1.Classification *INPE-COM.4/RPE*
C.D.U.: 533.9

2.Period

4.Distribution Criterion

internal [ ]

external [X]

3.Key Words (selected by the author)

PINCH EFFECT
SAUSAGE INSTABILITY
KINK INSTABILITY

5.Report No.
INPE-1520-RPE/056

6.Date
July, 1979

7.Revised by
Y.Sahai

8. Title and Sub-title

THE PINCH EFFECT

9.Authorized by
Nelson de Jesus Parada
Director

10.Sector *DCE/DGA/GIO* Code *30.371*

11.No. of Copies *13*

12.Authorship *J.A.Bittencourt*

14.No. of Pages *46*

15.Price

13.Signature of first author Bittencourt

16.Summary/Notes

This is the thirteenth chapter, in a series of twenty two, written as an introduction to the fundamentals of plasma physics. It presents a detailed treatment of the pinch effect for the special case in which the confinement is produced by an azimuthal (Θ) self-magnetic field, due to an axial current in the pinched plasma column. A detailed analysis is given for the equilibrium pinch, the Bennett pinch and the dynamic pinch. The geometrical instabilities which are commonly present in a pinched plasma column, known as the sausage instability and the kink instability, are discussed. The addition of stabilizing fields to the pinched plasma column, and some convex field configurations, which provide a stable magnetic field confinement, are also described.

17.Remarks

INDEX

CHAPTER 13

THE PINCH EFFECT

LIST OF FIGURES

CHAPTER 13

THE PINCH EFFECT

1. INTRODUCTION

In view of the the importance of plasma confinement by a magnetic field in controlled thermonuclear research, as well as in other applications, we present in this chapter a detailed treatment of plasma confinement for the special case in which the confinement is produced by an azimuthal (Θ) self-magnetic field, due to an axial current in the plasma generated by an appropriately applied electric field.

Consider an infinite cylindrical column of conducting fluid with an axial current density $\underset{\sim}{J} = J_z(r)\ \hat{\underset{\sim}{z}}$ and a resulting azimuthal magnetic induction $\underset{\sim}{B} = B_\Theta(r)\ \hat{\underset{\sim}{\Theta}}$, as depicted in Fig.1. The $\underset{\sim}{J} \times \underset{\sim}{B}$ force, acting on the plasma, forces the plasma column to contract laterally. This lateral constriction of the plasma column is known as the *pinch effect*. The isobaric surfaces, for which p = constant, are, in this case, concentric cylinders.

As the plasma is compressed laterally, the number density and the temperature of the plasma increase. The plasma kinetic pressure counteracts to hinder the constriction of the plasma column, whereas the magnetic forces act to confine the plasma. When these counteracting forces are balanced, a steady state condition results, in which the material is mainly confined within a certain radius R, which remains constant in time. This situation is commonly referred to as the *equilibrium pinch*. When the self-magnetic pressure exceeds the plasma kinetic pressure, the radius of the plasma column changes with time resulting in a situation known as the *dynamic pinch*. In what follows we investigate first the equilibrium pinch and afterwards the dynamic pinch.

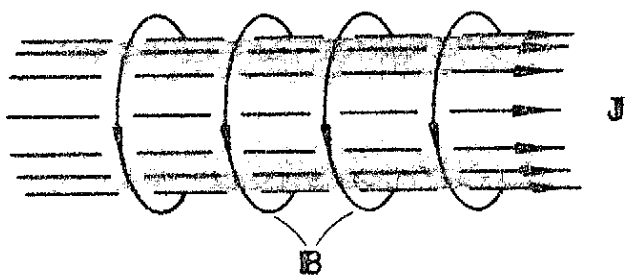

Fig. 1 - Pinch configuration in which a magnetoplasma is confined by azimuthal magnetic fields generated by axial currents flowing along the plasma column.

2. THE EQUILIBRIUM PINCH

For simplicity, the current density, the magnetic field and the plasma kinetic pressure are assumed to depend only on the distance from the cylinder axis. For steady state conditions, none of the variables change with time. The various parameters of the equilibrium pinch are schematically shown in Fig.2. Since the system is cylindrically symmetric, only the radial component of Eq. (12.5.1) must be considered,

$$\frac{dp(r)}{dr} = - J_z(r)\ B_\Theta(r) \tag{2.1}$$

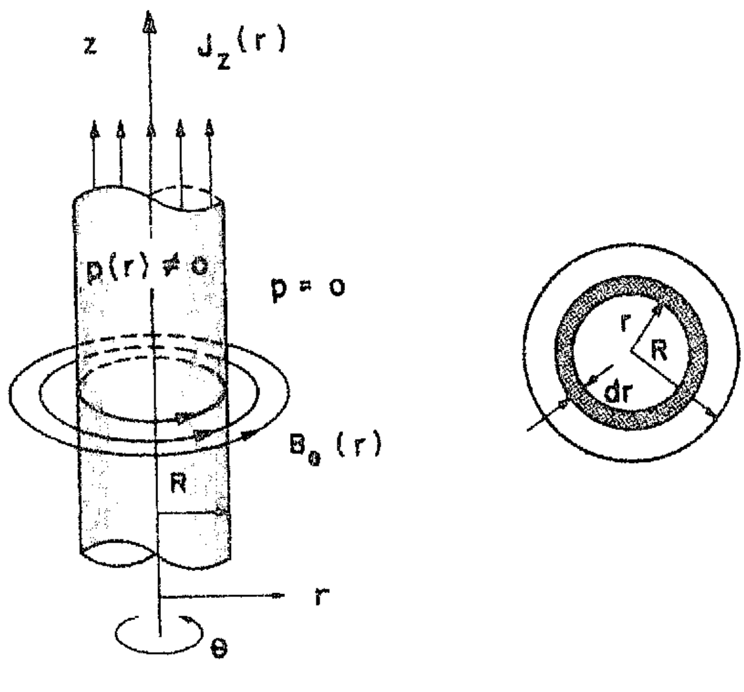

Fig. 2 - Schematic diagram illustrating the various parameters relevant to the study of the equilibrium longitudinal pinch configuration.

Inside a cylinder of general radius r, the total enclosed current, $I_z(r)$, is

$$I_z(r) = \int_0^r J_z(r) \quad 2 \pi r \quad dr \tag{2.2}$$

Note that the variable r inside the integrand is a dummy variable. From (2.2) we obtain

$$\frac{dI_z(r)}{dr} = 2 \pi r J_z(r) \tag{2.3}$$

Ampēre's law in integral form relates $B_\Theta(r)$ to the total enclosed current, giving for for the magnetic induction

$$B_\Theta(r) = \frac{\mu_o}{2\pi r} \quad I_z(r)$$

$$= \frac{\mu_o}{r} \int_0^r J_z(r) \quad r \quad dr \tag{2.4}$$

A number of results can be obtained even without specifying the precise form of $J_z(r)$. If the conducting fluid lies almost entirely inside $r = R$, then the magnetic induction $B_\Theta(r)$ outside the plasma is

$$B_\Theta(r) = \frac{\mu_o \ I_o}{2\pi r} \qquad (r \geq R) \tag{2.5}$$

where

$$I_o = \int_0^R J_z(r)\ 2\pi r\ dr = I_z(R) \tag{2.6}$$

which is the total current flowing inside the cylindrical plasma column. The substitution of $B_\Theta(r)$ and $J_z(r)$, from Eqs. (2.4) and (2.3), respectively, into Eq. (2.1), gives

$$\frac{dp(r)}{dr} = -\frac{\mu_o}{4\pi^2 r^2}\ I_z(r)\ \frac{d\ I_z(r)}{dr} \tag{2.7}$$

which can be written as

$$4\pi^2 r^2\ \frac{dp(r)}{dr} = -\frac{d}{dr}\left[\frac{\mu_o}{2}\ I_z^2(r)\right] \tag{2.8}$$

