1. Classification *INPE-COM.4/RPE* *C.D.U.: 533.9*	2. Period	4. Distribution Criterion internal [ ] external [X]
3. Key Words (selected by the author) *WAVE PROPAGATION* *MAGNETOIONIC THEORY* *COLD PLASMA*		
5. Report Nº *INPE-1546-RPE/066*	6. Date *August, 1979*	7. Revised by *Rene A. Medrano - B -*
8. Title and Sub-title *WAVES IN COLD PLASMAS*		9. Authorized by *Nelson de Jesus Parada* *Director*
10. Sector *DCE/DGA/GIO*	Code	11. Nº of Copies *10*
12. Authorship *J. A. Bittencourt*		14. Nº of Pages *86*
13. Signature of the responsible *Bittencourt*		15. Price

16. Summary/Notes

This is the sixteenth chapter, in a series of twenty two, written as an introduction to the fundamentals of plasma physics. This chapter analyses the problem of wave propagation in a cold electron plasma. Initially, the basic equations of magnetoionic theory, and their solution in terms of plane waves, are presented. The characteristics of high frequency wave propagation in isotropic electron plasmas, in the absence of an externally applied magnetic field, are analysed. Next, it is considered the problem of wave propagation in a cold, anisotropic plasma, immersed in a magnetic field. The general dispersion relation and the Appleton-Hartree equation are derived, and the various modes of wave propagation in directions parallel and perpendicular to the magnetic field are investigated. The properties of the CMA diagram are also described. Finally, it is presented an analysis of atmospheric whistler propagation, helicons, and the phenomenon of Faraday rotation in plasmas.

17. Remarks

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WAVES IN COLD PLASMAS

1. INTRODUCTION

In this chapter we analyze the problem of wave propagation in cold plasmas. In a cold plasma the thermal kinetic energy of the particles is ignored and the corresponding velocity distribution function is the Dirac delta function. The study of waves in plasmas is very useful for plasma diagnostics, since it provides information on the plasma properties. The theory of wave propagation in a cold homogeneous plasma immersed in a magnetic field is known as *magnetoionic theory*.

There are two main different methods of approach that can be used in analyzing the problem of waves in plasmas. In the first approach, the plasma is characterized as a medium having either a conductivity or a dielectric constant and the wave equation for this medium is derived from Maxwell equations. In the presence of an externally applied magnetostatic field, the plasma is equivalent to an anisotropic dielectric characterized by a dielectric tensor or dyad. In the second main method of approach, Maxwell equations are solved simultaneously with the equations describing the motion of the plasma particles. In this case, we do not explicitly derive a wave equation, and expressions for the dielectric or conductivity dyad are not obtained directly. Instead, we obtain a *dispersion relation*, which relates the wave number $\underset{\sim}{k}$ to the wave frequency ω. All the information about the propagation of a given plasma mode is contained in the appropriate dispersion relation. This second method of approach is often straightforward and simpler than the first one, and is the method we shall adopt in this treatment.

The neglect of the pressure term for a cold plasma is justified if the thermal velocity of the particles (v_{th}) is small when compared with the phase velocity (v_{ph}) of the wave ($v_{th} << v_{ph}$). Therefore, the cold plasma model gives a satisfactory description except for waves with extremely small phase velocities. For waves with such small phase velocities, the pressure term becomes important and must be considered for a correct description. In this case, the plasma is said to be warm. The propagation of waves in warm plasmas is the subject of the next chapter.

The study presented here is restricted to *small amplitude waves*, so that the analysis will be based on a *linear perturbation theory* under the assumption that the variations in the plasma parameters, due to the presence of waves, are small (to the first order) as compared to the undisturbed parameters. The plasma is assumed to be homogeneous and infinite (no boundary effects), and the externally applied magnetostatic field is assumed to be uniform. This medium is called a *magnetoionic medium*.

Because of mathematical simplicity, the analysis will be made in terms of *plane waves*. This does not imply in loss of generality, since any physically realizable wave motion can always be synthesized in terms of plane waves.

In magnetoionic theory, only the motion of the electrons is considered. This is valid for *high frequency* waves, i.e., for frequencies large compared to the ion cyclotron frequency. The theory of high-frequency small-amplitude plane waves propagating in an arbitrary direction with respect to the magnetostatic field, in a magnetoionic medium, is known as the Appleton-Hartree theory, in honor to E. V. Appleton and D. R. Hartree, who developed this theory when studying the problem of wave propagation in the Earth's ionosphere. At frequencies of the order of the ion cyclotron frequency, and smaller, the motion of the ions must be considered. The theory of wave propagation in a cold multicomponent plasma is commonly referred to as the hydromagnetic extension of magnetoionic theory.

2. BASIC EQUATIONS OF MAGNETOIONIC THEORY

In a cold electron gas the two hydrodynamic variables involved are the electron number density $n(\underset{\sim}{r}, t)$ and the average electron velocity $\underset{\sim}{u}(\underset{\sim}{r}, t)$. They satisfy the equation of continuity

$$\frac{\partial n(\underset{\sim}{r}, t)}{\partial t} + \underset{\sim}{\nabla} \cdot \left[n(\underset{\sim}{r}, t)\, \underset{\sim}{u}(\underset{\sim}{r}, t) \right] = 0 \tag{2.1}$$

and the Langevin equation of motion

$$m \frac{D\underset{\sim}{u}(\underset{\sim}{r}, t)}{Dt} = q \left[\underset{\sim}{E}(\underset{\sim}{r}, t) + \underset{\sim}{u}(\underset{\sim}{r}, t) \times \underset{\sim}{B}(\underset{\sim}{r}, t) \right] - m\nu \underset{\sim}{u}(r, t) \tag{2.2}$$

These two equations are complemented by Maxwell equations

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{E}(\underset{\sim}{r}, t) = \rho_c(\underset{\sim}{r}, t)/\varepsilon_o \tag{2.3}$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{B}(\underset{\sim}{r}, t) = 0 \tag{2.4}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{E}(\underset{\sim}{r}, t) = - \frac{\partial \underset{\sim}{B}(\underset{\sim}{r}, t)}{\partial t} \tag{2.5}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{B}(\underset{\sim}{r}, t) = \mu_o \left[\underset{\sim}{J}(\underset{\sim}{r}, t) + \varepsilon_o \frac{\partial \underset{\sim}{E}(\underset{\sim}{r}, t)}{\partial t} \right] \tag{2.6}$$

where the electron charge density in given by

$$\rho_c(\underset{\sim}{r}, t) = - en(\underset{\sim}{r}, t) \tag{2.7}$$

and the electric current density by

$$\underset{\sim}{J}(\underset{\sim}{r}, t) = \rho_c(\underset{\sim}{r}, t) \;\; \underset{\sim}{u}(\underset{\sim}{r}, t) = - en(\underset{\sim}{r}, t) \;\; \underset{\sim}{u}(\underset{\sim}{r}, t) \tag{2.8}$$

As previously discussed, Eq. (2.4) is actually considered only as an initial condition for Eq. (2.5). Furthermore, Eqs. (2.3) and (2.6) can be combined to yield the conservation equation of electric charge density.

3. PLANE WAVE SOLUTIONS AND LINEARIZATION

Let us separate the total magnetic induction and the electron number density into two parts,

$$\underset{\sim}{B}(\underset{\sim}{r}, t) = \underset{\sim}{B}_o + \underset{\sim}{B}_1(\underset{\sim}{r}, t) \tag{3.1}$$

$$n(\underset{\sim}{r}, t) = n_0 + n_1(\underset{\sim}{r}, t) \tag{3.2}$$

where $\underset{\sim}{B}_0$ is a constant and uniform field, and n_0 is the undisturbed electron number density in the absence of waves. For harmonic plane wave solutions the quantities $\underset{\sim}{E}$, $\underset{\sim}{B}_1$, $\underset{\sim}{u}$ and n_1 are all proportional to $\exp\left[i(\underset{\sim}{k} \cdot \underset{\sim}{r} - \omega t)\right]$,

$$\underset{\sim}{E}(\underset{\sim}{r}, t) = \underset{\sim}{E} \exp\left[i(\underset{\sim}{k} \cdot \underset{\sim}{r} - \omega t)\right] \tag{3.3a}$$

$$\underset{\sim}{B}_1(\underset{\sim}{r}, t) = \underset{\sim}{B}_1 \exp\left[i(\underset{\sim}{k} \cdot \underset{\sim}{r} - \omega t)\right] \tag{3.3b}$$

$$\underset{\sim}{u}(\underset{\sim}{r}, t) = \underset{\sim}{u} \exp\left[i(\underset{\sim}{k} \cdot \underset{\sim}{r} - \omega t)\right] \tag{3.3c}$$

$$n_1(\underset{\sim}{r}, t) = n_1 \exp\left[i(\underset{\sim}{k} \cdot \underset{\sim}{r} - \omega t)\right] \tag{3.3d}$$

where $\underset{\sim}{k}$ is the wave propagation vector and ω is the wave frequency. The use of the same symbol to denote the complex amplitude as well as the entire expression in Eqs. (3.3) should lead to no confusion, because in linear wave theory the same exponential factor will occur on both sides of any equation and can be cancelled out.

Eq. (2.2) is not yet quite tractable because of the *nonlinear* terms $(\underset{\sim}{u} \cdot \underset{\sim}{\nabla})\, \underset{\sim}{u}$ and $\underset{\sim}{u} \times \underset{\sim}{B}$. This difficulty can be eliminated considering $\underset{\sim}{u}$ and $\underset{\sim}{B}_1$ as small first order quantities and neglecting second order terms. As discussed in section 3, of Chapter 10, when dealing with wave phenomena in plasmas the second order nonlinear term

$\underset{\sim}{u} \times \underset{\sim}{B}_1$ can be neglected provided the average electron velocity is much less than the phase velocity of the wave ($u \ll \omega/k$).

For harmonic plane wave solutions the differential operators $\underset{\sim}{\nabla}$ and $\partial/\partial t$ are replaced, respectively, by $i\underset{\sim}{k}$ and $-i\omega$. Consequently, the differential equations become simple algebraic equations. Therefore, Eqs. (2.2), (2.5) and (2.6) become, respectively, neglecting second order terms,

$$- i \omega m \underset{\sim}{u} = -e (\underset{\sim}{E} + \underset{\sim}{u} \times \underset{\sim}{B}_0) - m \nu \underset{\sim}{u} \tag{3.4}$$

$$\underset{\sim}{k} \times \underset{\sim}{E} = \omega \underset{\sim}{B}_1 \tag{3.5}$$

$$i \underset{\sim}{k} \times \underset{\sim}{B}_1 = \mu_0 (-e n_0 \underset{\sim}{u} - i \omega \varepsilon_0 \underset{\sim}{E}) \tag{3.6}$$

where use was made of (2.8). These three equations involving the three dependent variables $\underset{\sim}{u}$, $\underset{\sim}{E}$ and $\underset{\sim}{B}_1$ can be used to derive a *dispersion relation* for wave propagation in a cold electron gas. In order to keep matters as simple as possible, we investigate initially the characteristics of wave propagation in a cold isotropic plasma with $\underset{\sim}{B}_0 = 0$.

4. WAVE PROPAGATION IN ISOTROPIC ELECTRON PLASMAS

4.1 - Derivation of the dispersion relation

In the absence of an externally applied magnetic field ($\underset{\sim}{B}_o = 0$) the Langevin equation (3.4) gives

$$\underset{\sim}{u} = - \frac{e}{m(\nu - i\omega)} \underset{\sim}{E} \tag{4.1}$$

Combining Eqs. (3.5), (3.6) and (4.1), we obtain

$$\underset{\sim}{k} \times (\underset{\sim}{k} \times \underset{\sim}{E}) = - \frac{i\,\omega\,\mu_o e^2 n_o}{m\,(\nu - i\,\omega)} \underset{\sim}{E} - \frac{\omega^2}{c^2} \underset{\sim}{E} \tag{4.2}$$

It is convenient to separate the electric field vector in a *longitudinal component* $\underset{\sim}{E}_\ell$ (parallel to $\underset{\sim}{k}$) and a *transverse component* $\underset{\sim}{E}_t$ (perpendicular to $\underset{\sim}{k}$),

$$\underset{\sim}{E} = \underset{\sim}{E}_\ell + \underset{\sim}{E}_t \tag{4.3}$$

as shown in Fig. 1. Therefore,

$$(\underset{\sim}{k} \times \underset{\sim}{E}_\ell) = 0 \tag{4.4}$$

$$\underset{\sim}{k} \times (\underset{\sim}{k} \times \underset{\sim}{E}_t) = - k^2 \underset{\sim}{E}_t \tag{4.5}$$

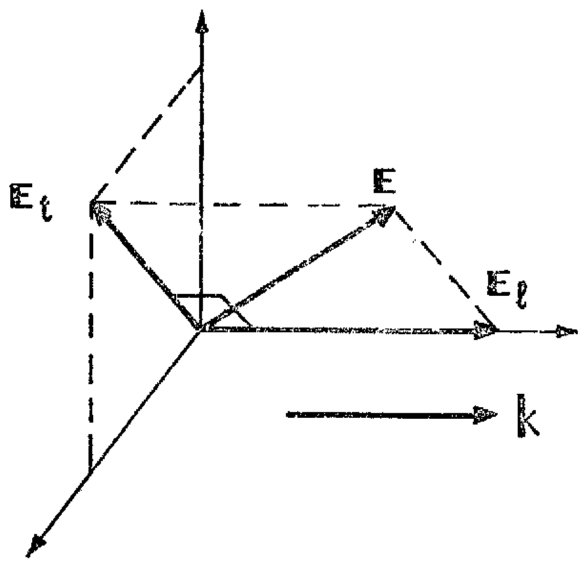

Fig. 1 - Longitudinal and transverse components of the electric field vector with respect to the propagation vector $\underset{\sim}{k}$.

and (4.2) becomes

$$- k^2 \underset{\sim}{E}_t = - \left[\frac{i\omega\mu_o e^2 n_o}{m(\nu - i\omega)} + \frac{\omega^2}{c^2} \right] (\underset{\sim}{E}_\ell + \underset{\sim}{E}_t) \qquad (4.6)$$

This equation can be separated in a *longitudinal component*,

$$\left[\frac{\omega_{pe}^2}{c^2(1 + i\nu/\omega)} - \frac{\omega^2}{c^2} \right] \underset{\sim}{E}_\ell = 0 \qquad (4.7)$$

and a *transverse component*,

$$- k^2 \underset{\sim}{E}_t = \left[\frac{\omega_{pe}^2}{c^2(1 + i\nu/\omega)} - \frac{\omega^2}{c^2} \right] \underset{\sim}{E}_t \tag{4.8}$$

where we have introduced the electron plasma frequency, $\omega_{pe} = (n_o e^2/m_e \varepsilon_o)^{1/2}$. Eq. (4.7) yields the following dispersion relation for a *longitudinal mode* ($\underset{\sim}{E}_\ell \neq 0$),

$$\omega^2(1 + i\nu/\omega) - \omega_{pe}^2 = 0 \tag{4.9}$$

For a *transverse mode* ($\underset{\sim}{E}_t \neq 0$) the dispersion relation is, from (4.8),

$$(\omega^2 - k^2c^2)\ (1 + i\nu/\omega) - \omega_{pe}^2 = 0 \tag{4.10}$$

4.2 - Collisionless plasma

For simplicity we consider first the case in which the collision frequency is much less than the wave frequency ($\nu \ll \omega$), so that the effect of collisions can be ignored. In sub-section 4.4 we shall take into consideration the effect of collisions. Thus, for $\nu = 0$ the dispersion relation (4.9), for *longitudinal waves*, becomes

$$\omega^2 = \omega_{pe}^2 \tag{4.11}$$

while, for *transverse waves*, (4.10) becomes

$$k^2c^2 = \omega^2 - \omega_{pe}^2 \tag{4.12}$$

Eq. (4.11) shows that *longitudinal oscillations* ($\underset{\sim}{E}_\ell \neq 0$) can occur just at the plasma frequency ω_{pe}. These longitudinal oscillations are just the *plasma oscillations* discussed in section 1, of Chapter 11. It is seen, from (4.1), that the electrons oscillate with a velocity given by

$$\underset{\sim}{u} = -\frac{ie}{m\omega}\underset{\sim}{E}_\ell \tag{4.13}$$

In virtue of (4.4) and (3.5) it is clear that $\underset{\sim}{B}_1 = 0$, so that there is no magnetic field associated with these longitudinal oscillations. Further, there is no wave propagation, since there is no relative phase variation from point to point in space, so that the longitudinal oscillations do not constitute a propagating mode . These oscillations are, therefore, longitudinal, electrostatic and stationary. In the next chapter we consider the wave propagation problem in a *warm* plasma, in which the thermal effects are included, where we show that these electron plasma oscillations correspond to the limit, when the electron temperature goes to zero, of the longitudinal mode of propagation called the *electron plasma wave*.

Considering now the dispersion relation (4.12) for *transverse waves* ($\underset{\sim}{E}_t \neq 0$), it is seen that k^2 is positive for $\omega > \omega_{pe}$ and negative for $\omega < \omega_{pe}$. Hence, for travelling waves (with ω real) k becomes imaginary for frequencies below the plasma frequency, ω_{pe}. Writing $k = \beta + i\alpha$, where β and α are real quantities, it is seen, from (4.12), that for $\omega > \omega_{pe}$ ($k = \beta$; $\alpha = 0$) the transverse wave propagates with a *phase velocity* (ω divided by the real part of k) given by

$$v_{ph} = \frac{\omega}{k} = \frac{c}{(1 - \omega_{pe}^2/\omega^2)^{1/2}} \qquad (\omega > \omega_{pe}) \tag{4.14}$$

Also, for $\omega > \omega_{pe}$ the *group velocity* of the transverse wave can be obtained differentiating (4.12) with repect to k,

$$v_g = \frac{\partial\omega}{\partial k} = \frac{c^2}{v_{ph}} \qquad (\omega > \omega_{pe}) \tag{4.15}$$

For $\omega < \omega_{pe}$, k is imaginary ($k = i\alpha$) and the transverse wave is exponentially damped, since

$$E_t \propto \exp\,(ik\zeta - i\omega t) = \exp\,(-\alpha\zeta)\,\exp\,(-i\omega t) \tag{4.16}$$

so that the wave dies out with increasing values of ζ. Such exponentially damped fields are called *evanescent waves* and do not transport any time-averaged power. Since $\beta = 0$, it is easily seen that, in this case,

$$v_{ph} = \infty \qquad (\omega < \omega_{pe}) \tag{4.17}$$

$$v_g = 0 \qquad (\omega < \omega_{pe}) \tag{4.18}$$

Also, from (4.12) we find, for $\omega < \omega_{pe}$,

$$\alpha = \mathrm{Im}(k) = \frac{(\omega_{pe}^2 - \omega^2)^{1/2}}{c} \tag{4.19}$$

where Im denotes the "imaginary part of".

A plot of phase velocity and group velocity as a function of frequency is shown in Fig. 2(a), and the frequency dependence of the attenuation factor, α, is shown in Fig. 2(b). Note that the phase velocity is always greater than the velocity of light c, but the group velocity, which is the velocity at which a signal propagates, is always less than c, in agreement with the requirements of the theory of relativity. For $\omega >> \omega_{pe}$ we find, from (4.14) and (4.15),

$$v_{ph} = v_g = c \qquad (\omega >> \omega_{pe}) \tag{4.20}$$

which shows that for very high frequencies the plane wave characteristics of a plasma degenerate to those of free space. This is expected on a physical basis, since in the limiting case of infinite frequency even the electrons are unable to respond to the oscillating electric field.

