an electron of energy E_0 at a radial distance r_0 ($r_0 > R_E$). Calculate its cyclotron frequency and gyroradius.

(c) Assuming that the electron is confined to move in the equatorial plane, calculate its gradient and curvature drift velocity, and determine the time it takes to drift once around the Earth at the radial distance r_0.

(d) Calculate the period of the bounce motion of the electron, as it gets reflected back and forth between the magnetic mirrors near the poles. What is the altitude of the reflection points? Assume that $W_\parallel = W_\perp$ at the magnetic equatorial plane.

(e) Recalculate (b), (c) and (d), considering $E_0 = 1$ MeV and $r_0 = 4\ R_E$. Examine these results in terms of typical values for charged particles in the outer Van Allen radiation belt.

(f) Assuming that there is an isotropic population of 1 MeV protons and 100 keV electrons at about 4 R_E, each having a density $n_e = n_i = 10^7\ m^{-3}$ in the equatorial plane, calculate the *ring current density* in Ampère/m².

3.9 - Consider an infinite straight wire carrying a current I and electrically charged to a negative potential Ø. Analyse the motion of an electron in the vicinity of this wire using first-order orbit theory. Sketch the path described by the electron, indicating the relative directions of the electromagnetic, gradient and curvature drift velocities.

3.10 - The field of a *magnetic monopole* can be represented by

$$\underset{\sim}{B}(\underset{\sim}{r}) = \lambda \frac{\underset{\sim}{r}}{r^3}$$

where λ is a constant. Solve the equation of motion to determine the trajectory of a charged particle in this field. (You may refer to B. Rossi and S. Olbert, *Introduction to the Physics of Space*, Chapter 2, section 2.5, pg. 29, Mc Graw-Hill, 1970).

3.11 - Analyse the motion of a charged particle in the field of a *magnetic dipole*. Determine the two constants of the motion and analyse their physical meaning. (For this problem, you may refer to C.Störmer, *The polar Aurora,* University Oxford Press, 1955, or to B. Rossi and S.Olbert, *Introduction to the Physics of Space,* Chapter 3, pg. 45, McGraw-Hill, 1970).