If we now integrate this equation, from r = 0 to r = R, and simplify the left-hand side by an integration by parts, we obtain

$$\left(4\pi^2 r^2\ p(r)\ \Big|_0^R\right) - 4\pi \int_0^R 2\pi r\ p(r)\ dr = -\frac{\mu_o}{2}\ I_o^2 \tag{2.9}$$

where $I_o = I_z(R)$ is the total current flowing through the entire cross section of the plasma column and, obviously, $I_z(0) = 0$ by Eq. (2.2). Considering the plasma column to be confined to the range $0 \leq r < R$, it follows that p(r) is zero for $r \geq R$, and finite for $0 \leq r < R$, so that the first term in the left-hand side of Eq.(2.9) vanishes. Therefore, we find that

$$I_o^2 = \frac{8\pi}{\mu_o} \int_0^R 2\pi r \, p(r) \, dr \qquad (2.10)$$

If the partial pressures of the electrons and the ions are governed by the ideal gas law,

$$p_e(r) = n(r) \, k \, T_e \qquad (2.11)$$

$$p_i(r) = n(r) \, k \, T_i \qquad (2.12)$$

assuming that the electron and ion temperatures, T_e and T_i, respectively, are constants throughout the column, we have

$$p(r) = p_e(r) + p_i(r) = n(r) \, k \, (T_e + T_i) \qquad (2.13)$$

Eq. (2.10) becomes, therefore,

$$I_o^2 = \frac{8\pi}{\mu_o} \, k \, (T_e + T_i) \int_0^R 2\pi r \, n(r) \, dr \qquad (2.14)$$

or

$$I_o^2 = \frac{8\pi}{\mu_o} \, k \, (T_e + T_i) \, N_\ell \qquad (2.15)$$

where

$$N_\ell = \int_0^R 2\pi r \; n(r) \; dr \tag{2.16}$$

is the number of particles per unit length of the cylindrical plasma column.

Eq.(2.15) is known as the *Bennett relation.* It gives the total current that must be discharged through the plasma column in order to confine a plasma at a specified temperature and a given number of particles, N_ℓ, per unit lenght of the plasma column. The current required for the confinement of hot plasmas is usually very large. As an example, suppose that $N_\ell = 10^{19}$ particles per meter and that the plasma temperature is such that $T_e + T_i = 10^8$ ^{0}K. Since $\mu_o = 4\pi \times 10^{-7}$ H/m and $k = 1.38 \times 10^{-23}$ J/^{0}K, it follows that the required current I_0 is of the order of one million Ampēres.

To obtain the radial distribution of p(r) in terms of $B_\Theta(r)$, it is convenient to start from Eq.(2.1) and proceed in a different way. First, we note that from Maxwell curl equation $\underset{\sim}{\nabla} \times \underset{\sim}{B}(\underset{\sim}{r}) = \mu_o \underset{\sim}{J}(\underset{\sim}{r})$ we have, in cylindrical coordinates, with only a radial dependence,

$$\frac{1}{r} \frac{d}{dr} \left[r \, B_\Theta(r) \right] = \mu_o \, J_z(r) \tag{2.17}$$

from which we get

$$J_z(r) = \frac{1}{\mu_o}\frac{d}{dr}B_\Theta(r) + \frac{1}{\mu_o}\frac{B_\Theta(r)}{r} \tag{2.18}$$

Substitution of this result for $J_z(r)$ into Eq.(2.1), yields

$$\frac{dp(r)}{dr} = -\frac{B_\Theta(r)}{\mu_o}\frac{d}{dr}\; B_\Theta(r) - \frac{B_\Theta^2(r)}{\mu_o r}$$

$$= -\frac{1}{2\mu_o r^2}\frac{d}{dr}\left[r^2 B_\Theta^2(r)\right] \tag{2.19}$$

We now integrate this equation from r=0 to a general radius r,

$$p(r) = p(0) - \frac{1}{2\mu_o}\int_0^r \frac{1}{r^2}\frac{d}{dr}\left[r^2\; B_\Theta^2(r)\right]dr \tag{2.20}$$

In particular, since for r=R we have p(R) = 0, we obtain

$$p(0) = \frac{1}{2\mu_o}\int_0^R \frac{1}{r^2}\frac{d}{dr}\left[r^2 B_\Theta^2(r)\right]dr \tag{2.21}$$

and substituting this result into Eq. (2.20),

$$p(r) = \frac{1}{2\mu_o}\int_r^R \frac{1}{r^2}\;\frac{d}{dr}\left[r^2 B_\Theta^2(r)\right]\; dr \tag{2.22}$$

The *average* pressure inside the cylinder can be related to the total current I_o and the column radius R without knowing the detailed radial dependence. The average value of the kinetic pressure inside the column is defined by

$$\bar{p} = \frac{1}{\pi R^2} \int_0^R 2\pi r\, p(r)\, dr \qquad (2.23)$$

Simplifying this expression by an integration by parts, yields

$$\bar{p} = \frac{1}{R^2} \left[r^2\, p(r) \Big|_0^R - \int_0^R r^2 \frac{d}{dr}\, p(r)\, dr \right]$$

$$= - \frac{1}{R^2} \int_0^R r^2 \frac{d}{dr}\, p(r)\, dr \qquad (2.24)$$

since the integrated term is zero, because p(R) = 0. Replacing dp(r)/dr, using Eq. (2.19), we get

$$\bar{p} = \frac{B_\Theta^2(R)}{2\mu_o} = \frac{\mu_o\, I_o^2}{8\pi^2\, R^2} \qquad (2.25)$$

This result shows that the average kinetic pressure in the equilibrium plasma column is balanced by the magnetic pressure at the boundary of the column.

From Eqs. (2.2), (2.4) and (2.22), we can deduce the radial distribution of $I_z(r)$, $B_\Theta(r)$, and $p(r)$ if we know the radial dependence of $J_z(r)$. So far, the radial dependence of $J_z(r)$ has not been discussed. In what follows, we will consider two simple possibilities, in order to illustrate the use of the above-mentioned equations.

As a simple example consider the case in which the current density, $J_z(r)$, is constant for $r < R$. Taking $J_z = I_o/\pi R^2$ in Eq. (2.4), we obtain for $r < R$,

$$B_\Theta(r) = \frac{\mu_o I_o}{\pi R^2 r} \int_0^r r\,dr$$

$$= \frac{\mu_o I_o}{2\pi R^2} r \qquad (r < R) \tag{2.26}$$

If we substitute this result into Eq. (2.22), we obtain a parabolic dependence for the pressure versus radius,

$$p(r) = \frac{1}{2\mu_o} \int_r^R \frac{1}{r^2} \frac{d}{dr} \left(\frac{\mu_o^2 I_o^2 r^4}{4\pi^2 R^4} \right) dr$$

$$= \frac{\mu_o I_o^2}{4\pi^2 R^2} \left(1 - \frac{r^2}{R^2} \right) \tag{2.27}$$

Note that, in this case, the axial pressure $p(r = 0)$ is twice the average pressure, $\bar{p}$, given in Eq. (2.25). The radial deoendence of the various quantities in shown in Fig. 3