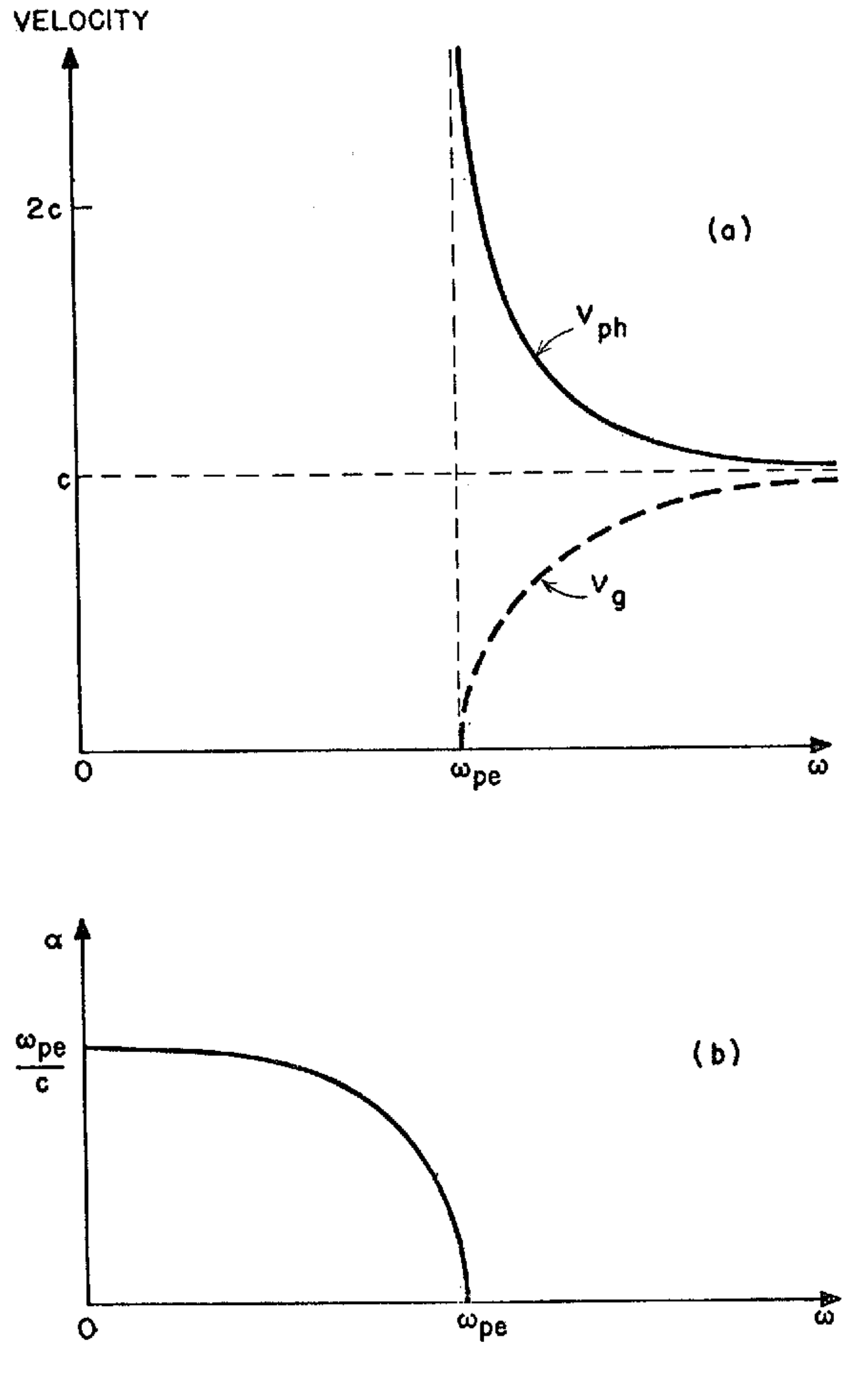

Fig. 2 - Frequency dependence of the phase velocity, group velocity and the attenuation factor α for transverse waves in a collisionless isotropic cold electron gas.

The dispersion relation (4.12) is plotted in Fig. 3 in terms of ω as a function of the real part of k. In this text we shall follow the usual graphic representation of dispersion relations, which is plotting ω versus k, rather than k versus ω. The frequency region in which the transverse wave is evanescent is the region for which $\omega < \omega_{pe}$.

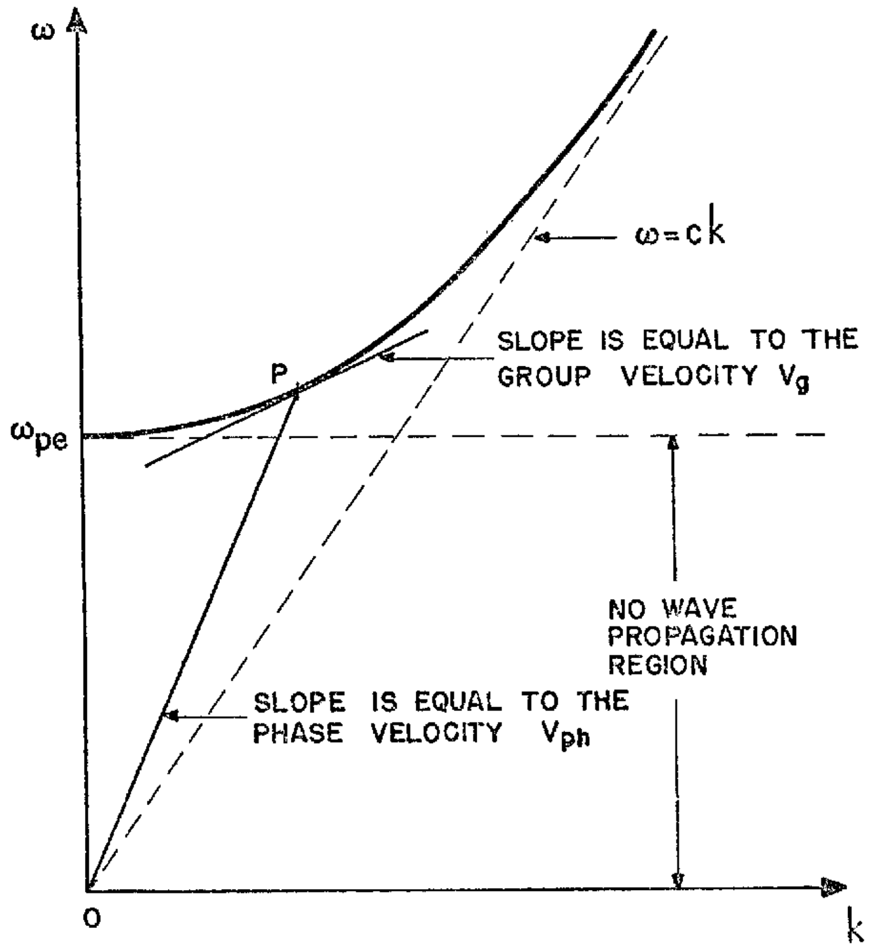

Fig. 3 - Dispersion relation, $\omega(k)$, for the transverse wave propagating in an isotropic cold electron plasma. Note the geometrical representation of the phase and group velocities at the point P.

From the relation $v_g = \partial\omega/\partial k$ we notice that at a given point in the $\omega(k)$ curve the group velocity is equal to the slope of the tangent to the curve at that point, whereas the phase velocity, ω/k, is equal to the slope of the line drawn from the origin to this point. This geometrical representation is illustrated in Fig. 3.

4.3 - Time-averaged Poynting vector

We evaluate next the time-averaged Poynting vector, $< \underset{\sim}{S} >$, which gives the time-averaged power flow for the transverse wave. From (3.5), taking $\underset{\sim}{B}_1 = \mu_o \underset{\sim}{H}_1$, we have

$$\underset{\sim}{H}_1 = \frac{\underset{\sim}{k} \times \underset{\sim}{E}}{\mu_o \omega} \tag{4.21}$$

and the expression for $< \underset{\sim}{S} >$, given in Eq. (14.5.13), becomes

$$< \underset{\sim}{S} > = \frac{1}{2} \mathrm{Re}\, (\underset{\sim}{E} \times \underset{\sim}{H}_1^*) = \frac{1}{2\mu_o \omega} \mathrm{Re}\, \left[\underset{\sim}{E} \times (\underset{\sim}{k}^* \times \underset{\sim}{E}^*)\right]$$

$$= \underset{\sim}{\hat{n}} \frac{1}{2\mu_o \omega} \mathrm{Re}\, \left[k^* E(\underset{\sim}{r}, t)\, E^*(\underset{\sim}{r}, t)\right] \tag{4.22}$$

where $\underset{\sim}{\hat{n}}$ is a unit vector in the direction of $\underset{\sim}{E} \times \underset{\sim}{H}_1$. Using (3.3a) and considering k to be a complex quantity, (4.22) becomes

$$< \underset{\sim}{S} > = \underset{\sim}{\hat{n}} \; \frac{E^2}{2\mu_o\omega} \; \mathrm{Re}\,\{ k^* \exp\left[i(k - k^*)\,\zeta \right] \} \tag{4.23}$$

Therefore, since k is either real or imaginary according to whether $\omega > \omega_{pe}$ or $\omega < \omega_{pe}$, respectively, it follows from (4.23) that

$$< \underset{\sim}{S} > \; = \; 0 \qquad \text{for} \quad \omega < \omega_{pe} \tag{4.24}$$

$$< \underset{\sim}{S} > \; = \; \underset{\sim}{\hat{n}} \, \frac{\varepsilon_o E^2}{2} \, v_g \qquad \text{for} \quad \omega > \omega_{pe} \tag{4.25}$$

where, in (4.25), we have used the relation $c^2k/\omega = v_g$ given in (4.15). Thus, for $\omega > \omega_{pe}$ the fields transport power in the direction $\underset{\sim}{\hat{E}} \times \underset{\sim}{B}_1$, whereas for $\omega < \omega_{pe}$ there is no power flow and the wave is evanescent. For this reason, the region $\omega > \omega_{pe}$ is called the *propagation region.* Since the wave is totally reflected for $\omega < \omega_{pe}$, the frequency ω_{pe} is often called a *reflection point* (where v_{ph} is infinite). It can be shown that the power transmitted into a *semi-infinite* slab of plasma is zero if $\beta = \mathrm{Re}(k)$ is zero, so that in a more general sense any frequency for which $\beta = 0$ ($v_{ph} = \infty$) is referred to as a reflection point. However, if the plasma medium is *finite*, some energy can be transmitted through the finite plasma slab even if $\beta = 0$. This effect is known as the *tunneling effect.*

4.4 - The effect of collisions

The principal effect of collisions is to produce a *damping* of the waves. Before considering the dispersion relations (4.9) and (4.10), it is useful to discuss some general results concerning dispersion relations of the form

$$k^2 = A + iB \tag{4.26}$$

where A and B are real quantities. If we separate k into its real and imaginary parts,

$$k = \beta + i\alpha \tag{4.27}$$

where β and α are both real, then it is a simple matter to verify that

$$A = \mathrm{Re}(k^2) = \beta^2 - \alpha^2 \tag{4.28}$$

$$B = \mathrm{Im}(k^2) = 2\,\beta\,\alpha \tag{4.29}$$

On the other hand, since the waves are proportional to $\exp(ik\zeta - i\omega t)$, we have

$$\exp(ik\zeta - i\omega t) = \exp(-\alpha\zeta) \quad \exp(i\beta\zeta - i\omega t) \tag{4.30}$$

Thus, the sign of β determines the *direction* of wave propagation, i.e., $\beta > 0$ implies propagation in the *positive* ζ direction, whereas $\beta < 0$

implies propagation in the *negative* ζ direction. The sign of α is related to *growing* or *damping* of the wave amplitude as the wave propagates. If both α and β are positive, then the wave travels in the positive ζ direction and is exponentially damped. If both α and β are negative, then the wave travels in the negative ζ direction and is also exponentially damped. On the other hand, if α and β have opposite signs, then the wave is exponentially growing (see Fig. 4). In any case, the sign of B determines whether the travelling wave is growing or decaying. For $B > 0$ the wave is *damped* with distance, whereas for $B < 0$ the wave *grows*.

Similarly, for a dispersion relation having the form

$$\omega^2 = A + iB \tag{4.31}$$

it can be easily verified that, for standing waves, if $B > 0$ the wave *grows* in time, whereas if $B < 0$ the wave is *damped*.

Consider now the dispersion relation (4.9) for the *longitudinal oscillation*,

$$\omega^2 + i\nu\omega - \omega_{pe}^2 = 0 \tag{4.32}$$

or

$$\omega = \frac{1}{2}\left[- i\nu \pm (4\omega_{pe}^2 - \nu^2)^{1/2}\right] \tag{4.33}$$

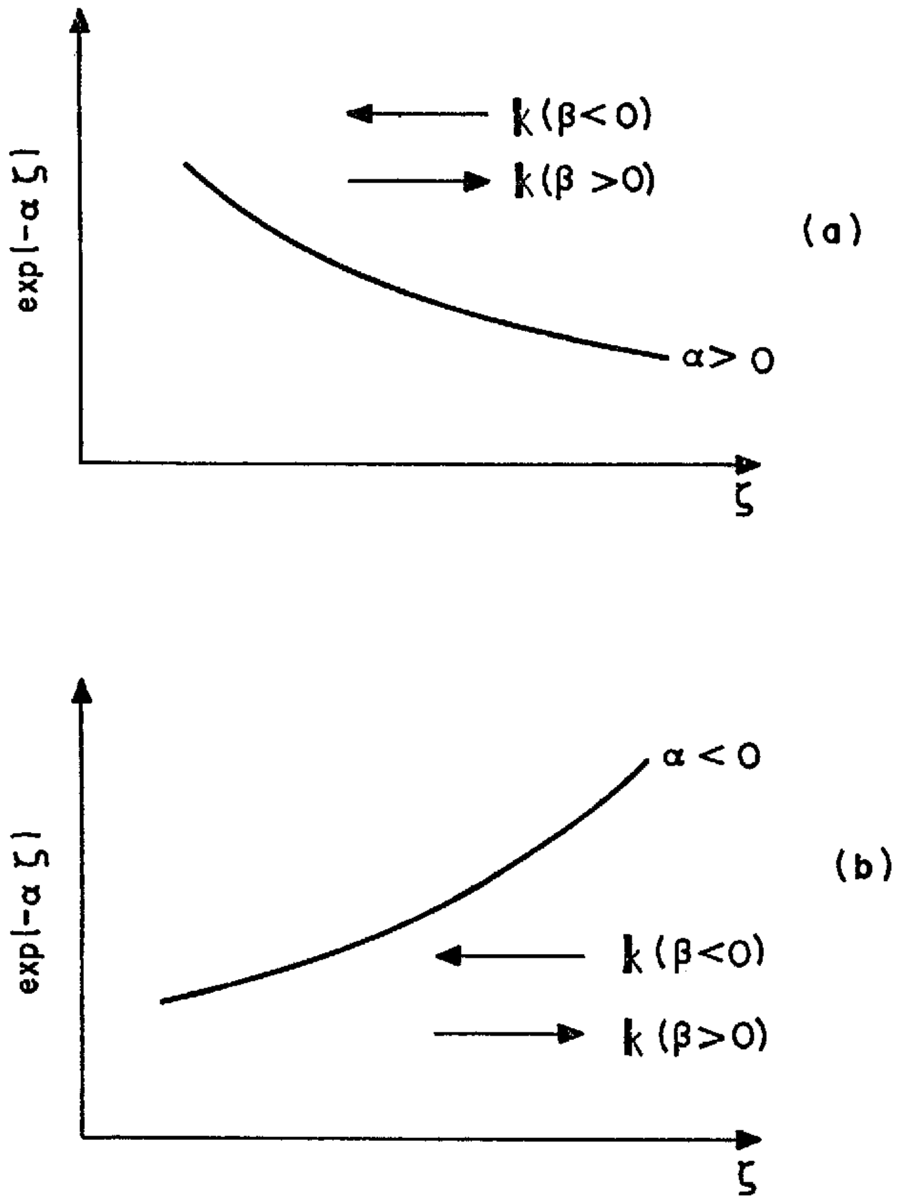

Fig. 4 - For $\alpha > 0$ (a) the wave is exponentially damped if it propagates in the positive ζ direction ($\beta > 0$), or exponentially growing if it propagates in the negative ζ direction ($\beta < 0$), whereas for $\alpha < 0$ (b) the opposite situation holds.

This equation shows that for any value of ν the imaginary part of the frequency ω is negative, so that the oscillation is *damped*, since it is proportional to $\exp(-i\omega t)$.

For the *transverse mode* the dispersion relation (4.10) gives

$$k^2c^2 = \omega^2 - \frac{\omega_{pe}^2}{1 + (\nu/\omega)^2} + \frac{i\omega_{pe}^2(\nu/\omega)}{1 + (\nu/\omega)^2} \tag{4.34}$$

Consequently, in the propagation band B is negative and the travelling waves are *damped* for all frequencies. A plot of the attenuation factor, $\alpha = \mathrm{Im}(k)$, as a function of the collision frequency, ν, is shown in Fig. 5, calculated from (4.34) for a given frequency satisfying $\omega \gg \omega_{pe}$. Fig. 6 shows the dispersion relation, plotted in terms of ω versus k, for the transverse mode of propagation

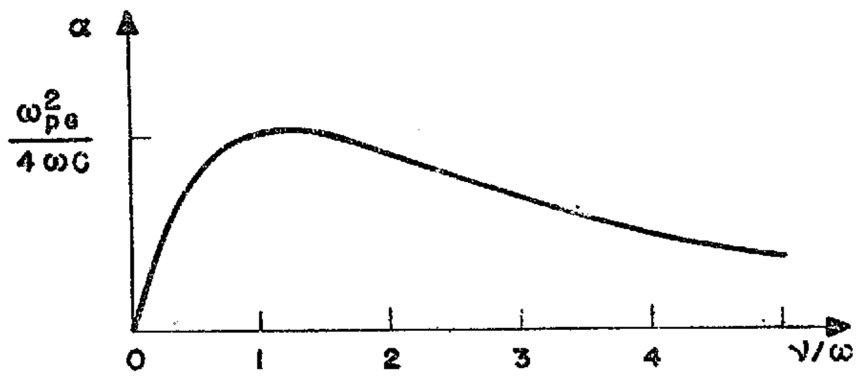

Fig. 5 - Attenuation factor, α, as a function of collision frequency for a given frequency such that $\omega \gg \omega_{pe}$, for the transverse wave.