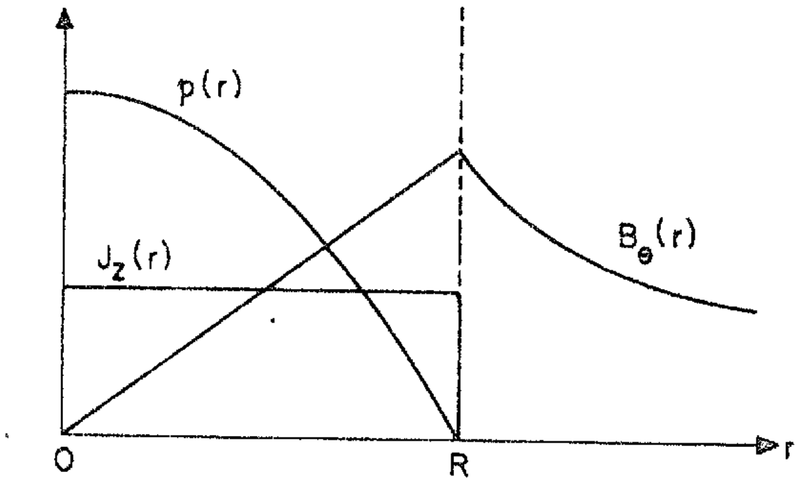

Fig.3 - Radial dependence of the azimuthal magnetic induction $B_\Theta(r)$ and plasma pressure $p(r)$ in a cylindrical plasma column with a constant current density $J_z(r)$. The radius of the column is R.

Another radial distribution of $J_z(r)$, which is also of interest in the investigation of the dynamic pinch, is the one in which the current density is confined to a very thin layer on the surface of the column. This model is appropriate for a highly conducting fluid or plasma. In a perfectly conducting plasma, the current cannot penetrate the plasma and exists only on the surface of the column. In this case, there is no magnetic field inside the column and $B_\Theta(r)$ exists only for $r > R$. From Eq. (2.5) the magnetic induction is given by

$$B_\Theta(r) = \frac{\mu_o I_o}{2\pi r} \qquad (r > R) \qquad (2.28)$$

where I_o is the total axial current. Therefore, from Eq. (2.20), we have

$$p(r) = p(0) \qquad (0 < r < R) \qquad (2.29)$$

so that the plasma kinetic pressure is constant inside the cylindrical column and equal to the average value given in Eq.(2.25)

$$p = \frac{\mu_o I_o^2}{8\pi^2 R^2} \qquad (0 < r < R) \qquad (2.30)$$

The radial dependence of the various quantities, for this model, is sketched in Fig. 4. Thus, for a perfectly conducting plasma column, the magnetic induction vanishes inside the column and falls off as 1/r outside the column. The plasma kinetic pressure is constant inside the column and vanishes outside it. The pinch effect, in this special case, can be thought of as due to an abrupt build up of the magnetic pressure $B_\Theta^2 / 2\mu_o$ in the region external to the column.

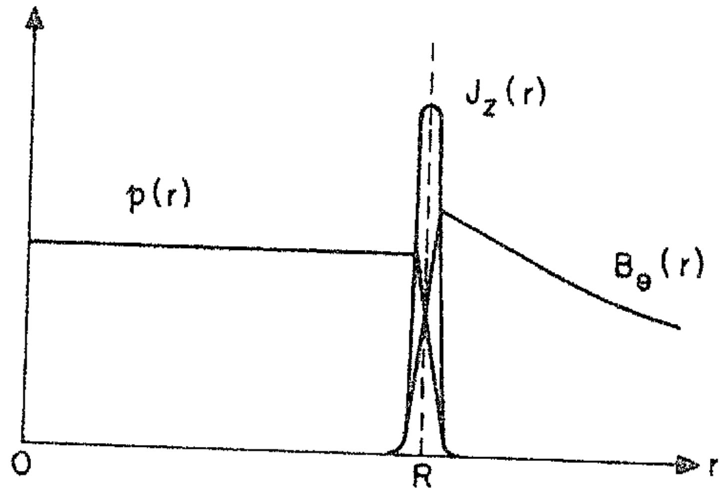

Fig. 4 - Radial dependence of the azimuthal magnetic induction, $B_\Theta(r)$, and plasma pressure, p(r), in a cylindrical plasma column with a current restricted to a very thin layer on the surface of the column.

3. THE BENNETT PINCH

W. H. Bennett, the discover of the pinch effect, has investigated a special model of the equilibrium longitudinal pinch, in which the radial distribution of the various quantitites corresponds to a situation in which the drift velocity of the plasma particles is constant throughout the cross section of the cylindrical plasma column.As an instructive application of the previous equations for the equilibrium pinch configuration, we investigate this particular model in what follows. In view of the fact that the ion mass is much larger than the electron mass, the drift velocity of the ions is much smaller than that of the electrons, since they are produced by the same magnetic field, and can, therefore, be neglected. Thus, we take the current density to be given by

$$\underset{\sim}{J}(r) = - e\, n(r)\, \underset{\sim}{u}_e \tag{3.1}$$

Since the applied electric field is in the z - direction, we have $\underset{\sim}{J}(r) = J_z(r)\, \hat{\underset{\sim}{z}}$ and $\underset{\sim}{u}_e = - u_{ez}\, \hat{\underset{\sim}{z}}$, where u_{ez} is positive and constant, independent of r. Therefore,

$$J_z(r) = e\, n(r)\, u_{ez} \tag{3.2}$$

Substitution of this equation for $J_z(r)$, and Eq.(2.13) for p(r),into the hydrostatic equation of motion (2.1), yields

$$k\ (T_e + T_i)\ \frac{dn(r)}{dr} = - e\, n(r)\, u_{ez}\, B_\Theta(r) \tag{3.3}$$

If we multiply this equation by $r/\left[n(r)\ k\ (T_e + T_i)\right]$ and differentiate it with respect to r, we obtain

$$\frac{d}{dr}\left[\frac{r}{n(r)}\ \frac{dn(r)}{dr}\right] = - \frac{e\, u_{ez}}{k\ (T_e+T_i)}\ \frac{d}{dr}\left[r\, B_\Theta(r)\right] \tag{3.4}$$

From Eqs. (2.17) and (3.2), we have

$$\frac{d}{dr}\left[r\, B_\Theta(r) \right] = \mu_o\, e\, u_{ez}\, r\, n(r) \tag{3.5}$$

and using this result in Eq. (3.4), it becomes

$$\frac{d}{dr}\left[\frac{r}{n(r)} \frac{dn(r)}{dr} \right] + \left[\frac{\mu_o\, e^2\, u_{ez}^2}{k\,(T_e+T_i)} \right] r\, n(r) = 0 \tag{3.6}$$

The solution of this nonlinear differential equation gives the radial dependence of the number density, n(r). Bennett has obtained the solution of this nonlinear equation subject to the boundary contition that n(r) is symmetric about the z-axis, where r = 0, and is a smoothly varying function of r, so that

$$\left.\frac{dn(r)}{dr}\right|_{r=0} = 0 \tag{3.7}$$

The solution of Eq. (3.6), subjected to the boundary condition (3.7), is known as the *Bennett distribution*, and is given by

$$n(r) = \frac{n_o}{(1+n_o\, br^2)^2} \tag{3.8}$$

where $n_o = n(r=0)$, which is the number density on the axis, and

$$b = \frac{\mu_o\, e^2\, u_{ez}^2}{8\, k(T_e+T_i)} \tag{3.9}$$

which has dimensions of length. This radial dependence of the number density is sketched in Fig. 5. From Eqs. (3.2) and (2.13) we see that the radial dependence of $J_z(r)$ and $p(r)$ is the same as that of $n(r)$. It can be used to determine $B_\Theta(r)$ according to Eq. (2.4).