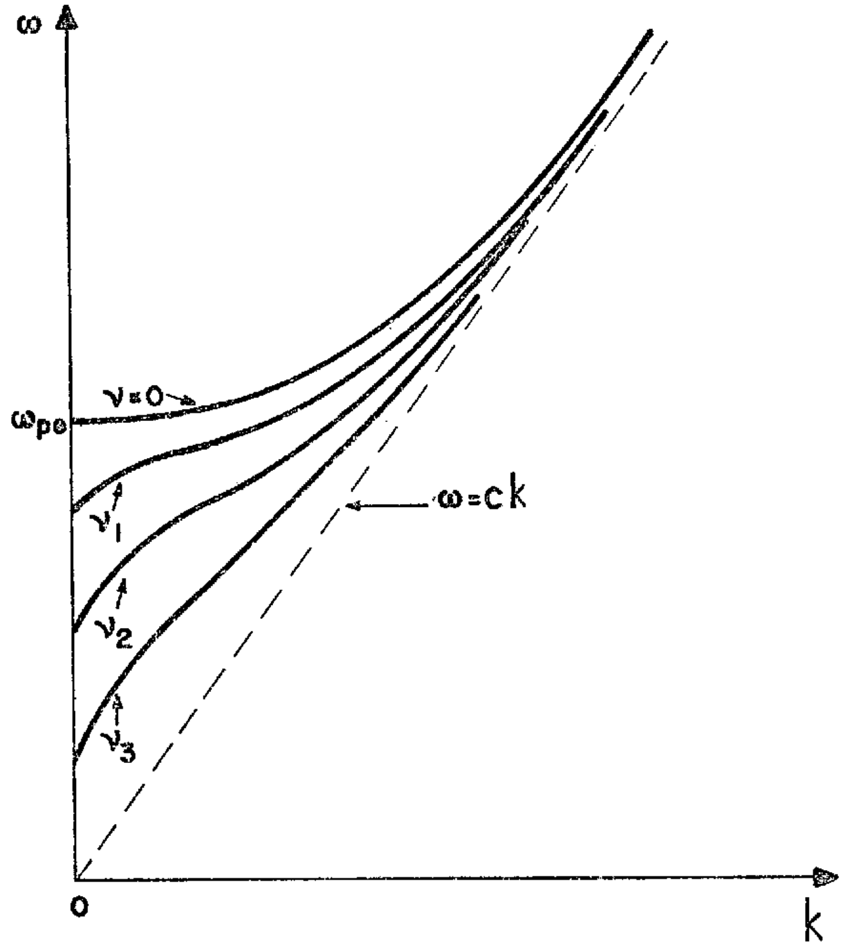

Fig. 6 - Plot of the dispersion relation, ω versus k, for transverse waves in an isotropic cold electron plasma, considering the effects of collisions ($\nu_3 > \nu_2 > \nu_1 > 0$).

in an isotropic cold electron plasma, for several values of the collision frequency, ν, such that $\nu_3 > \nu_2 > \nu_1 > 0$.

5. WAVE PROPAGATION IN ANISOTROPIC ELECTRON PLASMAS WITH $B_0 \neq 0$

We consider now the problem of wave propagation in a cold electron plasma when there is a uniform magnetostatic field externally applied. The presence of the magnetostatic field, $\underset{\sim}{B}_0$, introduces an anisotropy in the plasma.

5.1 - Derivation of the dispersion relation

To derive a dispersion relation for this case, we start from the coupled set of equations (3.4), (3.5) and (3.6). Combining (3.5) and (3.6), and rearranging, this set reduces to

$$\underset{\sim}{k} \times (\underset{\sim}{k} \times \underset{\sim}{E}) + \frac{\omega^2}{c^2} \underset{\sim}{E} = \frac{i\omega e n_0}{c^2 \varepsilon_0} \underset{\sim}{u} \tag{5.1}$$

$$(1 + \frac{i\nu}{\omega}) \underset{\sim}{u} + \frac{ie}{\omega m} (\underset{\sim}{u} \times \underset{\sim}{B}_0) = - \frac{ie}{\omega m} \underset{\sim}{E} \tag{5.2}$$

If we denote the angle between $\underset{\sim}{B}_0$ and $\underset{\sim}{k}$ by θ, and choose a Cartesian coordinate system in which $\hat{\underset{\sim}{z}}$ is in the direction of $\underset{\sim}{B}_0$, and $\hat{\underset{\sim}{y}}$ is perpendicular to the plane formed by $\underset{\sim}{B}_0$ and $\underset{\sim}{k}$ (Fig. 7), we have

$$\underset{\sim}{B}_0 = B_0 \hat{\underset{\sim}{z}} \tag{5.3}$$

$$\underset{\sim}{k} = k_\perp \hat{\underset{\sim}{x}} + k_\parallel \hat{\underset{\sim}{z}} = k \sin\theta \, \hat{\underset{\sim}{x}} + k \cos\theta \, \hat{\underset{\sim}{z}} \tag{5.4}$$

Note that the index symbols $\perp$ and $\parallel$ are used to denote components perpendicular and parallel to the direction of the magnetostatic field,

$\underset{\sim}{B}_o$, whereas the indices ℓ and t (used in the previous section) refer to components longitudinal and transverse with respect to the wave vector $\underset{\sim}{k}$, respectively.

With this choice of coordinate system, we have

$$\underset{\sim}{k} \times (\underset{\sim}{k} \times \underset{\sim}{E}) = (k_\perp \hat{\underset{\sim}{x}} + k_\parallel \hat{\underset{\sim}{z}}) \times \left[(k_\perp \hat{\underset{\sim}{x}} + k_\parallel \hat{\underset{\sim}{z}}) \times (E_x \hat{\underset{\sim}{x}} + E_y \hat{\underset{\sim}{y}} + E_z \hat{\underset{\sim}{z}}) \right]$$

$$= k^2 \cos\theta\, (\sin\theta\, E_z - \cos\theta\, E_x)\, \hat{\underset{\sim}{x}} - k^2 E_y \hat{\underset{\sim}{y}} +$$

$$+ k^2 \sin\theta\, (\cos\theta\, E_x - \sin\theta\, E_z)\, \hat{\underset{\sim}{z}} \tag{5.5}$$

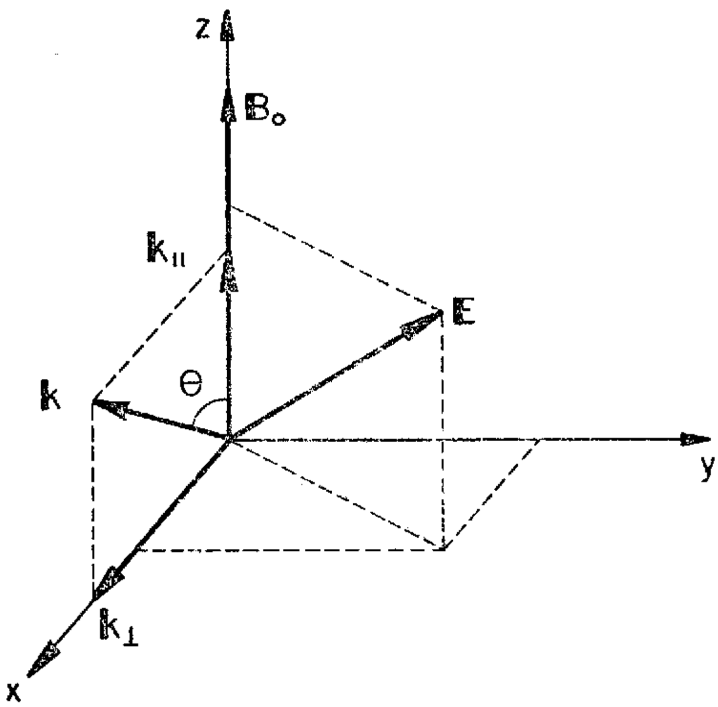

Fig. 7 - Set of retangular coordinates (x, y, z), chosen with $\hat{\underset{\sim}{z}}$ along $\underset{\sim}{B}_o$, and $\hat{\underset{\sim}{y}}$ perpendicular to the plane formed by $\underset{\sim}{B}_o$ and $\underset{\sim}{k}$.

Using this result in (5.1), we find the following relations for the x , y and z components of this equation,

$$\underset{\sim}{\hat{x}}: \quad \left(1 - \frac{k^2c^2}{\omega^2}\cos^2\theta\right)E_x + \left(\frac{k^2c^2}{\omega^2}\sin\theta\cos\theta\right)E_z = \left(\frac{ien_o}{\omega\varepsilon_o}\right)u_x \tag{5.6}$$

$$\underset{\sim}{\hat{y}}: \quad \left(1 - \frac{k^2c^2}{\omega^2}\right)E_y = \left(\frac{ien_o}{\omega\varepsilon_o}\right)u_y \tag{5.7}$$

$$\underset{\sim}{\hat{z}}: \quad \left(\frac{k^2c^2}{\omega^2}\sin\theta\cos\theta\right)E_x + \left(1 - \frac{k^2c^2}{\omega^2}\sin^2\theta\right)E_z = \left(\frac{ien_o}{\omega\varepsilon_o}\right)u_z \tag{5.8}$$

which can be written, in matrix form, as

$$\begin{pmatrix} \left(1 - \frac{k^2c^2}{\omega^2}\cos^2\theta\right) & 0 & \frac{k^2c^2}{\omega^2}\sin\theta\cos\theta \\ 0 & \left(1 - \frac{k^2c^2}{\omega^2}\right) & 0 \\ \frac{k^2c^2}{\omega^2}\sin\theta\cos\theta & 0 & \left(1 - \frac{k^2c^2}{\omega^2}\sin^2\theta\right) \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} =$$

$$= \frac{ien_o}{\omega\varepsilon_o}\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} \tag{5.9}$$

Note that the quantity kc/ω is the *index of refraction* of the medium.

Next, in order to write (5.2) also in matrix form, we first note that (Fig. 7)

$$\underset{\sim}{u} \times \underset{\sim}{B}_o = B_o \, (u_y \hat{\underset{\sim}{x}} - u_x \hat{\underset{\sim}{y}}) \tag{5.10}$$

Using this result in (5.2), and after some algebraic manipulations, we obtain for the x, y, and z components of this equation

$$\hat{\underset{\sim}{x}}: \quad (1 + \frac{i\nu}{\omega})\, u_x + (\frac{i\omega_{ce}}{\omega})\, u_x = - (\frac{ie}{\omega m}) E_x \tag{5.11}$$

$$\hat{\underset{\sim}{y}}: \; - (\frac{i\omega_{ce}}{\omega})\, u_x + (1 + \frac{i\nu}{\omega})\, u_y = - (\frac{ie}{\omega m}) E_y \tag{5.12}$$

$$\hat{\underset{\sim}{z}}: \quad (1 + \frac{i\nu}{\omega})\, u_z = - (\frac{ie}{\omega m}) E_z \tag{5.13}$$

Introducing now the notation

$$U = 1 + \frac{i\nu}{\omega} \tag{5.14}$$

$$Y = \frac{\omega_{ce}}{\omega} \tag{5.15}$$

$$X = \frac{\omega_{pe}^2}{\omega^2} \tag{5.16}$$

we can write Eqs. (5.11) to (5.13) in matrix form as

$$\begin{pmatrix} U & iY & 0 \\ -iY & U & 0 \\ 0 & 0 & U \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = - \frac{ie}{m\omega} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \qquad (5.17)$$

Inverting the 3 x 3 matrix of (5.17), and multiplying this equation by the inverted matrix, we find

$$- \left(\frac{ie}{m\omega}\right) \frac{1}{U(U^2 - Y^2)} \begin{pmatrix} U^2 & -iUY & 0 \\ iUY & U^2 & 0 \\ 0 & 0 & (U^2 - Y^2) \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} \qquad (5.18)$$

Eqs. (5.9) and (5.18) can now be combined to eliminate the velocity components u_x, u_y and u_z, yielding the following component equations involving only the electric field,

$$\underset{\sim}{\hat{x}}: \quad \left(- \frac{XU}{U^2 - Y^2} + 1 - \frac{k^2c^2}{\omega^2} \cos^2\theta \right) E_x + \left(\frac{iXY}{U^2 - Y^2}\right) E_y +$$

$$+ \left(\frac{k^2c^2}{\omega^2} \sin\theta \ \cos\theta\right) E_z = 0 \tag{5.19}$$

$$\underset{\sim}{\hat{y}}: \quad - \left(\frac{i\,X\,Y}{U^2 - Y^2}\right) E_x + \left(- \frac{XU}{U^2 - Y^2} + 1 - \frac{k^2c^2}{\omega^2}\right) E_y = 0 \tag{5.20}$$

$$\underset{\sim}{\hat{z}}: \quad \left(\frac{k^2c^2}{\omega^2} \sin\theta \cos\theta\right) E_x + \left(- \frac{X}{U} + 1 - \frac{k^2c^2}{\omega^2} \sin^2\theta\right) E_z = 0 \tag{5.21}$$

For reasons to become apparent later, it is appropriate to define the following quantities

$$S = 1 - \frac{XU}{U^2 - Y^2} \tag{5.22}$$

$$D = - \frac{XY}{U^2 - Y^2} \tag{5.23}$$

$$P = 1 - \frac{X}{U} \tag{5.24}$$

With this notation, Eqs. (5.19) to (5.21) can be written in matrix form as

$$\begin{pmatrix} \left(S - \frac{k^2c^2}{\omega^2} \cos^2\theta\right) & -iD & \frac{k^2c^2}{\omega^2} \sin\theta \cos\theta \\ iD & \left(S - \frac{k^2c^2}{\omega^2}\right) & 0 \\ \frac{k^2c^2}{\omega^2} \sin\theta \cos\theta & 0 & \left(P - \frac{k^2c^2}{\omega^2} \sin^2\theta\right) \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0 \tag{5.25}$$

In order to have a non-trivial solution ($\underset{\sim}{E} \neq 0$), the determinant of the 3 x 3 matrix in (5.25) must vanish. This condition gives the following *dispersion relation*, by direct calculation of the determinant,

$$(S \sin^2\theta + P \cos^2\theta)\left(\frac{k^2c^2}{\omega^2}\right)^2 - \left[RL \sin^2\theta + SP(1+\cos^2\theta)\right]\left(\frac{k^2c^2}{\omega^2}\right) +$$

$$+ PRL = 0 \tag{5.26}$$

where

$$R = S + D \qquad\qquad S = (R + L)/2 \tag{5.27}$$

or

$$L = S - D \qquad\qquad D = (R - L)/2 \tag{5.28}$$

Since (5.26) is a quadratic equation in k^2c^2/ω^2, there will be two solutions, that is, at each frequency there can be in general two types of waves that can propagate (two values for k^2c^2/ω^2), or, two *modes of propagation*. Note, however, that if we take the square root of k^2c^2/ω^2, we have two values for kc/ω which correspond to opposite directions of propagation and not to two different modes.

5.2 - The Appleton - Hartree equation

This well known equation is used with considerable success to study radio wave propagation in the ionosphere, taking account of the Earth's magnetic field. It is just the dispersion relation (5.26), but written in a different form. In order to obtain the Appleton-Hartree equation, we first write (5.26) as

$$A\left(\frac{k^2c^2}{\omega^2}\right)^2 - B\left(\frac{k^2c^2}{\omega^2}\right) + C = 0 \tag{5.29}$$

where

$$A = (S\sin^2\theta + P\cos^2\theta) \tag{5.30}$$

$$B = RL\sin^2\theta + SP(1+\cos^2\theta) \tag{5.31}$$

$$C = PRL \tag{5.32}$$

Solving (5.29) for k^2c^2/ω^2, we find

$$\frac{k^2c^2}{\omega^2} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} \tag{5.33}$$

Now, we add the quantity $A(k^2c^2/\omega^2)$ to both sides of (5.29) and rearrange, to obtain

$$\frac{k^2c^2}{\omega^2} = \frac{A(k^2c^2/\omega^2) - C}{A(k^2c^2/\omega^2) + A - B} \tag{5.34}$$

Next, we substitute k^2c^2/ω^2 from (5.33), into the right-hand side of (5.34) and manipulate, obtaining

$$\frac{k^2c^2}{\omega^2} = 1 - \frac{2(A - B + C)}{2A - B \pm \sqrt{B^2 - 4AC}} \tag{5.35}$$

Finally, we substitute the appropriate expressions which define the quantities A, B, C and S, D, P, R, L, to obtain

$$\frac{k^2c^2}{\omega^2} = 1 - \frac{X}{U - \dfrac{Y^2\sin^2\theta}{2(U-X)} \pm \left[\dfrac{Y^4\sin^4\theta}{4(U-X)^2} + Y^2\cos^2\theta\right]^{1/2}} \tag{5.36}$$

This is the *Appleton - Hartree equation*. It is valid for high wave frequencies as compared to the *ion cyclotron* frequency, since ion motion was neglected in the analysis presented here.

Because of the complexity of either (5.36) or (5.26), in order to simplify matters we shall first analyse the wave propagation problem when $\underset{\sim}{k}$ is either parallel to, or perpendicular to $\underset{\sim}{B}_0$. Afterwards, it will be easier to analyze some important aspects of wave propagation at an arbitrary angle θ with respect to $\underset{\sim}{B}_0$ using the dispersion relation (5.36) or (5.26).

6. PROPAGATION PARALLEL TO $\underset{\sim}{B}_0$

For wave propagation in the direction of $\underset{\sim}{B}_0$ ($\underset{\sim}{k} \parallel \underset{\sim}{B}_0$) we have $\theta = 0$, so that (5.25) simplifies to

$$\begin{pmatrix} (S - \frac{k^2c^2}{\omega^2}) & -iD & 0 \\ iD & (S - \frac{k^2c^2}{\omega^2}) & 0 \\ 0 & 0 & P \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0 \tag{6.1}$$

For a non-trivial solution ($\underset{\sim}{E} \neq 0$), we must require the determinant of the 3 x 3 matrix in (6.1) to vanish. Thus, by direct calculation of the determinant we find three independent conditions

$$P = 0 \ ; \ (\underset{\sim}{E}_{\parallel} \equiv \underset{\sim}{E}_{\ell} \neq 0) \tag{6.2}$$

$$(\frac{k^2c^2}{\omega^2}) = S + D \equiv R \ ; \ (\underset{\sim}{E}_{\perp} \equiv \underset{\sim}{E}_t \neq 0) \tag{6.3}$$

$$(\frac{k^2c^2}{\omega^2}) = S - D \equiv L \ ; \ (\underset{\sim}{E}_{\perp} \equiv \underset{\sim}{E}_t \neq 0) \tag{6.4}$$

Using Eqs. (5.22) to (5.24), and (5.14) to (5.16), which define S, D, P, and U, Y, X, respectively, we obtain from (6.2), neglecting collisions ($\nu = 0$),

$$\omega^2 = \omega_{pe}^2 \tag{6.5}$$

which corresponds to the *longitudinal electron plasma oscillations* discussed previously in section 4. Thus, these oscillations are not affected by the presence of the magnetostatic field, $\underset{\sim}{B}_o$, in the direction of the oscillations (z-axis). Since there is no wave propagation in this case, these plasma oscillations do not constitute a mode of propagation.