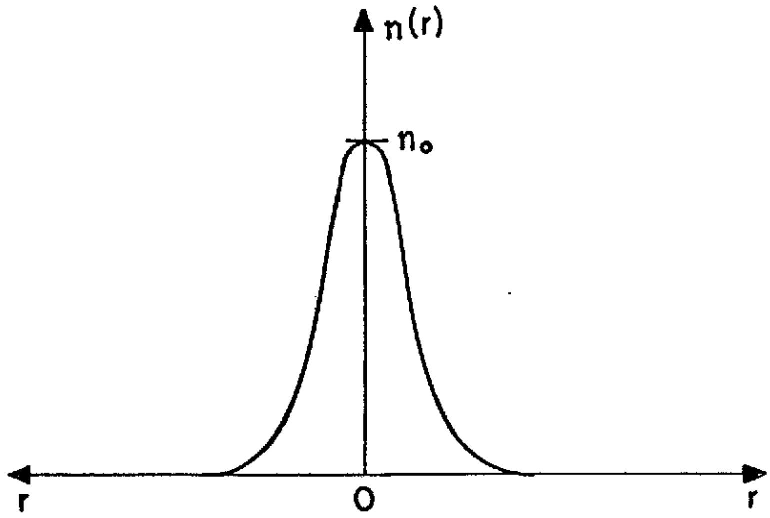

Fig. 5 - The Bennett distribution of the number density, $n(r)$,of the particles in an equilibrium pinched plasma column.

The Bennett distribution (3.8) shows that particles are present up to infinity but, since $n(r)$ falls off very rapidly with increasing values of r, we can consider, for all practical purposes, that the plasma is essentially concentrated symmetrically in a small cylindrical region about the z-axis. Using Eq. (3.8) we can obtain the number of particles $N_\ell(R)$, per unit lenght, contained in a cylindrical column of radius R,

$$N_\ell(R) = \int_0^R n(r)\ 2\pi r\ dr$$

$$= 2\pi n_o \int_0^R \frac{r}{(1+n_o br^2)^2} dr \qquad (3.10)$$

Evaluating the integral yields

$$N_\ell(R) = \frac{n_o \pi R^2}{(1+n_o bR^2)} \qquad (3.11)$$

Since particles are present up to infinity, the *total* number of particles per unit length can be obtained from Eq. (3.11) by taking the limit as $R \to \infty$, which gives

$$N_\ell(\infty) = \frac{\pi}{b} \qquad (3.12)$$

If we let α denote the fraction of the number of particles per unit length that is contained in a cylinder of radius R, that is,

$$\alpha = \frac{N_\ell(R)}{N_\ell(\infty)} = \frac{bN_\ell(R)}{\pi} \qquad (3.13)$$

and use Eq. (3.11), we obtain, after some rearrangement,

$$R(n_0 b)^{1/2} = \left(\frac{\alpha}{1-\alpha} \right)^{1/2} \tag{3.14}$$

Therefore, if 90% of the plasma particles are confined within the cylindrical plasma column of radius R, that is $\alpha = 0.9$, we have

$$R(n_0 b)^{1/2} = 3 \tag{3.15}$$

Thus, even though the particles extend up to infinity, the major portion of them lies in a small neighborhood around the z-axis. Note that, since $(n_0 b)^{1/2}$ has dimensions of an inverse length, we can think of $R(n_0 b)^{1/2}$ as a normalized radius of the cylindrical plasma column. It is convenient to assume, arbitrarily, that a plasma is confined within a cylindrical surface of radius R if 90% of the particles are within this cylindrical column. Therefore, the radius R of the cylindrical surface, within which the plasma is confined, is given by (3.15).

4. DYNAMIC MODEL OF THE PINCH

The simple theory of the equilibrium pinch, considered in section 2, is valid when the radius of the plasma column is constant in time or when it is varying very slowly compared with the time required for the plasma to attain a constant temperature. In actual practice, however, static or quasi-static situations do not arise and it is necessary to consider the dynamical behavior of the pinch effect. Initially, when the current starts flowing down the plasma column, the kinetic pressure is generally too small to resist the force due to the external magnetic pressure, so that the radius of the cylinder of plasma is forced inwards and the plasma column is pinched. The essential dynamic features of the time-varying pinch are illustred by the following simple model.

Suppose that a fully ionized plasma fills the interior region ($0 < r < R_0$) of a hollow dielectric cylinder of radius R_0 and length L. A voltage difference V is applied between the ends of the cylinder, so that a current I flows in the plasma. This current produces an azimuthal magnetic induction $B_\Theta(r)$, which causes the plasma to pinch inwards. The plasma is assumed to be perfectly conducting, so that all the current flows on the surface and there is no magnetic flux inside the plasma. Also, the plasma kinetic pressure is neglected. Let R(t) be the radius of the plasma column at time t (Fig. 6). The magnitude of the azimuthal magnetic induction just adjacent to the current sheath at radius R(t), is given by

$$B_\Theta(R) = \frac{\mu_0 \; I(t)}{2\pi R} \qquad (4.1)$$

where I(t) is the total axial current at the instant t. In particular, for t = 0, we have $R = R_0$, and this equation gives the inital value $B_\theta(R_o)$ of the magnetic induction.

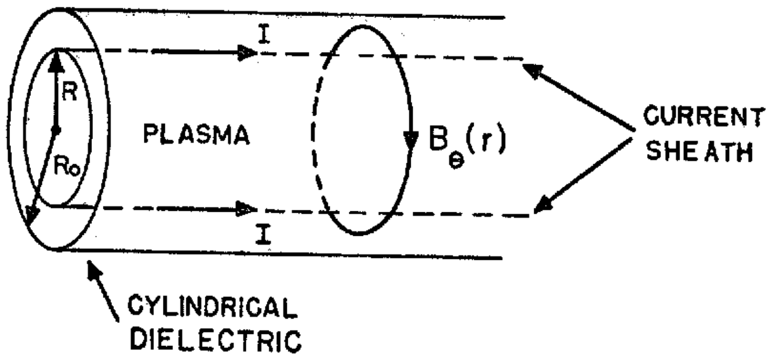

Fig. 6 - Plasma column of infinite conductivity, inside a hollow cylindrical dielectric, with a current sheath on its surface.

The magnetic pressure, $p_m(R)$, produced by this magnetic induction, acting on the current sheath radially inwards, is given by

$$p_m(R) = B_\theta^2(R)/2\mu_o$$

$$= \frac{\mu_o I^2(t)}{8\pi^2 R^2} \tag{4.2}$$

The force per unit length of the current sheath, acting radially inwards, is obtained from Eq. (4.2) as

$$\underset{\sim}{F}(R) = \hat{\underset{\sim}{r}}\, F(R) = -\, 2\pi R\, p_m(R)\, \hat{\underset{\sim}{r}}$$

$$= -\,\frac{\mu_o\, I^2(t)}{4\pi R}\, \hat{\underset{\sim}{r}} \tag{4.3}$$

To set up the equation of motion, relating $I(t)$ to the instantaneous radius $R(t)$ of the pinch discharge, we must make some assumption about the plasma. We shall consider a model, known as the *snowplow model*, in which the current sheath is imagined to carry along with it all the material which it hits as it moves inward. If ρ is the original mass density of the plasma, then the mass per unit length carried by the interface as it moves in, at time t, when the radius of the current sheath is R, is given by

$$M(R) = \pi(R_0^2 - R^2)\rho \tag{4.4}$$

Fig. 7. illustrates the cross sectional area swept by the current sheath as it moves from the radius R_0 to $R(t)$.