Eq. (6.3) corresponds to *transverse right-hand circularly polarized waves* (RCP), with the dispersion relation

$$\left(\frac{k^2c^2}{\omega^2}\right)_R = 1 - \frac{X}{(U - Y)} = R \tag{6.6}$$

or, neglecting collisions ($\nu = 0$),

$$\left(\frac{k^2c^2}{\omega^2}\right)_R = 1 - \frac{\omega_{pe}^2}{\omega(\omega - \omega_{ce})} \tag{6.7}$$

Eq. (6.4) corresponds to *transverse left-hand circularly polarized waves* (LCP), with the dispersion relation

$$\left(\frac{k^2c^2}{\omega^2}\right)_L = 1 - \frac{X}{(U + Y)} = L \tag{6.8}$$

or, neglecting collisions ($\nu = 0$),

$$\left(\frac{k^2c^2}{\omega^2}\right)_L = 1 - \frac{\omega_{pe}^2}{\omega(\omega + \omega_{ce})} \tag{6.9}$$

The *polarization* of these two modes of propagation can be obtained from the x-component of (6.1), which gives

$$\frac{E_x}{E_y} = \frac{iD}{(S - k^2c^2/\omega^2)} \tag{6.10}$$

Thus, for the *RCP wave*, substituting $k^2c^2/\omega^2 = R$,

$$\frac{E_x}{E_y} = -i \tag{6.11}$$

whereas for the *LCP wave*, substituting $k^2c^2/\omega^2 = L$,

$$\frac{E_x}{E_y} = i \tag{6.12}$$

Since the time dependence of $\underset{\sim}{E}$ is of the form $\exp(-i\omega t)$, if we take $E_x \propto \cos(\omega t)$ then for the RCP wave we have $E_y \propto \sin(\omega t)$, whereas for the LCP wave we have $E_y \propto -\sin(\omega t)$. Therefore, for an observer looking at the outgoing wave, as time passes the transverse electric field vector $\underset{\sim}{E}_t$ rotates in the clockwise direction for the RCP wave, and in the counterclockwise direction for the LCP wave. This is illustrated in Fig. 8. Note that the RCP wave rotates in the same direction as the electrons about the $\underset{\sim}{B}_0$ field. This means that, when $\omega = \omega_{ce}$, the RCP wave is in *resonance* with the cyclotron motion of the electrons, and therefore energy is transferred from the wave to the electrons. This

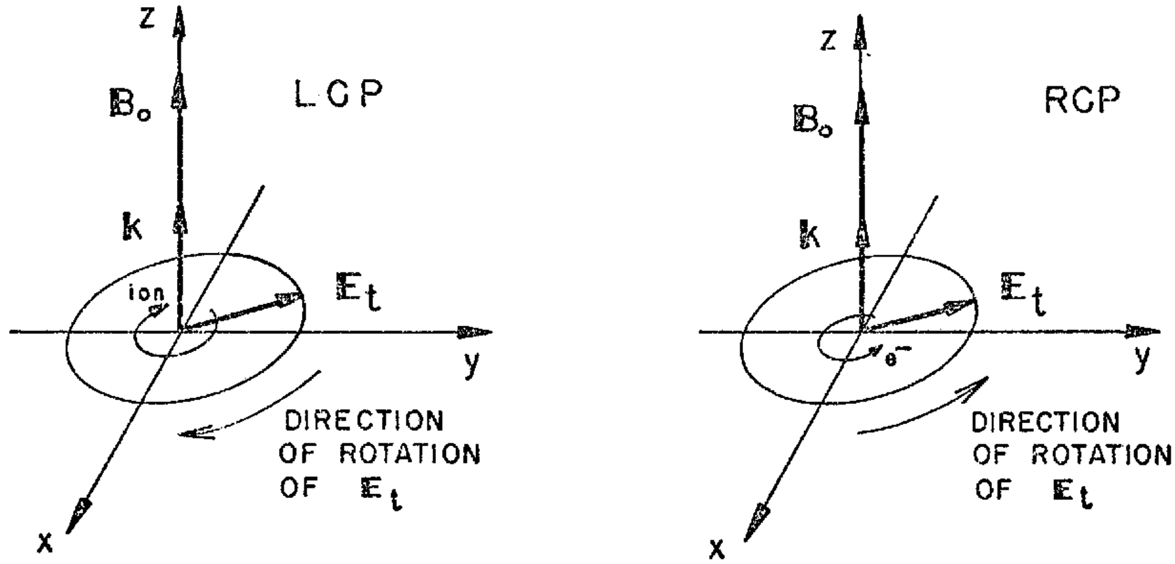

Fig. 8 - For propagation along the magnetostatic field ($\theta = 0$), the LCP wave rotates in the counterclockwise direction, and the RCP wave rotates in the clockwise direction, for an observer looking at the outgoing wave.

absorption of energy by the electrons, from the RCP electromagnetic wave, at the electron cyclotron frequency, is used as a means of heating the plasma electrons. When the motion of the ions is taken into account, a resonance exists between the cyclotron motion of the ions and the LCP wave, at $\omega = \omega_{ci}$, since the ions gyrate in the same direction as the $\underset{\sim}{E}_t$ vector of the LCP wave.

The phenomenon of *resonance* occurs when the phase velocity goes to zero, $v_{ph} = 0$ (or $kc/\omega \to \infty$), whereas *reflection* occurs when $v_{ph} \to \infty$ (or $kc/\omega = 0$). Thus, it is clear from (6.7) and from the physical argument just given that the RCP wave has a resonance at $\omega = \omega_{ce}$, whereas (6.9) indicates no resonance for the LCP wave (for the case when ion motion is included the LCP wave has a resonance at ω_{ci}). Also, from (6.9) it is easily verified that the LCP wave has a reflection point when

$$\omega = \omega_{01} \equiv \frac{1}{2}\left(-\omega_{ce} + \sqrt{\omega_{ce}^2 + 4\omega_{pe}^2}\right) \qquad (L = 0) \tag{6.13}$$

and, from (6.7), the RCP wave has a reflection point when

$$\omega = \omega_{02} \equiv \frac{1}{2}\left(\omega_{ce} + \sqrt{\omega_{ce}^2 + 4\omega_{pe}^2}\right)$$

$$= \omega_{01} + \omega_{ce} \qquad (R = 0) \tag{6.14}$$

The *phase velocity* of the *LCP wave* is obtained, from (6.9), as

$$(v_{ph})_L = \left(\frac{\omega}{k}\right)_L = \frac{c(1 + \omega_{ce}/\omega)^{1/2}}{(1 + \omega_{ce}/\omega - \omega_{pe}^2/\omega^2)^{1/2}} \qquad (\omega > \omega_{01}) \tag{6.15}$$

For $\omega < \omega_{01}$, the wave number k is imaginary and the LCP wave is evanescent. Thus, the LCP wave propagates only for $\omega > \omega_{01}$.

Similarly, the *phase velocity* of the *RCP wave* is obtained, from (6.7), as

$$(v_{ph})_R = \left(\frac{\omega}{k}\right)_R = \frac{c(1 - \omega_{ce}/\omega)^{1/2}}{(1 - \omega_{ce}/\omega - \omega_{pe}^2/\omega^2)^{1/2}}; \quad (\omega < \omega_{ce};\ \omega > \omega_{02}) \tag{6.16}$$

Thus, the RCP wave propagates in two frequency ranges: $0 < \omega < \omega_{ce}$ and $\omega_{02} < \omega < \infty$; it is evanescent for $\omega_{ce} < \omega < \omega_{02}$.

The *group velocity* for the LCP and RCP waves in the propagation bands are given, respectively, by

$$(v_g)_L = \left(\frac{\partial\omega}{\partial k}\right)_L = \frac{2c(\omega + \omega_{ce})^{3/2}\left[\omega(\omega^2 + \omega\omega_{ce} - \omega_{pe}^2)\right]^{1/2}}{2\omega(\omega + \omega_{ce})^2 - \omega_{ce}\,\omega_{pe}^2} \qquad (6.17)$$

$$(v_g)_R = \left(\frac{\partial\omega}{\partial k}\right)_R = \frac{2c(\omega - \omega_{ce})^{3/2}\left[\omega(\omega^2 + \omega\omega_{ce} - \omega_{pe}^2)\right]^{1/2}}{2\omega(\omega - \omega_{ce})^2 + \omega_{ce}\,\omega_{pe}^2} \qquad (6.18)$$

A plot of phase velocity and group velocity as a function of frequency for these two transverse modes of propagation is shown in Fig. 9. The same dispersion relations (6.7) and (6.9) are plotted, in a different form, in Figs. 10 and 11, respectively, where it is shown the frequency, ω, as a function of the real part of the wave number, k. The frequency bands for which there is no wave propagation are indicated.

The RCP waves in the lower branch which have $\omega \leq \omega_{ce}$ are commonly known as *electron cyclotron waves*. Similarly, when ion motion is taken into account, the LCP mode has also a lower branch of propagation for $0 < \omega < \omega_{ci}$ with a resonance at ω_{ci}. The LCP waves having $\omega \leq \omega_{ci}$ are commonly known as *ion cyclotron waves*.

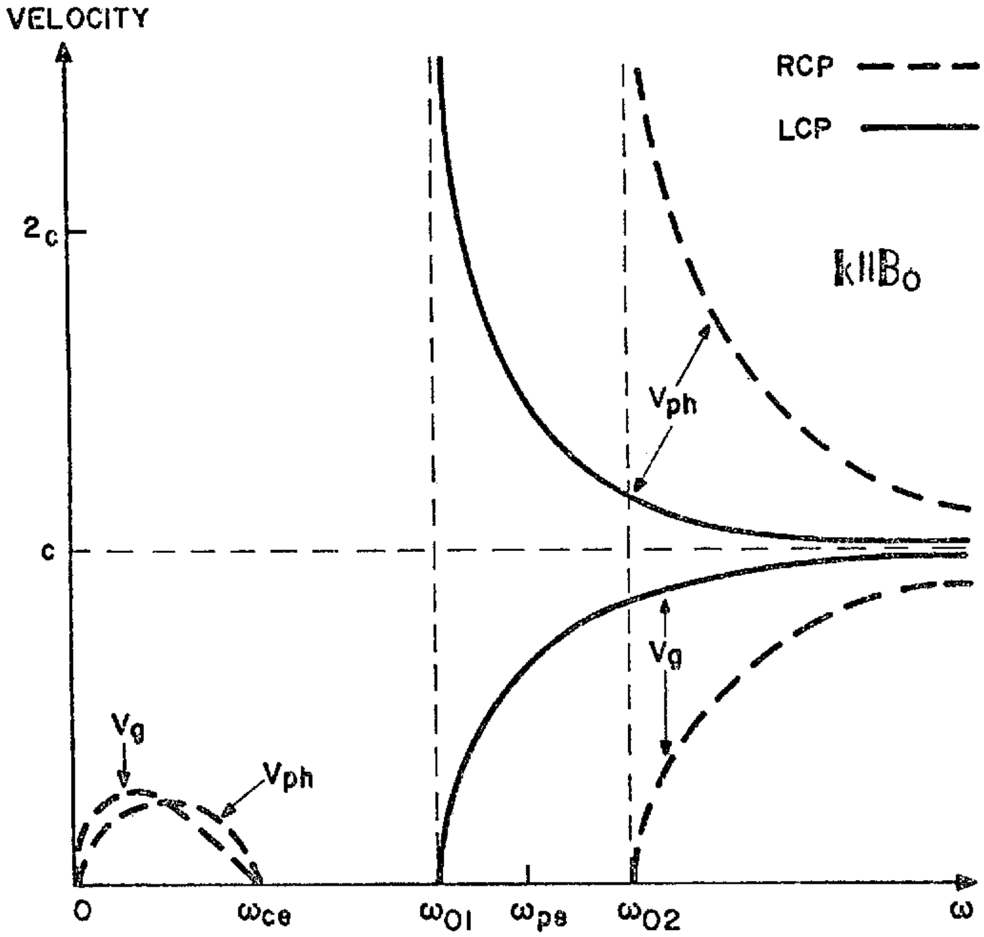

Fig. 9 - Phase velocity and group velocity as a function of frequency for the transverse RCP and LCP waves propagating along the magnetostatic field ($\underset{\sim}{k} \parallel \underset{\sim}{B}_o$).

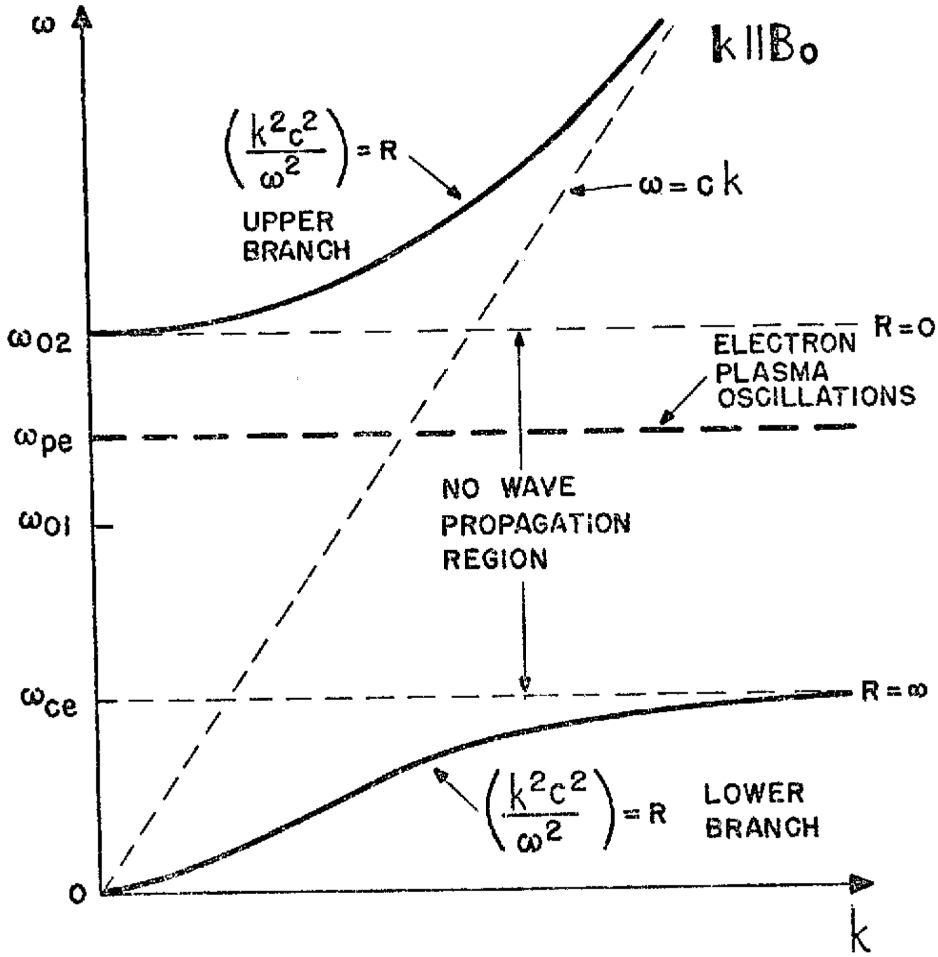

Fig. 10 - Dispersion plot for the RCP wave propagating along the magnetostatic field ($\underset{\sim}{k} \parallel \underset{\sim}{B}_o$).

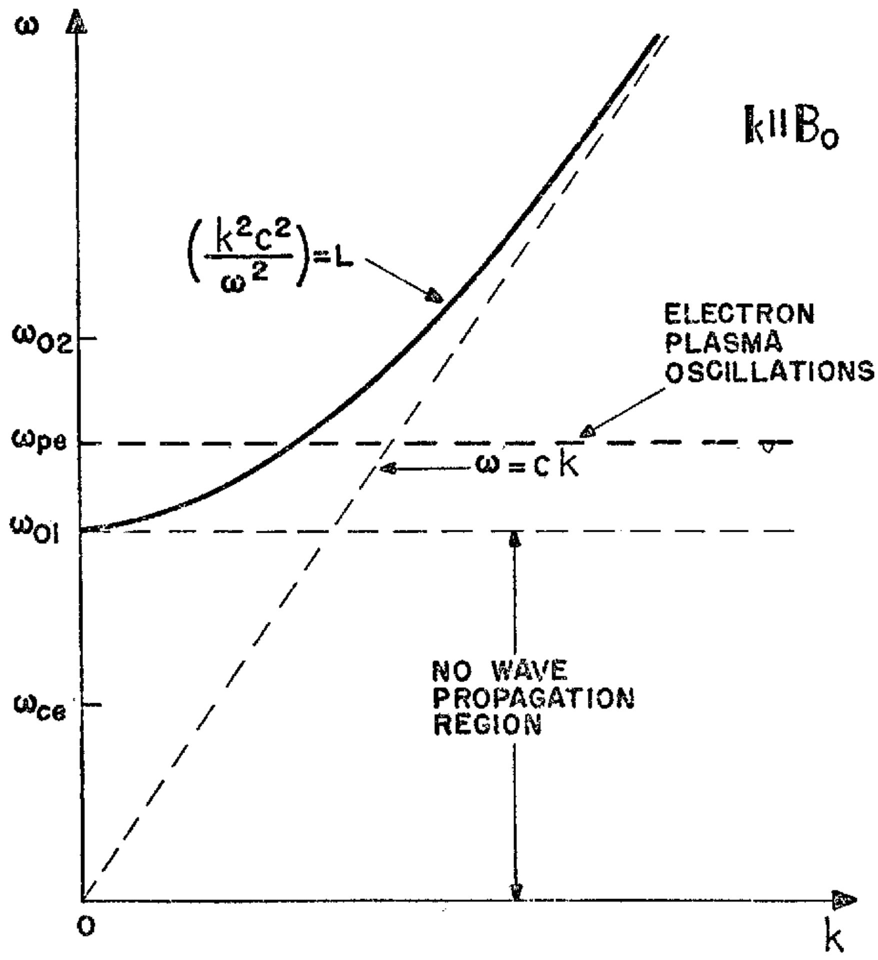

Fig. 11 - Dispersion plot for the LCP wave propagating along the magnetostatic field ($\underset{\sim}{k} \parallel \underset{\sim}{B}_o$).

7. PROPAGATION PERPENDICULAR TO $\underset{\sim}{B}_0$

We consider now wave propagation in the direction perpendicular to $\underset{\sim}{B}_0$ ($\underset{\sim}{k} \perp \underset{\sim}{B}_0$). For $\theta = 90^0$, (5.25) simplifies to

$$\begin{pmatrix} S & -iD & 0 \\ iD & (S - \frac{k^2c^2}{\omega^2}) & 0 \\ 0 & 0 & (P - \frac{k^2c^2}{\omega^2}) \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0 \tag{7.1}$$

Again , for a non-trivial solution ($\underset{\sim}{E} \neq 0$) the determinant of the 3×3 matrix in (7.1) must vanish. Direct calculation of this determinant yields the following two independent modes of propagation:

$$\left(\frac{k^2c^2}{\omega^2}\right)_0 = P \qquad (\underset{\sim}{E}_{\parallel} \neq 0) \tag{7.2}$$

$$\left(\frac{k^2c^2}{\omega^2}\right)_X = \frac{RL}{S} \qquad (\underset{\sim}{E}_{\perp} \neq 0) \tag{7.3}$$

The indices 0 and X refer to the *ordinary* and *extraordinary* modes, respectively, as will be explained shortly.

From (7.2), and using (5.24), we obtain the dispersion relation

$$\left(\frac{k^2c^2}{\omega^2}\right)_0 = 1 - \frac{X}{U} \tag{7.4}$$

or using (5.14) and (5.16), neglecting collisions ($\nu = 0$),

$$\left(\frac{k^2c^2}{\omega^2}\right)_0 = 1 - \frac{\omega_{pe}^2}{\omega^2} \tag{7.5}$$

This relation is identical to that in (4.12) for *transverse waves* in an isotropic plasma. Hence, this mode of propagation is not affected by the presence of the magnetic field $\tilde{B}_0$ and, for this reason, it is called an *ordinary wave.* For this mode propagating perpendicular to $\tilde{B}_0$, the electric field of the wave ($\tilde{E}_\parallel \neq 0$) is parallel to $\tilde{B}_0$, so that it involves electron velocities solely in the direction of $\tilde{B}_0$. Consequently, the magnetic force term $\tilde{u} \times \tilde{B}_0$ is zero and the wave propagates as if $\tilde{B}_0$ were zero. The *ordinary mode* is also called a *TEM (Transverse Electric-Magnetic) mode,* since both the electric and the magnetic fields are transverse to the direction of propagation ($\tilde{E}_\parallel \perp \tilde{k}$, $\tilde{B} \perp \tilde{k}$; see Fig. 12). The electric field is *linearly polarized* in the direction of the $\tilde{B}_0$ field.

The other mode of propagation ($\tilde{E}_\perp \neq 0$) is called the *extraordinary mode,* since it depends on the $\tilde{B}_0$ field, with the dispersion relation given by (7.3),

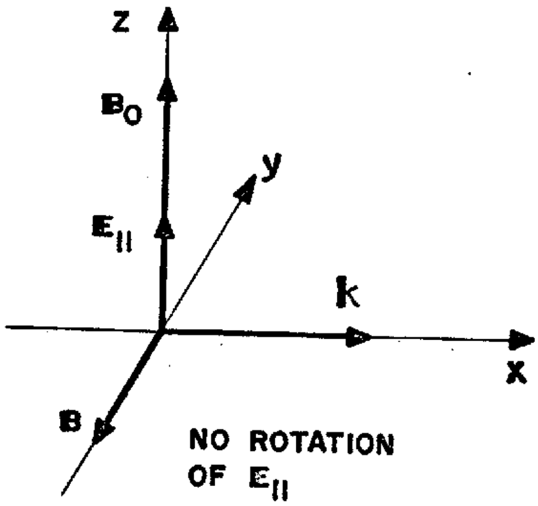

Fig. 12 - Vector diagram for the ordinary wave propagating perpendicular to $\underset{\sim}{B}_o$ ($\theta = \pi/2$).

$$\left(\frac{k^2c^2}{\omega^2}\right)_X = \frac{RL}{S} = \frac{1}{\left[1 - XU/(U^2 - Y^2)\right]}\left[\left(1 - \frac{XU}{U^2 - Y^2}\right)^2 - \left(\frac{XY}{U^2 - Y^2}\right)^2\right] \tag{7.6}$$

or, using (5.14), (5.15) and (5.16), after neglecting collisions ($\nu = 0$),

$$\left(\frac{k^2c^2}{\omega^2}\right)_X = \frac{(\omega^2 + \omega\,\omega_{ce} - \omega_{pe}^2)\ (\omega^2 - \omega\,\omega_{ce} - \omega_{pe}^2)}{\omega^2(\omega^2 - \omega_{ce}^2 - \omega_{pe}^2)}$$

$$= \frac{(\omega^2 - \omega_{01}^2)\,(\omega^2 - \omega_{02}^2)}{\omega^2(\omega^2 - \omega_{UH}^2)} \tag{7.7}$$

where ω_{01} and ω_{02} are given by (6.13) and (6.14), respectively, and where ω_{UH} denotes the *upper hybrid frequency*, defined by

$$\omega_{UH} = (\omega_{pe}^2 + \omega_{ce}^2)^{1/2} \tag{7.8}$$

For the *extraordinary mode*, the electric field ($\underset{\sim}{E}_\perp \neq 0$) of the wave has in general a longitudinal component (along $\underset{\sim}{k}$) and a transverse component (normal to $\underset{\sim}{k}$), as shown in Fig. 13. Hence, these waves are

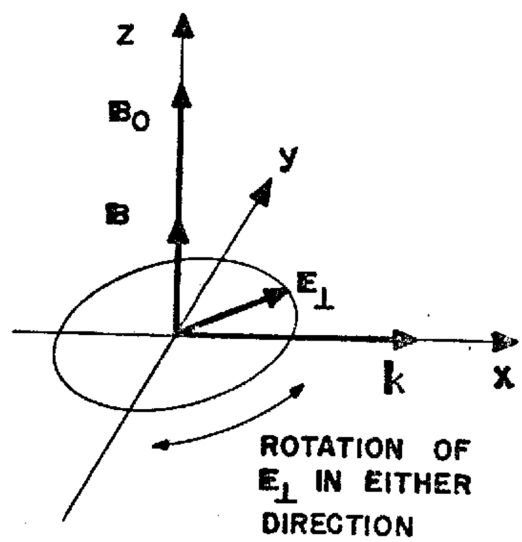

Fig. 13 - Vector diagram for the extraordinary wave propagating perpendicular to $\underset{\sim}{B}_0$ ($\theta = \pi/2$).

partially longitudinal and *partially transverse*. From (7.1), the *polarization* of the extraordinary mode is determined by

$$\frac{E_x}{E_y} = i\,\frac{D}{S} \tag{7.9}$$

so that this mode is in general *elliptically polarized*. The extraordinary mode is also called a *TM (Transverse Magnetic) mode*, since the magnetic field of this wave is transverse to the direction of propagation (Fig. 13).

From (7.5) it is clear that the *ordinary wave* has a *reflection point* ($v_{ph} \to \infty$ or $kc/\omega = 0$) at $\omega = \omega_{pe}$, and no *resonances* ($v_{ph} = 0$ or $kc/\omega \to \infty$). For the *extraordinary wave*, (7.7) indicates a *resonance* at the upper hybrid frequency $\omega_{UH} = (\omega_{pe}^2 + \omega_{ce}^2)^{1/2}$, and *reflection points* at ω_{01} and ω_{02} (when ion motion is included it turns out that the extraordinary wave has also a resonance at the *lower hybrid frequency*, given approximatelly by $\omega_{LH}^2 \simeq \omega_{ce}\,\omega_{ci}$). The dispersion plot for the ordinary wave is the same as that presented in Fig. 3 for the transverse wave in an isotropic plasma. For the extraordinary mode, the dispersion plot shown in Fig. 14 (in terms of ω as a function of the real part of k) indicates that there is wave propagation only for $\omega > \omega_{02}$ and for ω in a band of frequencies between ω_{01} and ω_{UH}; for other frequencies k is imaginary and the phase velocity is infinite.

The *phase velocities* of the ordinary and extraordinary waves are obtained from (7.5) and (7.7), respectively, as

$$(v_{ph})_0 = \left(\frac{\omega}{k}\right)_0 = \frac{c}{(1 - \omega_{pe}^2/\omega^2)^{1/2}} \quad ; \quad (\omega > \omega_{pe}) \qquad (7.10)$$

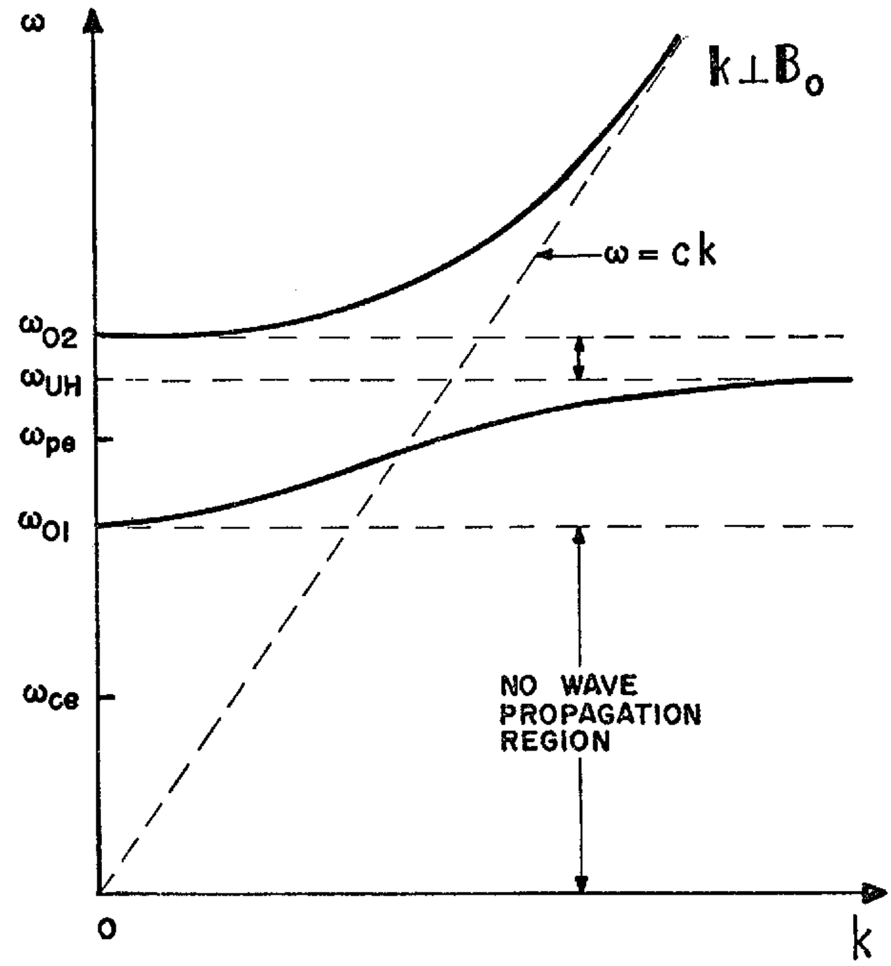

Fig. 14 - Dispersion relation for the extraordinary wave (k^2c^2/ω^2 = RL/S) propagating perpendicular to the magnetostatic field ($\underset{\sim}{k} \perp \underset{\sim}{B}_o$).

$$(v_{ph})_X = \left(\frac{\omega}{k}\right)_X = c\left[\frac{\omega^2(\omega^2-\omega_{UH}^2)}{(\omega^2-\omega_{01}^2)(\omega^2-\omega_{02}^2)}\right]^{1/2} \quad ; \ (\omega > \omega_{02} \ ; \ \omega_{01} < \omega < \omega_{UH})$$

(7.11)

Expressions for the *group velocities* of these two modes can be derived with the help of (7.5) and (7.7),

$$(v_g)_0 = \left(\frac{\partial\omega}{\partial k}\right)_0 = c\,(1-\omega_{pe}^2/\omega^2)^{1/2} \quad ; \quad (\omega > \omega_{pe}) \tag{7.12}$$

$$(v_g)_X = \left(\frac{\partial\omega}{\partial k}\right)_X = \frac{c\,(\omega^2-\omega_{UH}^2)^2}{\omega\left[\omega^4 - 2\omega^2(\omega_{ce}^2+\omega_{pe}^2)+\omega_{ce}^4+3\omega_{ce}^2\,\omega_{pe}^2+\omega_{pe}^4\right]} \,\cdot$$

$$\cdot\left[\frac{(\omega^2-\omega_{01}^2)\,(\omega^2-\omega_{02}^2)}{(\omega^2-\omega_{UH}^2)}\right]^{1/2} \quad ; \ (\omega > \omega_{02} \ ; \omega_{01} < \omega < \omega_{UH})$$

(7.13)

The plot of phase velocity and group velocity vs. frequency for the extraordinary (TM) mode has the form depicted in Fig. 15. A similar plot for the ordinary (TEM) mode is shown in Fig. 2 (the same one for the transverse wave in an isotropic plasma).

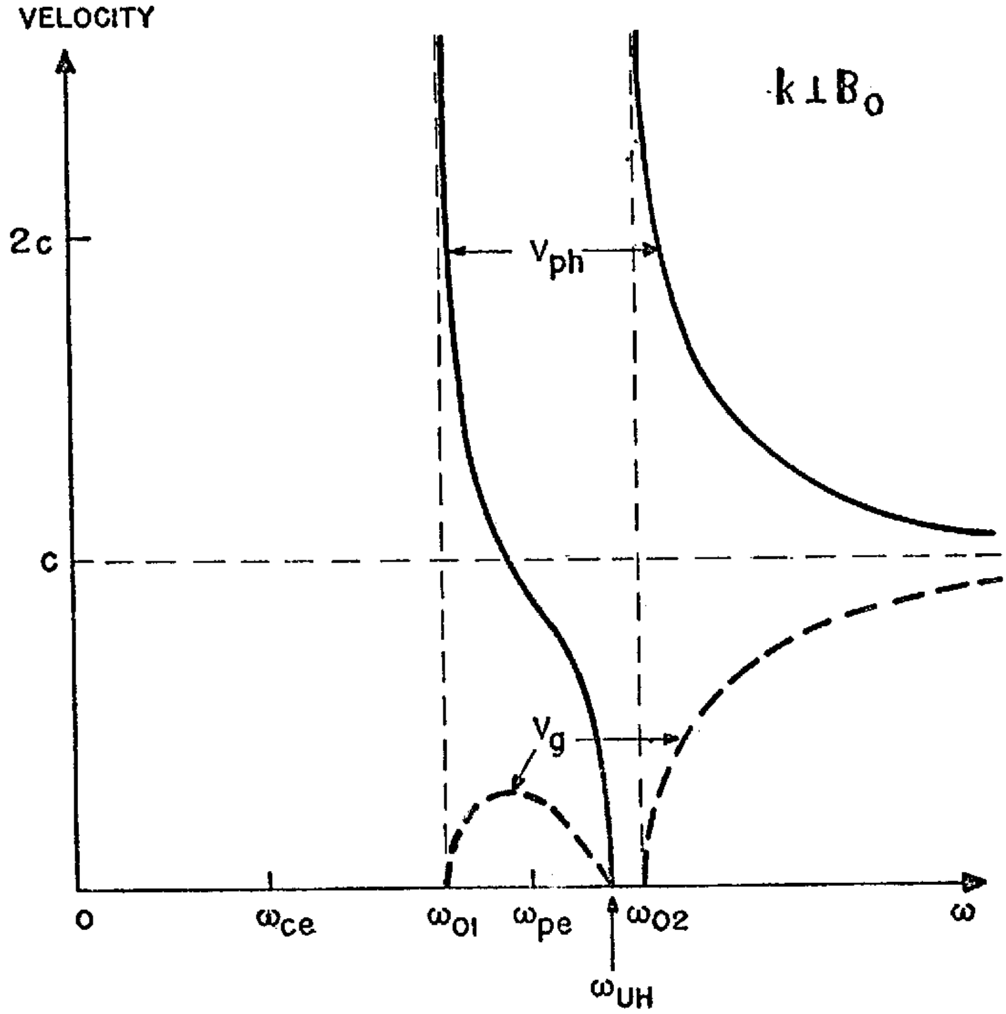

Fig. 15 - Phase velocity and group velocity as a function of frequency for the extraordinary (TM) mode propagating perpendicular to the magnetic field ($\underset{\sim}{k} \perp \underset{\sim}{B}_o$).

8. PROPAGATION AT ARBITRARY DIRECTIONS

8.1 - Resonances and reflection points

Going back now to Eq. (5.26), we shall first determine the resonances ($v_{ph} = 0$ or $kc/\omega \to \infty$) and the reflection points ($v_{ph} \to \infty$ or $kc/\omega = 0$) for arbitrary angles of propagation with respect to $\underline{B}_0$. From (5.33) and (5.30) it is seen that *resonance* occurs when

$$S \sin^2 \theta + P \cos^2 \theta = 0 \tag{8.1}$$

or

$$\tan^2 \theta = - \frac{P}{S} \tag{8.2}$$

Using (5.22) and (5.24), and neglecting collisions ($\nu = 0$), (8.2) yields

$$1 - X = Y^2(1 - X \cos^2 \theta) \tag{8.3}$$

or, using (5.15) and (5.16),

$$\omega^4 - \omega^2(\omega_{pe}^2 + \omega_{ce}^2) + \omega_{pe}^2\omega_{ce}^2\cos^2\theta = 0 \tag{8.4}$$

Thus, the *resonance frequencies* as functions of θ are given by

$$\omega_{0\pm}^2 = \frac{\omega_{pe}^2 + \omega_{ce}^2}{2} \pm \left[\left(\frac{\omega_{pe}^2 + \omega_{ce}^2}{2}\right)^2 - \omega_{pe}^2\, \omega_{ce}^2 \cos^2\theta \right]^{1/2} \qquad (8.5)$$

These two resonance frequencies are plotted against the angle θ in Fig. 16. From (8.5) it is clear that the sum of the square of these two frequencies $(\omega_{0+}^2 + \omega_{0-}^2)$ is always equal to $(\omega_{pe}^2 + \omega_{ce}^2)$ for any angle θ. From Fig. 16 we see that the high-frequency resonance increases with increasing θ, from the larger of ω_{pe} and ω_{ce}, at $\theta = 0^0$, to the upper hybrid resonance frequency, $(\omega_{pe}^2 + \omega_{ce}^2)^{1/2}$, at $\theta = 90^0$. The low-frequency resonance decreases correspondingly from the frequency which is the smaller between ω_{pe} and ω_{ce}, at $\theta = 0^0$, to zero, at $\theta = 90^0$. The resonances at $\theta = 0^0$ and $\theta = 90^0$ are called the *principal resonances*. At $\theta = 0^0$ the principal resonances are given by $S = \infty$ and $P = 0$ (Eq.8.2); at $\theta = 90^0$ the principal resonance is given by $S = 0$.