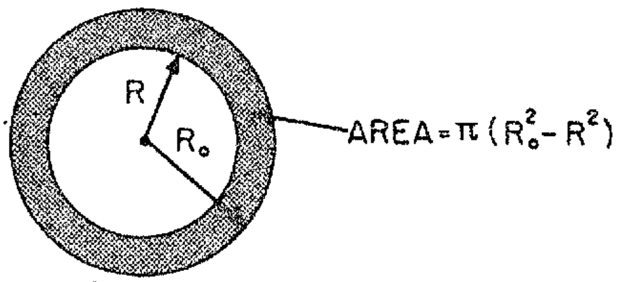

Fig. 7 - Area swept by the current sheath as it moves inward from the radius R_0 to $R(t)$.

From Newton's second law, the magnetic pressure force and the rate of change of momentum are related by

$$\frac{d}{dt}\left[M(R)\ \frac{dR}{dt} \right] = F(R)$$

or

$$\frac{d}{dt}\left[\pi\ \rho\ (R_o^2 - R^2)\ \frac{dR}{dt} \right] = -\ \frac{\mu_o\ I^2(t)}{4\pi R} \tag{4.5}$$

where we used Eqs.(4.3) and (4.4). If the functional dependence of the pinch current I(t) is known, Eq.(4.5) permits the evaluation of the radius of the pinch discharge as a function of time.

A standard inductive relation between the applied voltage, the current and the dimensions (inductance) of the plasma column can be obtained using Faraday's law of induction. For this purpose, consider the closed loop shown in Fig. 8, in which the inner arm lies on the interface and moves inward with it. Applying Faraday's law to this dotted loop,

$$\oint \underset{\sim}{E} \cdot d\underset{\sim}{\ell} = -\ \frac{d}{dt}\ \left(\int_S \underset{\sim}{B} \cdot d\ \underset{\sim}{S} \right) \tag{4.6}$$

and noting that the only contribution to the line integral of $\underset{\sim}{E}$ comes from the side of the loop lying in the conducting wall, we obtain

$$- \frac{V}{L} = - \frac{d}{dt} \int_{R(t)}^{R_o} B_\Theta(r)\, dr \tag{4.7}$$

Using Eq. (4.1), and performing the integral, yields

$$\frac{V}{L} = \frac{\mu_o}{2\pi} \frac{d}{dt} \left[I(t)\, \ell n \left(\frac{R_o}{R(t)} \right) \right] \tag{4.8}$$

If we denote the applied electric field, V/L, by $E_o\, f(t)$, where the function f(t) is assumed known and is normalized so that the peak value of the applied electric field is E_o, Eq. (4.8) becomes

$$I(t)\, \ell n \left(\frac{R_o}{R(t)} \right) = \frac{2\pi}{\mu_o} E_o \int_0^t f(t')\, dt' \tag{4.9}$$

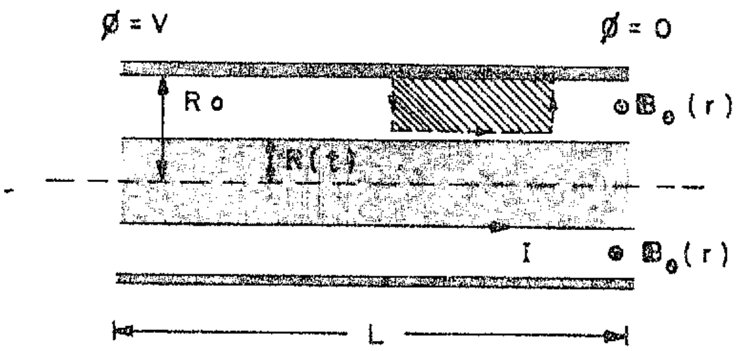

Fig. 8 - Sheamatic representation of the closed loop, for application of Faraday's law, with the inner side lying on the interface and moving inwards with it.

This equation can be used to eliminate I(t) from the equation of motion (4.5), resulting in the following equation for the rate of change of R(t)

$$\frac{d}{dt}\left[(R_0^2 - R^2)\frac{dR}{dt} \right] = - \frac{E_0^2 \left[\int_0^t f(t')\, dt' \right]^2}{\mu_0\, \rho\, R \left[\ell n\, (R_0/R) \right]^2} \tag{4.10}$$

It is convenient to introduce the following dimensionless variables

$$x = R/R_0 \tag{4.11}$$

$$\tau = \left(\frac{E_0^2}{\mu_0\, \rho\, R_0^4} \right)^{1/4} t \tag{4.12}$$

and recast Eq. (4.10), in normalized form, as

$$\frac{d}{d\tau}\left[(1-x^2)\, \frac{dx}{d\tau} \right] = - \frac{\left[\int_0^\tau f(\tau')\, d\tau' \right]^2}{x\, (\ell n\, x)^2} \tag{4.13}$$

This equation cannot be solved without knowing the function f(t). However, some idea of the results can be obtained, without solving this equation, by noting that x changes significantly for time periods such that $\tau = 1$. Thus, from Eq. (4.12), the scaling law for the radial velocity of the pinch is, approximately,

$$\left| \frac{dR}{dt} \right| \simeq v_0 = \left(\frac{E_0^2}{\mu_0 \rho} \right)^{1/4} \tag{4.14}$$

The typical experimental conditions involved in a small scale pinch column of hydrogen or deuterium plasma are initial densities of the order of 10^{-8} gm/cm^3 and applied electric fields of the order of 10^3 Volts/cm, which give a velocity v_0 of the order of 10^7 cm/sec. For these conditions, in a tube of 10 cm radius, the current measured is of the order of 10^5 or 10^6 Ampères.

It is instructive to consider a particular case in which the pinch current varies in time according to

$$I(t) = I_0 \sin(\omega t) \simeq I_0 \,\omega t \tag{4.15}$$

Then, from Eq. (4.5), we obtain directly

$$\frac{d}{d\tau} \left[(1 - x^2) \frac{dx}{d\tau} \right] = - \frac{\tau^2}{x} \tag{4.16}$$

with x as given by Eq. (4.11), and

$$\tau = \left(\frac{\mu_0 \, I_0^2 \, \omega^2}{4\pi^2 \, \rho \, R^4} \right)^{1/4} t \tag{4.17}$$

Equation (4.16) has to be solved numerically to determine $x(\tau)$. The resulting relation between the normalized radius of the dynamic pinch and the normalized time is sketched in Fig. 9. This simplified model indicates that the radius of the plasma column goes to zero in a time slightly greater than τ. This is a consequence of neglecting the kinetic pressure of the plasma. The above discussion is, therefore, valid only for very short times after the onset of the current flow.

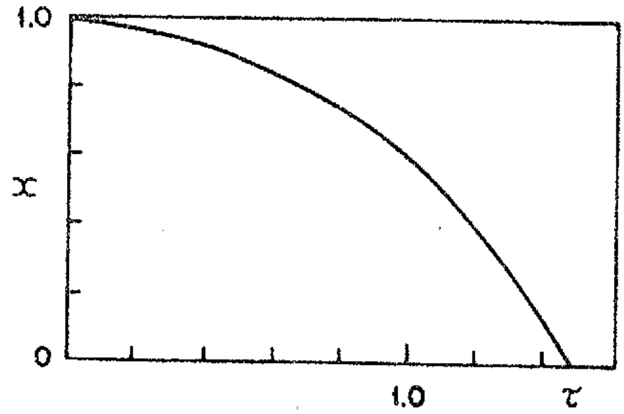

Fig. 9 - Normalized radius $x = R/R_0$ of the dynamic pinch column as a function of the normalized time τ, according to Eq.(4.16).

An important phenomenon that usually occurs in the dynamic pinch has not been considered in this analysis. As the current sheath moves radially inwards, compressing the plasma, the behavior just discussed is modified. A radial wave motion is usually set up by the pinch and this wave travels faster than the current sheath. These waves, travelling inwards, get reflected off the axis and move outwards striking the interface and retarding the inward motion of the current sheath or even reversing it. This phenomenon is known as *bouncing*. This sequence of events takes place periodically and the amplitude of each succeeding bounce becomes smaller, and the radius of the plasma column presumably reaches an equilibrium state at some radius less than R_0. Fig. 10 illustrates the general behavior expected for the radius R of the column as a function of time.

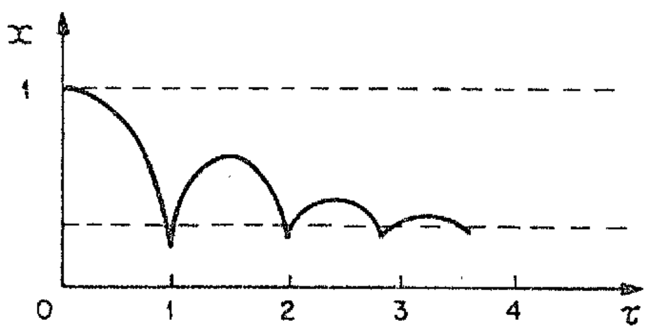

Fig. 10 - Normalized radius of the plasma column as a function of the normalized time, illustrating the phenomenon of bouncing.

5. INSTABILITIES IN A PINCHED PLASMA COLUMN

Although it is possible to achieve an equilibrium state for plasma confinement with the pinch effect, this equilibrium state is not stable. A small departure from the cylindrical geometry of the equilibrium state, results in the growth of the original perturbations with time and the desintegration of the plasma column. The growth of instabilities is the reason why it is difficult to sustain reasonably long-lived pinched plasmas in the laboratory.

A detailed mathematical treatment of these instabilities is out of the scope of this text. For simplicity, in the following discussion of instabilities,we shall consider a perfectly diamagnetic plasma column confined by a static magnetic field. Since the plasma is perfectly diamagnetic, there is no magnetic field and, consequently, no magnetic pressure inside the plasma column. The plasma kinetic pressure is assumed to be uniform inside the plasma and vanishes outside it. In the *equilibrium state*, the magnetic pressure at the plasma surface, p_{mo}, must be equal to the kinetic pressure p of the plasma,

$$p = p_{mo} = \frac{B_o^2}{2\mu_o} \qquad (5.1)$$

where B_o is the magnitude of the magnetic flux density at the plasma surface. This situation of a sharp plasma boundary is an idealized one, and is difficult to create in laboratory, since the plasma particles diffuse through the magnetic field lines in a diffussion time of the order of $\mu_o \sigma_o L^2$, in view of the finite conductivity σ_o of the plasma, as discussed in section 3, of Chapter 12.

In the cylindrical pinch column, the confining magnetic field lines have a curvature such that they are concave toward the plasma and the field strength decreases whith increasing distance from the center of curvature of the field lines (Fig. 11) According to

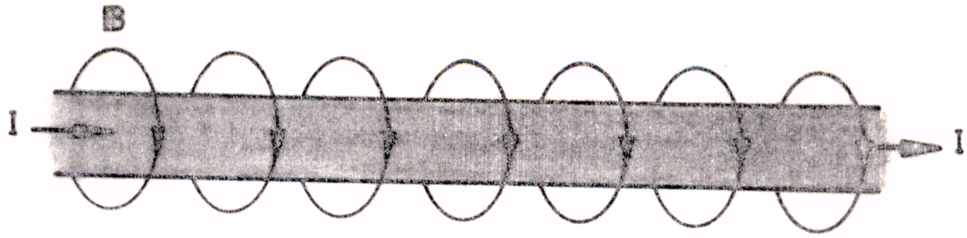

Fig. 11 - Unstable equilibrium configuration of a cylindrical plasma column. The azimuthal $\underset{\sim}{B}$ field decreases radially outwards.

Ampère's law, this azimuthal magnetic field is inversely proportional to the radial distance r from the axis of the cylindrical plasma column.

6. THE SAUSAGE INSTABILITY

Suppose that the equilibrium state of the pinched plasma column, shown in Fig. 11, is disturbed by a wave-like pertubation, with the crests and throughs on the surface of the plasma column and cylindrically symmetric about the column axis, as indicated in Fig.12.

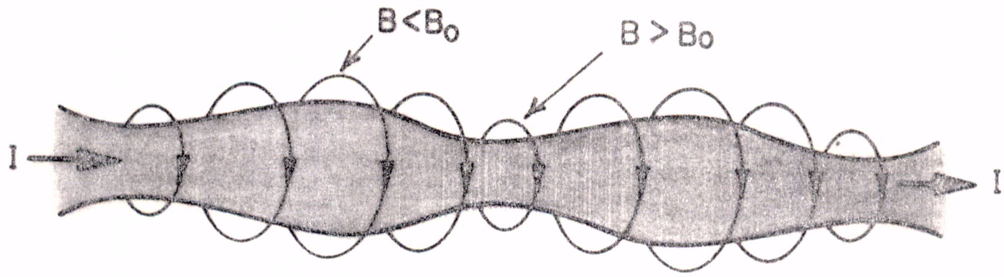

Fig. 12 - The sausage instability.

We shall consider that the plasma is constricted in some locations and expanded at others, in such a way that its volume does not change. Consequently, the uniform kinetic pressure of the plasma is left unchanged. However, in view of the $1/r$ radial dependence of the azimuthal magnetic field, the magnitude of this field at the surface of the disturbed plasma column will vary from place to place on the surface. At the locations where the radius has decreased, in relation with the equilibrium value, the magnetic pressure at the constricted plasma surface will be larger than the plasma kinetic pressure, and will force the plasma surface radially inward, thus enhancing the constriction. At the locations where the radius has become larger than the equilibrium value, the plasma kinetic pressure will be larger than the magnetic pressure at the expanded plasma surface and will force the surface radially outward, increasing the local expansion of the plasma. Therefore, the troughs will become deeper and the crests higher. The initial perturbation gives rise to forces that tend to further increase the initial disturbance, so that the initial equilibrium state is unstable. When the constrictions reach the axis, the column appears like a string of sausages and, for this reason, this type of instability has been called a *sausage* instability.

The sausage instability can be inhibited by a longitudinal magnetic field applied inside the plasma column. This longitudinal magnetic field is produced by passing a current through a solenoidal coil wound around the column. Because of the high electric conducticity of the plasma, the longitudinal field lines are frozen in the plasma. When the sausage distortion starts to grow, the longitudinal magnetic field lines are compressed at the constrictions, causing an increase in the pressure inside the plasma that opposes the increased magnetic pressure of the azimuthal field at the constricted surface, and forces the constriction to expand. At the locations where the column radius has increased, the longitudinal field lines move apart with the plasma expansion, thus decreasing the internal pressure, with the result that the net pressure forces the plasma surface radially inwards. This is illustrated schematically in Fig. 13.