The *reflection points* are seen, from (5.26), to be given by

$$PRL = 0 \qquad (8.6)$$

This equation is satisfied whenever $P = 0$, or $R = 0$, or $L = 0$. However, for $\theta = 0^0$ (5.26) simplify to

$$\left(\frac{k^2c^2}{\omega^2}\right)^2 - 2S\left(\frac{k^2c^2}{\omega^2}\right) + RL = 0 \qquad (8.7)$$

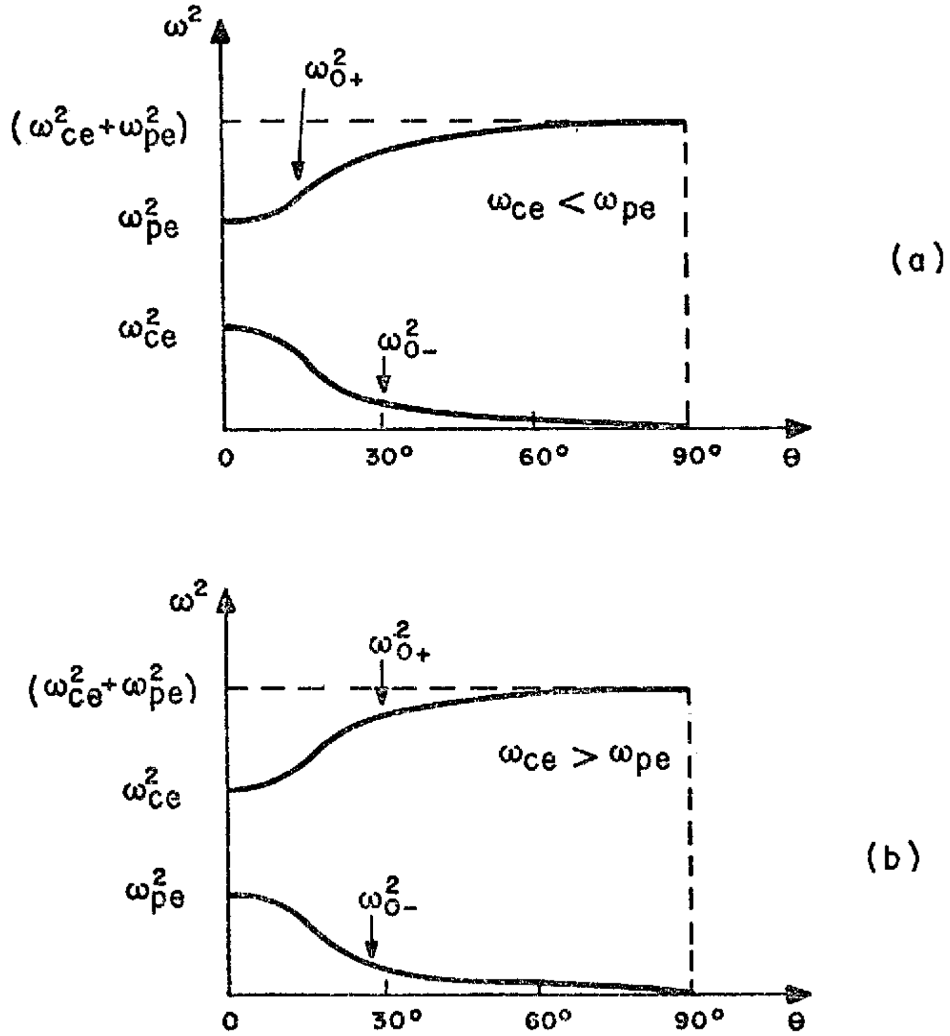

Fig. 16 - Resonance frequencies as functions of the angle θ between $\underset{\sim}{B}_o$ and the direction of wave propagation in a cold electron plasma, for (a) $\omega_{ce} < \omega_{pe}$ and (b) $\omega_{ce} > \omega_{pe}$.

so that P = 0 is no longer a reflection point for $\theta = 0^0$. Thus, for propagation exactly along the $\underset{\sim}{B}$ field the reflection points are given by R = 0 and L = 0 . But for $\theta \neq 0^0$, irrespective of how small θ is, P = 0 also corresponds to a reflection point. Note that these cut-off frequencies are, otherwise, *independent* of θ. Therefore, as we have seen in section 6, they are given by (neglecting collisions) $\omega = \omega_{pe}$ ($P = 0$; $\theta \neq 0^0$), $\omega = \omega_{01}$ [L = 0; see Eq.(6.13)] and $\omega = \omega_{02}$ [R = 0; see Eq. (6.14)]. The cut-off frequencies and the principal resonances are summarized in Table 1.

Expressions for the phase velocity and group velocity for arbitrary angles of propagation can be obtained from the dispersion relation (5.26) or (5.36). Since this involves considerable algebra, they will not be presented here. For this case, the curves of k, v_{ph} , and v_g , as functions of ω, must lie somewhere between the corresponding curves for $\theta = 0^0$ (see Figs. 9, 10 and 11) and for $\theta = 90^0$ (see Figs. 2, 3, 14 and 15). If the angle θ is continuously changed from 0^0 to 90^0, then the curves for $\theta = 0^0$ must change continuously into those for $\theta = 90^0$.

TABLE 1

CUT-OFFS AND PRINCIPAL RESONANCES FOR WAVES

IN A COLD ELECTRON PLASMA

Cut-offs	Principal Resonances	
	$\theta = 0^0$	$\theta = 90^0$
$P = 0$ $(\theta \neq 0^0)$ $R = 0$ $L = 0$	$P = 0$ $\left.\begin{array}{l} R = \infty \\ L = \infty \end{array}\right\} S = \infty$	$S = 0$

Fig. 17 shows ω as a function of the real part of k, while Fig. 18 shows v_{ph} and v_g as functions of ω, for the two modes of propagation at an angle $\theta = 45^0$ with respect to $\underset{\sim}{B}_o$. It is interesting to note that the branch of mode 2, that propagates for $\omega_{01} < \omega < \omega_{0+}$, and the branch of mode 1, that propagates for $\omega > \omega_{pe}$, are transformed, as θ goes to zero, into the LCP wave and the electron plasma oscillations at ω_{pe}. This is indicated in Fig. 19.

Fig. 20 is a plot of the phase velocity versus frequency illustrating how the two modes of wave propagation when $\theta = 0^0$ (left circularly polarized wave and right circularly polarized

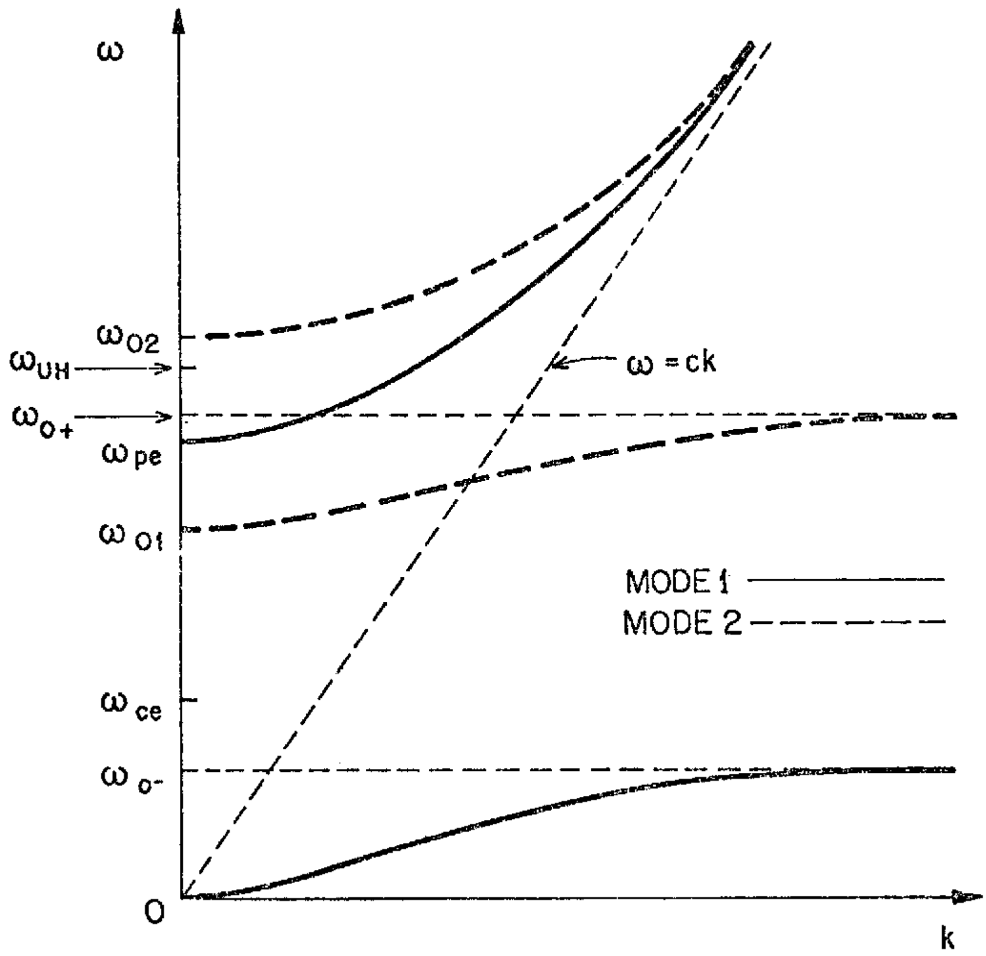

Fig. 17 - Dispersion relation for the two modes of propagation at an angle $\theta = 45^{0}$ with respect to the magnetostatic field in a cold electron plasma.

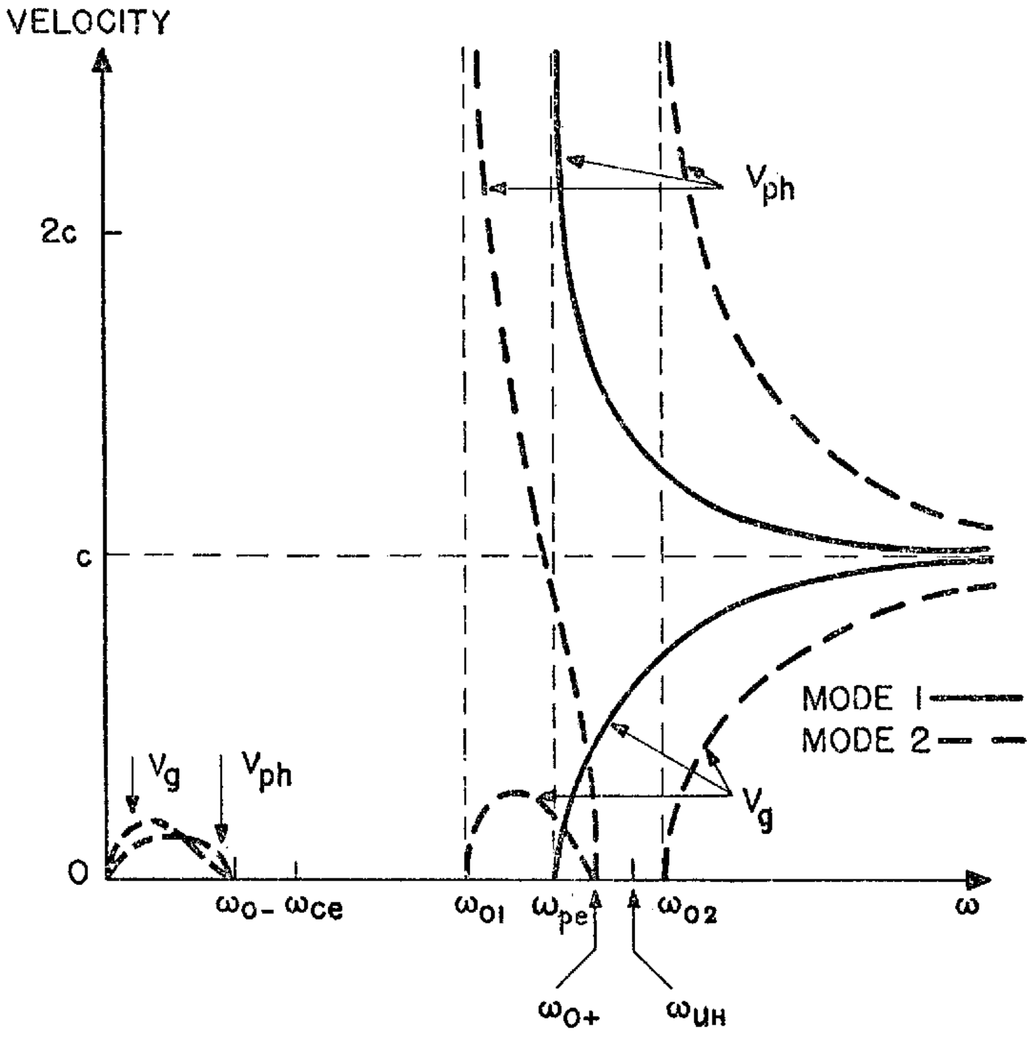

Fig. 18 - Phase velocity (v_{ph}) and group velocity (v_g) as a function of frequency for the two modes of propagation at an angle $\theta = 45^0$ with respect to the magnetostatic field in a cold electron plasma.

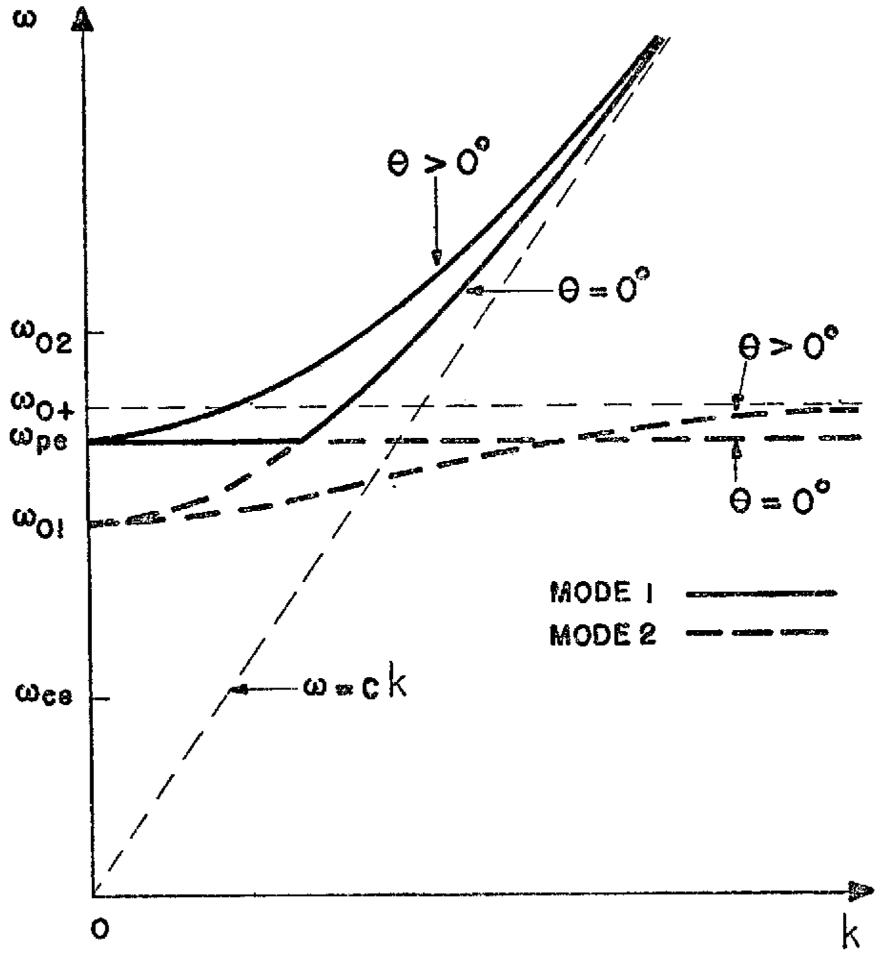

Fig. 19 - Illustrating how the branch $\omega_{01} < \omega < \omega_{0+}$ of mode 2 and the branch $\omega > \omega_{pe}$ of mode 1, for $\theta > 0^o$, are related to the LCP wave and the electron plasma oscillations when $\theta = 0^o$.

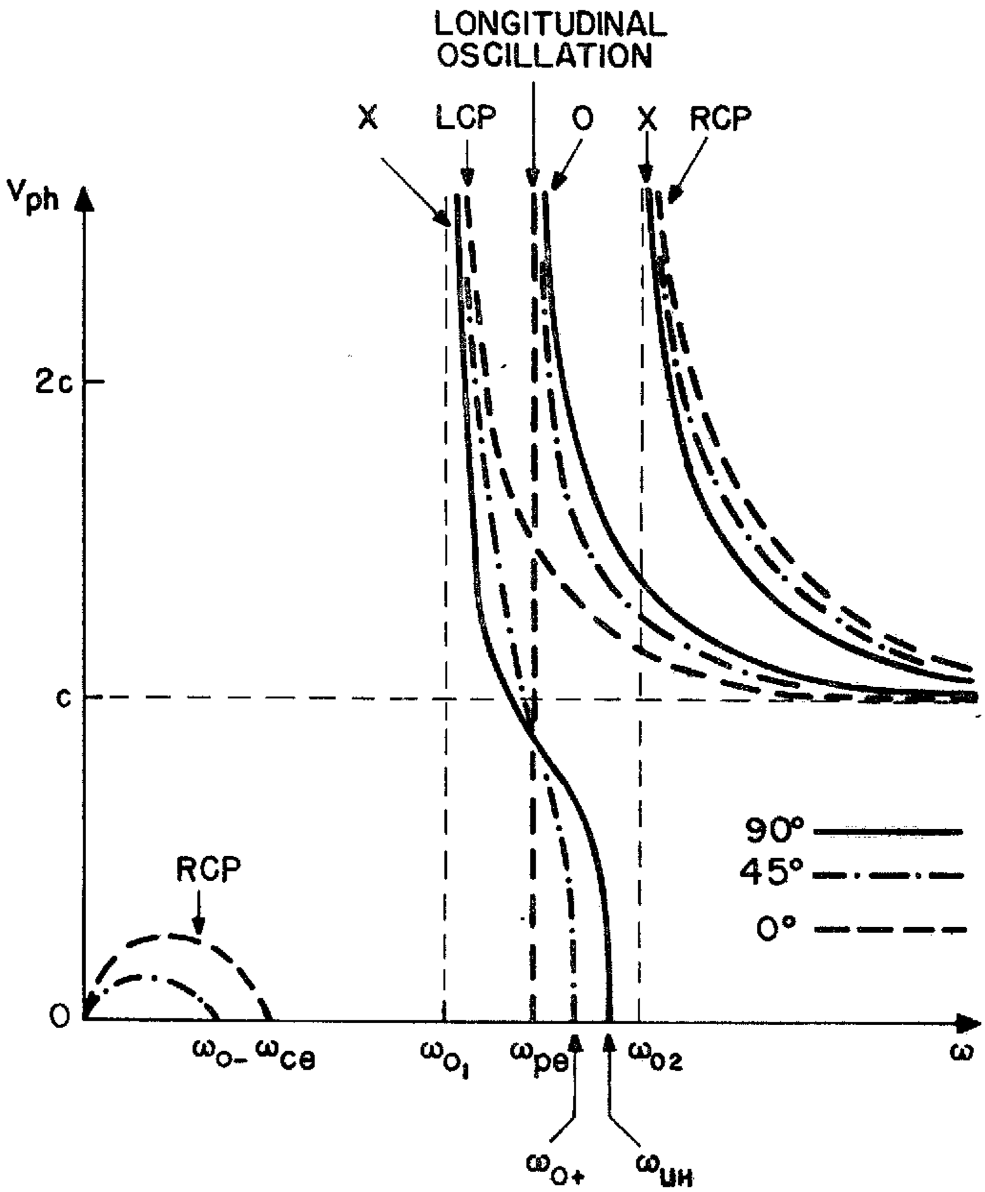

Fig. 20 - Phase velocity versus frequency for waves in a cold electron plasma, illustrating how the two modes of propagation for $\theta = 0^0$ (LCP and RCP) evolute into the two modes for $\theta = 90^0$ (O and X).

wave) evolve into the two modes of wave propagation when $\theta = 90^{o}$ (ordinary wave and extraordinary wave).