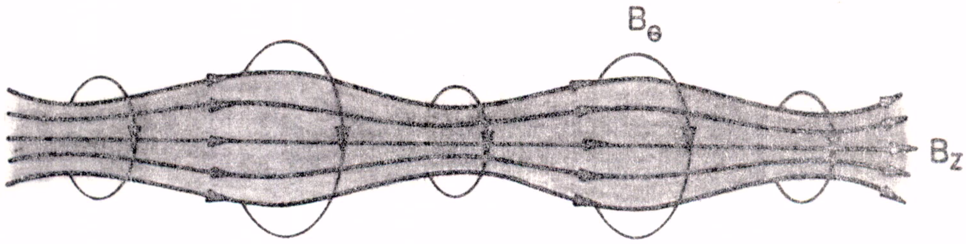

Fig. 13 - A longitudinal magnetic flux density B_z can be used to inhibit the sausage instability.

We shall, next, determine what must be the magnitude B_z of the longitudinal magnetic flux density, as compared to the magnitude of the azimuthal B_Θ field, in order that the longitudinal field be able to stabilize the plasma column against the setting of the sausage instability. If the radius r of the column, at the constriction, is decreased by an amount dr, and considering that the magnetic flux ($\Phi_m = B_z \pi r^2$) through the cross sectional area of the column remains constant during compression, then we have

$$d\Phi_m = \pi r^2 \, dB_z + B_z \, 2\pi \, r \, dr = 0 \qquad (6.1)$$

Hence, the longitudinal magnetic flux density, B_z, is increased by the amount

$$dB_z = - 2 \, B_z \frac{dr}{r} \qquad (6.2)$$

Consequently, the corresponding internal magnetic pressure increases by the amount

$$dp_z = \frac{(B_z + dB_z)^2}{2\mu_o} - \frac{B_z^2}{2\mu_o} = \frac{1}{\mu_o} \, B_z \, dB_z \qquad (6.3)$$

or, using (6.2),

$$dp_z = - \frac{2B_z^2}{\mu_o} \frac{dr}{r} \qquad (6.4)$$

Considering now the azimuthal magnetic flux density, B_Θ, it is easily seen, from Ampère's law, that external to the column we have

$$r \, B_\Theta(r) = \text{constant} \qquad (6.5)$$

so that the azimuthal magnetic flux density, at the constriced surface, increases by the amount

$$dB_\Theta = - B_\Theta \frac{dr}{r} \tag{6.6}$$

Hence, the corresponding increase in the external magnetic pressure is

$$dp_\Theta = \frac{B_\Theta \, dB_\Theta}{\mu_o} = - \frac{B_\Theta^2}{\mu_o} \frac{dr}{r} \tag{6.7}$$

Therefore, in order that the plasma column be stable against the sausage distortion, we must have $dp_z > dp_\Theta$, or, using (6.4) and (6.7),

$$B_z^2 > \frac{1}{2} B_\Theta^2 \tag{6.8}$$

7. THE KINK INSTABILITY

Another type of instability of the pinched plasma column is the so-called *kink instability*. The kink distortion consists of a perturbation in the form of a bend or kink in the column, but with the disturbed column maintaining its uniform circular cross section, as shown in Fig.14. Usually, there may be several kinks along the length of the plasma column. In the neighborhood of the column, where the kink has developed, the magnetic field lines are brought closer together on the concave side, and separated on the convex side, so that the external magnetic pressure is increased on the concave side

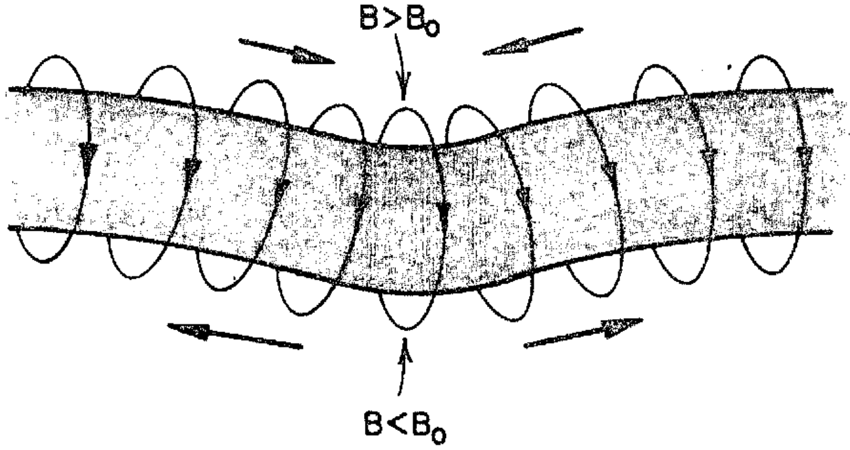

Fig. 14 - The kink instability.

and decreased on the convex side. Therefore, the changes in the external magnetic pressure are in such a way as to accentuate the distortion still further. This type of distortion is, therefore, unstable.

The kink instability can be hindered by the application of a longitudinal magnetic field within the plasma column, as in the case of the sausage instability. In the kink distortion, the longitudinal magnetic field lines, frozen inside the plasma column, are stretched, and the increased tension acting along the longitudinal magnetic field lines opposes the external forces. The net result is the stabilization of the column (Fig.15).

In actual pratice, however, the plasma is not perfectly

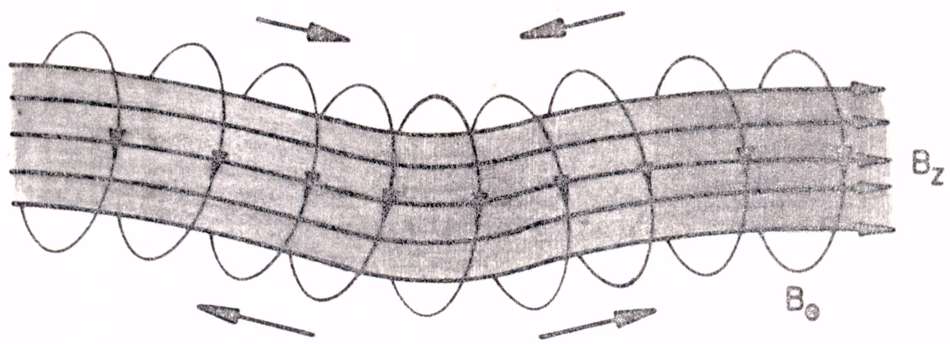

Fig. 15 - The increased tension of a longitudinal magnetic field, applied inside the column, inhibits the kink instability.

diamagnetic and other fields may also be present. The calculation of the stability of the pinched plasma column is not, in general, a very simple task.

8. CONVEX FIELD CONFIGURATIONS

In the linear pinch configuration, the azimuthal magnetic field confining the plasma column is produced by a longitudinal current flowing along the column. The configuration of this field is such that the magnetic flux lines are *concave* towards the plasma. Configurations of this type are unstable, as we have seen with the sausage and the kink instabilities.

Configurations for which the field lines are *convex* towards the plasma lead to a stable equilibrium, since the magnetic field strength increases in a direction away from the plasma. If the plasma surface is perturbed by a wave-like disturbance, the magnetic pressure at the crests will be larger than the internal kinetic pressure, and the plasma is forced to return to its equilibrium configuration (assuming that the kinetic pressure is not affected by the perturbations). At the troughs, the internal kinetic pressure will be larger than the magnetic pressure acting on the plasma surface, and will force the plasma to expand. Therefore, for plasma confinement, it is desirable to use a magnetic field configuration in which the magnetic flux lines are everywhere convex towards the plasma. An example of this type of configuration is the *cusp field*, which can be produced by an array of four current-carrying wires, as shown in Fig. 16. The presence of sharp edges and cusps at the plasma boundary, however, can lead to escape of the plasma particles. Although edges and cusps are characteristics of these configurations, modifications of the cusp field geometry are commonly employed for confinement of high temperature plasmas. Higher order cusp fields can be produced by lining up several pairs of current-carrying wires as,for example, in the *picket-fence* field geometry illustrated in Fig. 17.

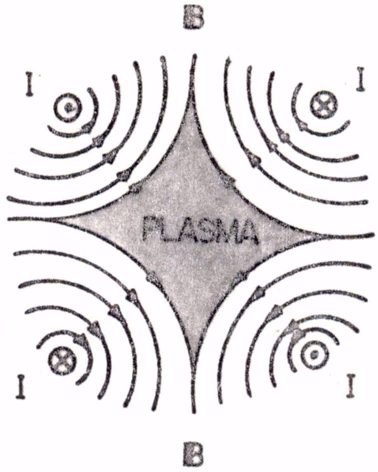

Fig. 16 - Plasma confinement by a cusped magnetic field, produced by four current-carrying wires.

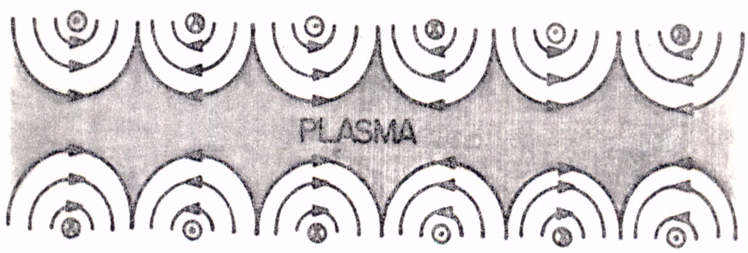

Fig. 17 - Picket-fence field configuration for magnetic confinement of a plasma.

PROBLEMS

13.1 - The minimum intensity of the magnetic induction ($\underset{\sim}{B}_o$) necessary to confine a plasma at an internal pressure of 100 atm is (see Problem 12.3) 5 Weber/m^2. Assuming that this field is produced by an axial current flowing in a cylindrical plasma column (as in the longitudinal pinch effect) of 10 cm radius, show (applying Ampère's law) that the total current, necessary to produce this magnetic field ($\underset{\sim}{B}_o$) at the surface of the cylindrical plasma column, is 2.5×10^6 Ampères. (1 atm = 10^5 Newton/m^2; $\mu_o = 4\pi \times 10^{-7}$ Henry/m).

13.2 - For the equilibrium Bennett pinch with cylindrical geometry, calculate $B_\theta(r)$ using Eq. (2.4) and the expression for $n(r)$ given in Eq. (3.8). Make a plot showing the radial distribution of $p(r)$, $J_z(r)$ and $B_\theta(r)$.

13.3 - For the *equilibrium* theta-pinch produced by an azimuthal current in the theta-direction ($\underset{\sim}{J}_\theta$), as illustrated in Fig.6 of Chapter 12, determine expressions for the radial distributions of $J_\theta(r)$ and $p(r)$ in terms of $B_z(r)$. Draw a diagram illustrating these radial distributions for the special case when B_z = constant.

13.4 - Consider the *dynamical* theta-pinch. Derive the differential equation which specifies the time-dependence of the radius, $R(t)$, of the plasma column using the snowplow model.

13.5 - Use the equation for the fluid velocity component ($\underset{\sim}{u}_{\perp}$) normal to $\underset{\sim}{B}$, derived in Problem 9.7, to determine the relative orientations of $\underset{\sim}{u}$, $\underset{\sim}{B}$, $\underset{\sim}{E}$, $\underset{\sim}{J}$ and $\underset{\sim}{\nabla} p$ in a theta-pinch device.

13.6 - In the longitudinal equilibrium pinch shown schematically in Fig. 1, assume that the radial dependence of the current density $J_z(r)$ is such that

$$J_z = 0 \quad \text{for} \quad 0 < r < a$$

$$J_z = J_o = \text{constant for } a < r < b$$

$$J_z = 0 \quad \text{for} \quad r > b$$

Calculate $p(r)$ and $B_\theta(r)$ and make a plot showing the radial dependence. Show that, as $a \to b$, the magnetic pressure $B_\theta^2/2\mu_o$ at $r = b$ becomes equal to p at $r = 0$, while as $a \to 0$, B_θ^2/μ_o at $r = b$ becomes equal to p at $r = 0$.

13.7 - (a) Show that a *force-free* magnetic field satisfies the relation

$$(\underset{\sim}{\nabla} \times \underset{\sim}{B}) \times \underset{\sim}{B} = 0$$

(b) Let

$$\underset{\sim}{\nabla} \times \underset{\sim}{B} = \alpha(\underset{\sim}{r})\, \underset{\sim}{B}$$

and show that

$$\underset{\sim}{B} \cdot \underset{\sim}{\nabla}\alpha = 0$$

(c) Verify that the surfaces $\alpha = \text{constant}$ are made up of magnetic field lines.

(d) Show that $\alpha(\underset{\sim}{r})$, as defined in part (b), for the force-free field, can be expressed as

$$\alpha = \underset{\sim}{\hat{B}} \cdot (\underset{\sim}{\nabla} \times \underset{\sim}{\hat{B}})$$

(e) Prove that for the force-free field $\underset{\sim}{\nabla} B$ lies on the osculating plane ($\underset{\sim}{\hat{B}}$, $\underset{\sim}{\hat{n}}$ plane, where $\underset{\sim}{\hat{n}}$ is the principal normal to the field line).

13.8 - Consider the following basic equation for the equilibrium of a plasma column with cylindrical geometry (see Problem 12.9)

$$\frac{d}{dr}\left(p + \frac{B_z^2}{2\mu_o} + \frac{B_\theta^2}{2\mu_o} \right) = - \frac{1}{\mu_o} \frac{B_\theta^2}{r}$$

(a) Verify that, for the θ-pinch, this equation reduces to

$$p + \frac{B_z^2}{2\mu_o} = \text{constant}$$

whereas, for the longitudinal pinch, it becomes

$$\frac{d}{dr}\left(p + \frac{B_\theta^2}{2\mu_o} \right) = - \frac{1}{\mu_o} \frac{B_\theta^2}{r}$$

(b) For the cylindrical screw-pinch, in which both B_θ and B_z are nonzero, assume that the longitudinal current density and the kinetic pressure are given, respectively, by

$$J_z(r) = J_o \left(1 - \frac{r^2}{a^2} \right) \qquad (r \leq a)$$

$$p(r) = p_o = \text{constant} \qquad (r < a)$$

Verify that

$$B_\theta(r) = \frac{\mu_o J_o}{2} r\left(1 - \frac{r^2}{2a^2}\right) \qquad (r \leq a)$$

$$B_\theta(r) = \frac{\mu_o J_o a^2}{4} \frac{1}{r} \qquad (r \geq a)$$

Show that $B_z(r)$ is given by

$$\frac{B_z^2}{2\mu_o} = - \frac{B_\theta^2}{2\mu_o} - \frac{1}{\mu_o} \int_r \frac{B_\theta^2(r)}{r} dr$$

From this equation determine $B_z(r)$ and make a plot showing $p(r)$, $J_z(r)$, $B_\theta(r)$ and $B_z(r)$ as a function of r.