8.2 - Wave normal surfaces

The *wave normal surface*, also known as the normalized *phase velocity surface*, is a polar plot of v_{ph}/c as a function of θ. Because of the symmetry in the azimuthal angle ϕ, it is a surface of revolution about $\underset{\sim}{B}_0$. For any direction of propagation, the "length" (properly normalized) of the line drawn from the origin to intersect this surface is v_{ph}/c. This surface is, therefore, the loci of points of constant phase emitted from the origin. The shape of the wave normal surface is, generally, not the same as the shape of a wave front. A typical wave normal surface is presented in Fig. 21, in which the velocity of light is shown as a dashed circle. The two solutions

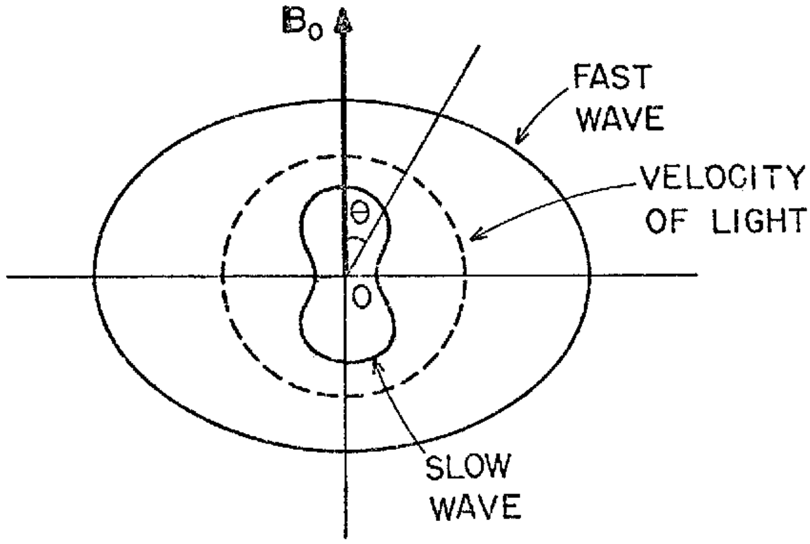

Fig. 21 - Typical wave normal surface, or phase velocity surface.

line $Y^2 = 1 - X$. The loci of the *reflection* points, as determined from (8.6), can be shown to be the curves $Y^2 = (1 - X)^2$ for any angle θ, and $X = 1$ for any angle except $\theta = 0^0$. The two reflection point curves and the two principal resonance curves divide the (X, Y^2) plane into eight regions. In each of these regions, a polar plot of the normalized phase velocity (v_{ph}/c) as a function of θ (wave normal surface) is presented for each mode of propagation.

Fig. 22 shows the CMA diagram for wave propagation in a cold electron plasma. The dashed lines are the loci of the reflection points and the solid lines are the loci of the principal resonances (the dotted line indicates the loci of the resonances when $\theta = 30^0$). The dashed circles represent the wave normal surface corresponding to the velocity of light. The "slow" and "fast" wave notation, used in Fig.21, becomes now apparent. The labels R (right-hand polarization) and L (left-hand polarization) appear on the phase velocity surface only along the magnetic field axis (up in the diagram). The labels O (ordinary) and X (extraordinary) appear only at 90^0 with respect to the magnetic field axis.

In some regions of the CMA diagram certain modes are present and others are not. As the boundaries of these regions are crossed, the wave normal surfaces for the modes change shape, and a given mode may appear or disappear. For instance, in region I both modes are present, but when we move to region II the fast wave disappears. Similarly, if the parameters are changed so as to move along a path that goes from region VIII to VII (decreasing electron

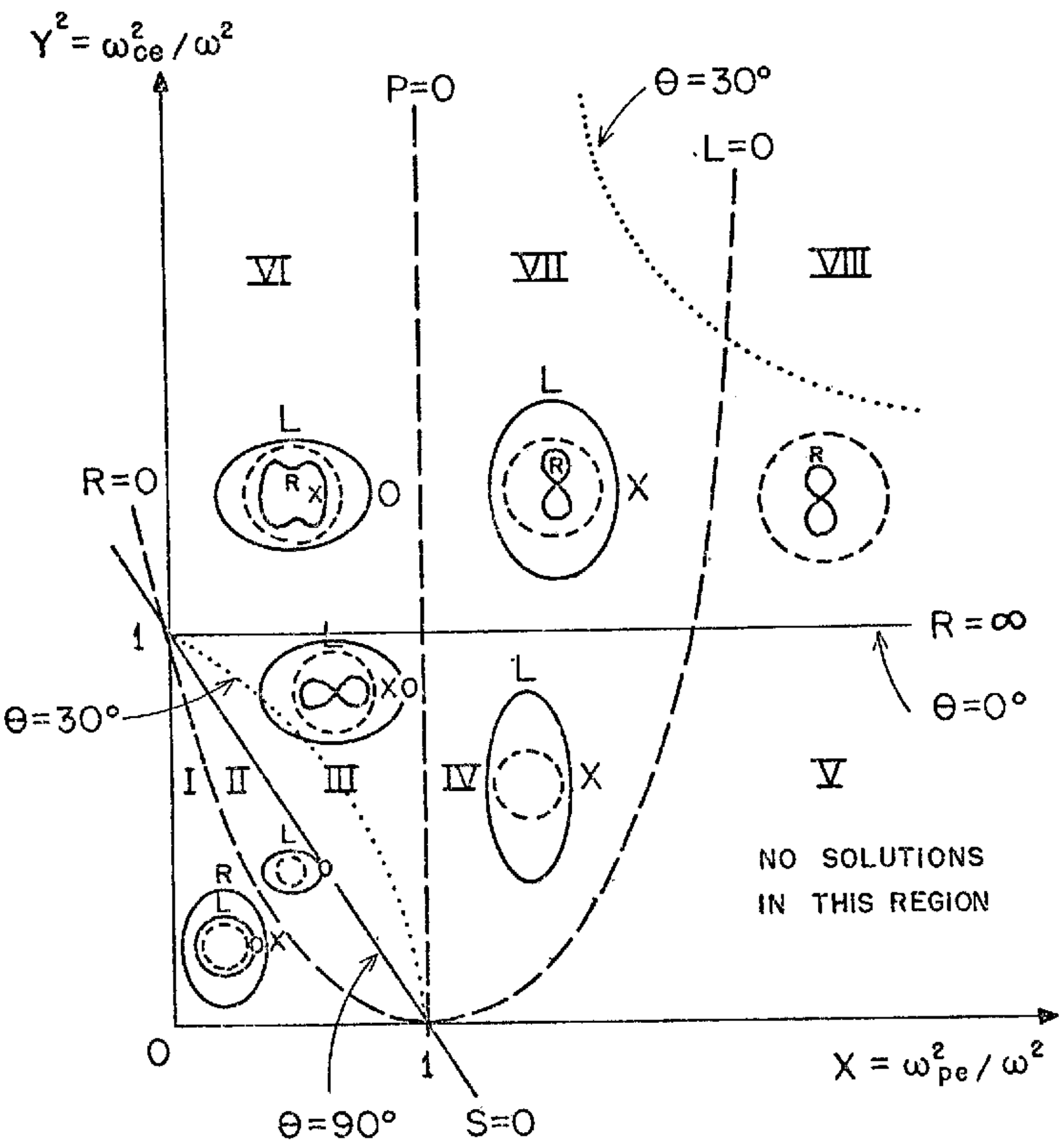

Fig. 22 - The CMA diagram for waves in a cold electron gas. The solid lines represent the principal resonances, and the dashed lines the reflection points.

density), the fast wave appears as the boundary $L = 0$ is crossed, and so on. Note that the same frequency may appear in the modes of different regions, depending upon the values of electron density and magnetic field. Note also that, although the characteristic shapes of the wave normal surfaces remain the same inside each bounded region, their magnitudes may change. A detailed examination of the CMA diagram shows that it provides a very broad picture of the nature of the waves that propagate in a cold electron plasma.

9. SOME SPECIAL WAVE PHENOMENA IN COLD PLASMAS

9.1 - Atmospheric whistlers

The propagation of *whistlers* is a naturally occurring phenomenon which can be originated by a lightning flash in the atmosphere. During thunderstorms and lightning, a pulse of electromagnetic radiation energy is produced which is rich in very low frequency components. This pulse, or wave packet, propagates through the ionosphere, being guided by ducts along the Earth's magnetic field to a distant point at the Earth's surface (the magnetic conjugate point), where it may be detected (Fig. 23). When the whistler is detected at this point it is called a *short whistler*. However, the electromagnetic signal may be reflected at the Earth's surface and guided back along the Earth's magnetic field to a point close to where it originated; if the whistler is detected at this point it is called a *long whistler*.

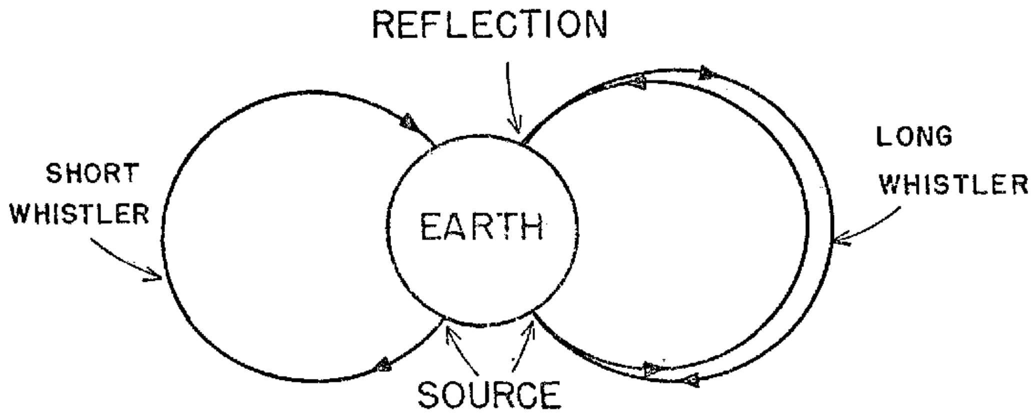

Fig. 23 - Atmospheric whistler propagation, illustrating the detection of a short whistler and a long whistler.

As the wave packet, rich in low frequencies, propagates through the ionosphere along the Earth's magnetic field, it gets dispersed in course of time in such a way that the higher frequencies move faster than the lower ones. The frequencies in a whistler are in the audio range, usually between about 100 H_z and 10 kH_z. Thus, at the point of detection, the high frequencies arrive at the receiver sooner than the low ones, and if the receiver is attached to a loudspeaker we hear a discending pitch whistle. These frequencies are much smaller than the electron cyclotron frequency ($\omega \ll \omega_{ce}$) in the Earth's ionosphere.

At various locations on the Earth there are stations that continuously record *sonograms* of whistler activity. A sonogram is a spectrum of the frequency versus time of arrival, as illustrated in Fig. 24. These sonograms are used as an effective diagnostic tool for studying the ionosphere conditions.

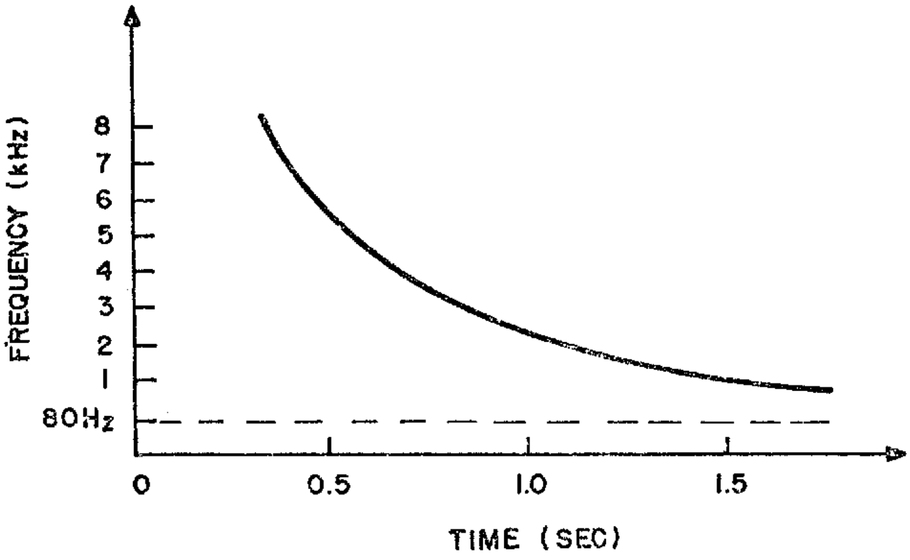

Fig. 24 - Typical sonogram of a whistler.

The phenomenon of atmospheric whistler propagation can be explained in terms of the very low frequency ($\omega \ll \omega_{ce}$) region of propagation of the *right circularly polarized wave* (see Fig. 20). For a simplified analysis of this phenomenon, consider the Appleton-Hartree equation (5.36), neglecting collisions ($U = 1$). For propagation nearly along the magnetic field lines, and for $\omega \ll \omega_{ce}$ and $\omega \ll \omega_{pe}$, we have $Y \cos\theta \gg Y^2 \sin^2\theta / [2(1 - X)]$, so that (5.36) simplifies to (using the "minus" sign),

$$\frac{k^2c^2}{\omega^2} = 1 - \frac{X}{(1 - Y \cos\theta)} \tag{9.1}$$

This equation is often referred to as the dispersion relation for the *quasi-longitudinal mode*. For $Y\cos\theta \gg 1$ (i.e., $\omega \ll \omega_{ce}\cos\theta$), (9.1) becomes

$$\frac{k^2c^2}{\omega^2} = 1 + \frac{X}{Y\cos\theta} \tag{9.2}$$

and considering, further, that $X \gg Y$ (i.e., $\omega_{pe}^2 \gg \omega\omega_{ce}$), we obtain

$$\frac{kc}{\omega} = \left(\frac{X}{Y\cos\theta}\right)^{1/2} \tag{9.3}$$

The *phase velocity* is found directly from (9.3),

$$v_{ph} = \frac{\omega}{k} = c\left(\frac{Y\cos\theta}{X}\right)^{1/2} \tag{9.4}$$

or, substituting $Y = \omega_{ce}/\omega$ and $X = \omega_{pe}^2/\omega^2$,

$$v_{ph} = c\,\frac{(\omega\omega_{ce}\cos\theta)^{1/2}}{\omega_{pe}} \tag{9.5}$$

Also, from (9.3) we obtain the *group velocity* as

$$v_g = \frac{\partial\omega}{\partial k} = 2c\,\frac{(\omega\omega_{ce}\cos\theta)^{1/2}}{\omega_{pe}} \tag{9.6}$$

Thus, the group velocity is proportional to the square root of the frequency and, consequently, the higher frequencies arrive at the receiver slightly ahead of the lower frequencies, producing a descending pitch whistle when received with a simple antenna and loudspeaker.

The characteristics of atmospheric whistler propagation are such that they are situated in region VIII of the CMA diagram. In this region, the wave normal diagram for the RCP wave is a lemniscate, as shown in Fig. 25. This wave normal surface has a

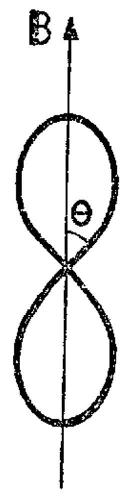

Fig. 25 - Wave normal surface for whistlers and helicons.

resonant cone, which gives the maximum value that the angle θ may have. The angle between the direction of propagation of the wave packet and the magnetic field also has a maximum value, which specifies the maximum angular deviation, from the magnetic field, of the direction in which a wave packet can propagate. It can be shown that the maximum value of this angle is about 19.5^{0}. Therefore, the wave packet is confined to a beam of less than 20^{0} about the magnetic field lines.

Experiments carried out on whistlers have verified the results presented here. In addition, when the frequency is near (but smaller than) the electron cyclotron frequency, it is possible to have the frequency increasing with the time of arrival, and these have been called *ascending frequency whistlers.* The whistlers in the frequency regime where they change from the ascending to the descending tone are known as the *nose whistlers.* These types of whistlers have also been observed experimentally.

9.2 - Helicons

The experimentally observed *helicon* waves, in a solid state plasma, is also a phenomenon related with the very low frequency propagation of the *right circularly polarized wave.* The reason for the name helicons comes from the fact the tip of the wave $\underset{\sim}{B}$ vector traces a helix.

Consider a solid state plasma slab of thickness d, the other two dimensions being very large, oriented perpendicularly to an externally applied $\underset{\sim}{B}$ field, as indicated in Fig. 26. Suppose that a low frequency ($\omega \ll \omega_{ce}$) right circularly polarized wave is launched in the direction of the $\underset{\sim}{B}$ field.

From the dispersion relation (6.7) for the RCP wave we obtain, approximately, for $\omega \ll \omega_{ce}$,

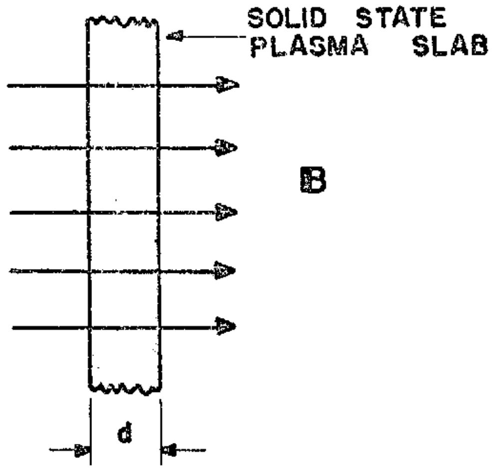

Fig. 26 - Geometrical arrangement for the detection of helicon waves.

$$k = \frac{\omega_{pe}}{c} \left(\frac{\omega}{\omega_{ce}}\right)^{1/2} \quad ; \qquad (\omega \ll \omega_{ce}) \tag{9.7}$$

Denoting the propagation coefficient of the electromagnetic wave in the medium *external* to the plasma slab by k_v the magnitude of the reflection coefficient, at the plasma boundary, is given by $(k_v - k)/(k_v + k) \simeq 1$, since $\omega \ll \omega_{ce}$. Consequently, the reflection of the waves, at the plasma boundary, is nearly complete. Therefore, the wave will be successively reflected at the boundaries of the plasma slab and will form a standing wave, whose resonances are given approximately by

$$n\lambda/2 = d \tag{9.8}$$

where λ is the wavelength inside the slab of thickness d and n is an integer.

Since $\lambda = 2\pi/k$, we can combine (9.7) and (9.8) to obtain

$$\frac{n}{2} \quad \frac{2\pi c}{\omega_{pe}} \left(\frac{\omega_{ce}}{\omega}\right)^{1/2} = d \tag{9.9}$$

This is the condition for resonance of the standing waves. It is appropriate to add the subscript n to ω, in order to identify the resonance frequency with the corresponding value of n which gives the number of the standing -wave pattern is the slab. Thus, (9.9) can be rearranged in the following convenient form

$$\omega_n = \left(\frac{n\,\pi\,c}{\omega_{pe}\,d}\right)^2 \omega_{ce} \tag{9.10}$$

In some experiments carried out on helicons, the frequency, ω, of the wave excited along the $\underset{\sim}{B}$ field in the plasma is continuously varied, maintaining constant the values of ω_{pe}, ω_{ce} and d. At the frequencies where $\omega = \omega_n$, given by (9.10), there are standing wave resonances inside the plasma slab, resulting in large wave amplitudes, which can be measured. A plot of wave amplitude inside the plasma slab, versus frequency, permits the identification of the resonant frequencies ω_n.

In sodium, which contains about 10^{28} electrons/m^3, the first (n = 1) standing-wave resonant frequency is of the order of 100 H_z. Note that ω_n is proporcional to n^2.

In some other experimental investigations, the parameters d, ω_{pe}, and ω are kept fixed, and the B field is varied. Then, the standing-wave resonant frequencies occur for those values of the $\underset{\sim}{B}$ field for which

$$\omega_{ce} = (\omega_{pe})_n = \left(\frac{\omega_{pe}\, d}{n\pi c}\right)^2 \omega \tag{9.11}$$

9.3 - Faraday rotation

We consider now a phenomenon, known as *Faraday rotation*, which occurs in the range of frequencies where both the right (RCP) and the left circularly polarized (LCP) waves propagate. When a plane polarized wave is sent along the magnetic field in a plasma, the plane of polarization of the wave gets rotated as it propagates in the plasma . Since a plane polarized wave can be considered as a superposition of RCP and LCP waves (Fig. 27), which propagate independently , this phenomenon can be understood in terms of the *difference in phase velocity* of the RCP and LCP waves.

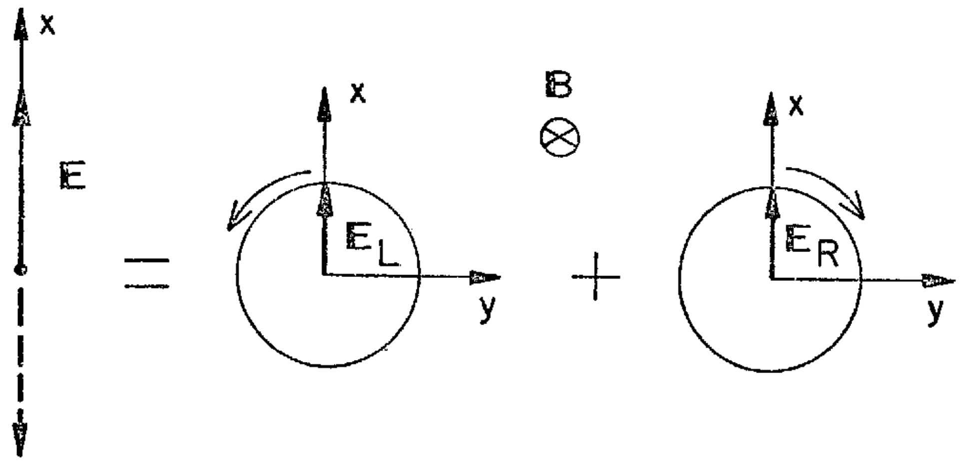

Fig. 27 - A plane polarized wave as a superposition of left and right circularly polarized waves, $\underset{\sim}{E} = \underset{\sim}{E}_L + \underset{\sim}{E}_R$.

If we take a look at Fig. 9, we see that the RCP wave (for frequencies greater than ω_{02}) propagates faster than the LCP wave. After travelling a given distance, in which the RCP wave has undergone N cycles, the LCP wave (which travels more slowly) will have undergone N + ε (with $\varepsilon > 0$) cycles. Obviously, both waves are considered to be at the same frequency. Therefore, the plane of polarization of the plane wave is rotated counterclockwise (looking along $\underset{\sim}{B}$), as shown in Fig. 28.

In order to obtain an expression for the angle of rotation θ_F, as the plane wave propagates a given distance in the plasma, let us consider a Cartesian coordinate system in which the wave propagates along the z-axis (also the direction of the magnetostatic field), and such that, at $z = 0$, the electric field has only the

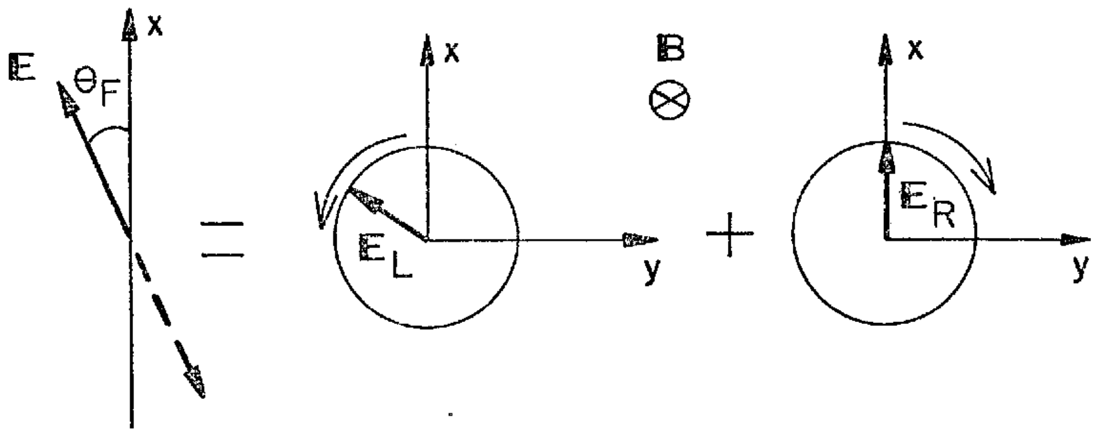

Fig. 28 - After travelling a given distance in the plasma, the plane of polarization of the plane wave is rotated, since the LCP wave moves slower than the RCP wave.

x-component, as indicated in Fig. 27. Therefore, without loss of generality, we take

$$\underset{\sim}{E}(z=0,t) = E_0\, \hat{\underset{\sim}{x}} \; \exp(-i\omega t) \tag{9.12}$$

This equation can be rewritten as

$$\underset{\sim}{E}(z=0,t) = \left[\frac{E_0}{2}\,(\hat{\underset{\sim}{x}} + i\hat{\underset{\sim}{y}}) + \frac{E_0}{2}\,(\hat{\underset{\sim}{x}} - i\hat{\underset{\sim}{y}}) \right] \exp\,(-i\omega t) \tag{9.13}$$

where the first and the second terms in the right-hand side are, respectively, the RCP and the LCP components. These two components propagate independently, so that, for any $z > 0$, the electric field vector is given by

$$\underset{\sim}{E}(z, t) = \left[\frac{E_o}{2}(\hat{\underset{\sim}{x}} + i\hat{\underset{\sim}{y}}) \exp(ik_R z) + \frac{E_o}{2}(\hat{\underset{\sim}{x}} - i\hat{\underset{\sim}{y}}) \exp(ik_L z)\right] \exp(-i\omega t) \tag{9.14}$$

where $\underset{\sim}{k}_R$ and $\underset{\sim}{k}_L$ denote the wave number vectors for the RCP and LCP waves, respectively, propagating along the z-direction. Eq. (9.14) can be rearranged as follows

$$\underset{\sim}{E}(z, t) = \frac{E_o}{2} \exp\left[i(k_R + k_L)z/2\right] \left\{(\hat{\underset{\sim}{x}} + i\hat{\underset{\sim}{y}}) \exp\left[i(k_R - k_L)z/2\right] + \right.$$

$$\left. + (\hat{\underset{\sim}{x}} - i\hat{\underset{\sim}{y}}) \exp\left[-i(k_R - k_L)z/2\right]\right\} \exp(-i\omega t)$$

$$= \frac{E_o}{2} \exp\left[i(k_R + k_L)z/2\right] \left\{\hat{\underset{\sim}{x}} \cos\left[(k_R - k_L)z/2\right] - \right.$$

$$\left. - \hat{\underset{\sim}{y}} \sin\left[(k_R - k_L)z/2\right]\right\} \exp(-i\omega t) \tag{9.15}$$

Eq. (9.12) represents a linearly polarized wave in the $\hat{\underset{\sim}{x}}$ direction at $z = 0$, and Eq. (9.15) is also a linearly polarized wave, but with the polarization direction rotated in the counterclockwise direction (looking along $\underset{\sim}{B}$) by the angle

$$\theta_F = \frac{1}{2} (k_R - k_L) z \tag{9.16}$$

Therefore, the angle of rotation per unit distance, θ_F/z, depends on the difference between the propagation coefficients of the RCP and LCP waves. Expressions for k_R and k_L are given in Eqs. (6.6) and (6.8), respectively.

The measurement of Faraday rotation is a useful tool in plasma diagnostic, and it has been widely used in the investigation of ionospheric properties. A linearly polarized wave, emitted by an orbiting satellite, has its plane of polarization rotated as it traverses the ionospheric plasma. A measurement of the rotation angle, θ_F, after the wave has traversed the plasma, provides information on the total electron content along the wave path.

PROBLEMS

16.1 - Consider a plane electromagnetic wave incident normally on a semi-infinite plasma occupying the semi-space $x \geqslant 0$, with vacuum for $x < 0$ (Fig.P 16.1) . Denote the incident, reflected and transmitted waves, respectively, by

$$\underset{\sim}{E}_i = \hat{\underset{\sim}{y}} \exp(i k_o x - i\omega t)$$

$$\underset{\sim}{E}_r = \hat{\underset{\sim}{y}} E_r \exp(-ik_o x - i\omega t)$$

$$\underset{\sim}{E}_t = \hat{\underset{\sim}{y}} E_t \exp(ik_1 x - i\omega t)$$

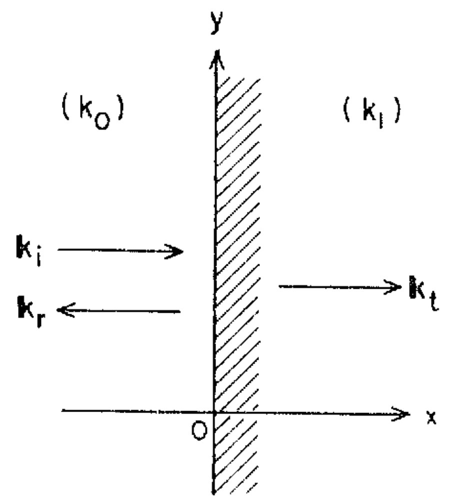

Fig. P 16.1

(a) Show that the associated magnetic fields are given by

$$\underset{\sim}{H}_i = \hat{\underset{\sim}{z}} \frac{k_o}{\omega\mu_o} \exp(i\, k_o\, x - i\omega t)$$

$$\underset{\sim}{H}_r = - \hat{\underset{\sim}{z}} \frac{k_o}{\omega\mu_o} E_r \exp(-i\, k_o\, x - i\omega t)$$

$$\underset{\sim}{H}_t = \hat{\underset{\sim}{z}} \frac{k_1}{\omega\mu_o} E_t \exp(i\, k_1\, x - i\omega t)$$

(b) From the continuity of E_y and H_z at the boundary $x = 0$, show that

$$E_r = \frac{k_o - k_1}{k_o + k_1} \quad ; \quad E_t = \frac{2k_o}{k_o + k_1}$$

(c) Prove that the ratio of the transmitted average power to the incident average power, at the boundary $x = 0$, is

$$T = \left. \frac{\mathrm{Re}\, \{\underset{\sim}{E}_t \times \underset{\sim}{H}_t^*\}}{\mathrm{Re}\, \{\underset{\sim}{E}_i \times \underset{\sim}{H}_i^*\}} \right|_{x=0} = \frac{E_t\, E_t^*}{k_o}\, \beta$$

where k_o is real and $\beta = \mathrm{Re}\, \{k_1\}$. Show that $T = 0$ both at a reflection point and a resonance.

16.2 - Consider a plane electromagnetic wave incident normally on an infinite plane plasma slab occupying the space $0 \leq x \leq L$, with vacuum for $x < 0$ and $x > L$ (Fig. P 16.2). Use the following representation for the wave electric field vector, as indicated in Fig. P 16.2:

$\underset{\sim}{E}_i = \hat{\underset{\sim}{y}} \exp(i k_o x - i\omega t)$ (incident wave)

$\underset{\sim}{E}_r = \hat{\underset{\sim}{y}} E_r \exp(-i k_o x - i\omega t)$ (reflected wave)

$\underset{\sim}{E}_f = \hat{\underset{\sim}{y}} E_f \exp(i k_1 x - i\omega t)$ (forward wave)

$\underset{\sim}{E}_b = \hat{\underset{\sim}{y}} E_b \exp(-i k_1 x - i\omega t)$ (backward wave)

$\underset{\sim}{E}_t = \hat{\underset{\sim}{y}} E_t \exp[i k_o(x-L) - i\omega t]$ (transmitted wave)

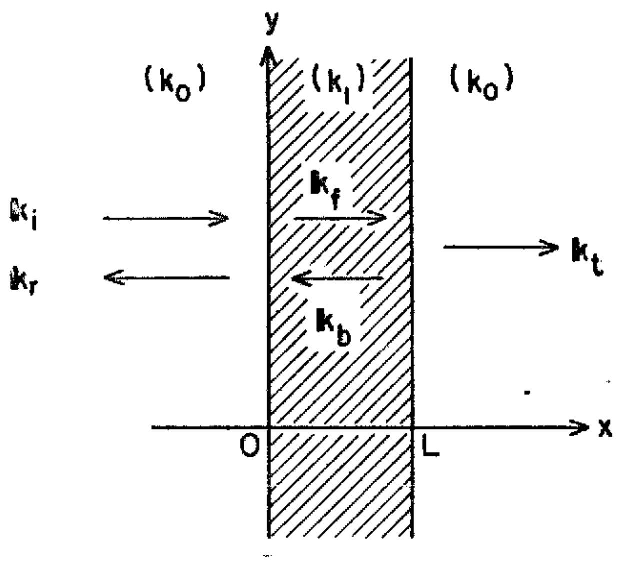

Fig. P 16.2

(a) Calculate the corresponding expressions for the associated magnetic fields.

(b) Calculate the amplitudes E_r, E_f, E_b and E_t by applying the condition of continuity of E_y and H_z at the boundaries $x = 0$ and $x = L$.

(c) Show that the ratio of the average power transmitted out of the plasma slab to the incident average power is given by

$$T = \frac{\mathrm{Re}\,\{\underset{\sim}{E}_t \times \underset{\sim}{H}_t^*\}_{x=L}}{\mathrm{Re}\,\{\underset{\sim}{E}_i \times \underset{\sim}{H}_i^*\}_{x=0}} = E_t\, E_t^*$$

where

$$E_t = 4\left[(2+k_o/k_1 + k_1/k_o)\, e^{-ik_1L} + (2-k_o/k_1-k_1/k_o)\, e^{ik_1L}\right]^{-1}$$

(d) For $\omega<\omega_{pe}$, where $k_1 = i\alpha$, with α real, show that

$$E_t = 4\,[4\cosh(\alpha L) + 2i(\alpha/k_o - k_o/\alpha)\sinh(\alpha L)]^{-1}$$

$$T = [\cosh^2(\alpha L) + (\alpha/k_o - k_o/\alpha)^2\,\sinh^2(\alpha L)/4]^{-1}$$

This result shows that some power is transmitted through the slab, even with $\beta = \mathrm{Re}\{k_1\} = 0$. This effect is known as the *tunneling effect*.

16.3 - Derive expressions for the phase velocity, v_{ph}, and the group velocity, v_g, from the dispersion relation (5.26), for wave propagation at arbitrary angles in a cold magnetoplasma.

16.4 - Use the dispersion relation (4.10), for the transverse mode of propagation in a cold isotropic electron gas (with $\underset{\sim}{B}_o = 0$) to calculate the damping factor $\alpha = \mathrm{Im}\{k\}$. Show that, when $\omega >> \omega_{pe}$, the damping factor is given approximately by

$$\alpha \simeq \frac{\omega_{pe}^2(\nu/\omega)}{2\omega c\left[1+(\nu/\omega)^2\right]}$$

16.5 - Consider the propagation of high-frequency waves in a solid state plasma with equal number of electrons and holes (considering $m_e = m_h$ and $\nu_e = \nu_h$), immersed in a magnetostatic field $\underset{\sim}{B}_o$. Let $\underset{\sim}{k} = k\hat{\underset{\sim}{x}}$ and $\underset{\sim}{B}_o = B_o\ (\cos\theta\ \hat{\underset{\sim}{x}} + \sin\theta\ \hat{\underset{\sim}{y}})$.
Use the Langevin equations for electrons and holes, and Maxwell equations, to show that

$$(-2UX - Y^2 \sin^2\theta + U^2)\, u_x + (Y^2 \sin\theta \cos\theta)\, u_y = 0$$

$$(Y^2 \sin\theta \cos\theta)\, u_x + (-2U\phi - Y^2 \cos^2\theta + U^2)\, u_y = 0$$

$$(-2U\phi - Y^2 + U^2)\, u_z = 0$$

where $\underset{\sim}{u} = \underset{\sim}{u}_e - \underset{\sim}{u}_h$ and

$$U = 1 + i(\nu/\omega)$$

$$X = \omega_{pe}^2/\omega^2$$

$$Y = \omega_{ce}/\omega$$

$$\phi = X/(1 - k^2c^2/\omega^2)$$

From these component equations derive the following dispersion relations

$$\phi = \frac{U}{2} - \frac{Y^2\cos^2\theta}{2U} - \frac{Y^4 \sin^2\theta \cos^2\theta}{2U(-2UX - Y^2\sin^2\theta + U^2)}$$

$$\phi = \frac{U}{2} - \frac{Y^2}{2U}$$

Obtain expressions for the reflection points and the resonances. In particular, for the collisionless case (ν=0; U=1) show that the conditions for resonance are

$$\omega^2 = \frac{1}{2}\left\{ \omega_{ce}^2 + 2\omega_{pe}^2 \pm \left[(\omega_{ce}^2 + 2\omega_{pe}^2)^2 - 8\omega_{pe}^2 \, \omega_{ce}^2 \cos^2\theta \right]^{1/2} \right\}$$

$$\omega^2 = \omega_{ce}^2$$

and the reflection points are given by

$$\omega^2 = \frac{1}{2}(\omega_{ce}^2 + 4\omega_{pe}^2 \pm \omega_{ce}^2)$$

$$\omega^2 = \omega_{ce}^2 + 2\omega_{pe}^2$$

16.6 - Use Eqs. (6.17) and (6.18) for the group velocities of the left and the right circularly polarized waves, respectively, propagating along $\underset{\sim}{B}_0$, to show that the group velocity vanishes at the resonances and reflection points.

16.7 - Consider the problem of wave propagation at an arbitrary direction in a cold magnetoplasma, but including the motion of the ions (one type only).

(a) Show that the dispersion relation is obtained from an equation identical to (5.25), except that now we have (neglecting collisions, $\nu = 0$)

$$S = 1 - \frac{X_e}{1 - Y_e^2} - \frac{X_i}{1 - Y_i^2}$$

$$D = - \frac{X_e Y_e}{1 - Y_e^2} + \frac{X_i Y_i}{1 - Y_i^2}$$

$$P = 1 - X_e - X_i$$

where (with α = e,i)

$$X_\alpha = \omega_{p\alpha}^2/\omega^2$$

$$Y_\alpha = \omega_{c\alpha}/\omega$$

(b) Obtain the dispersion relation and show that it can be written in the form

$$\tan^2\theta = - \frac{P(k^2c^2/\omega^2 - R)(k^2c^2/\omega^2 - L)}{(S\ k^2c^2/\omega^2 - RL)(k^2c^2/\omega^2 - P)}$$

where θ is the angle between $\underset{\sim}{k}$ and $\underset{\sim}{B}_o$,

$R = S + D$ and $L = S - D$.

(c) Determine and plot the resonances and reflection points as a function of θ.

(d) Analyse the various modes of propagation for the particular cases when $\theta = 0$ and $\theta = \pi/2$. Compare the results with those for a cold electron gas. Make a plot analogous to the one presented in Fig. 20.

16.8 - From Eq. (5.25) show that the polarization of the waves propagating at an angle θ with respect to $\underset{\sim}{B}_o$ is determined by

$$\frac{i\,E_x}{E_y} = \frac{k^2c^2/\omega^2 - S}{D}$$

From this result verify that for $\theta = 0$ the waves are left and right circularly polarized, whereas for $\theta = \pi/2$ the polarization is given by (for the extraordinary mode)

$$\left(\frac{i\,E_x}{E_y}\right)_X = -\frac{D}{S}$$

so that this mode is in general elliptically polarized.

16.9 - For a helicon wave, show that the tip of the wave magnetic field vector traces out a helix.

16.10 - Make a plot analogous to Fig. 20 for wave propagation in a cold magnetoplasma, but in terms of ω as a function of the real part of k.