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16.Summary/Notes

This is the final chapter, in a series of twenty two, written on the fundamentals of plasma physics. This chapter analyses some basic transport processes in weakly ionized plasmas, using the Boltzmann equation with the relaxation model, for a velocity-dependent collision frequency. Explicit expressions are derived for the electrical conductivity, the diffusion coefficient and the thermal conductivity for a weakly ionized plasma.

17.Remarks

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CHAPTER 22

TRANSPORT PROCESSES IN PLASMAS

CHAPTER 22

TRANSPORT PROCESSES IN PLASMAS

1. INTRODUCTION

In this chapter we analyze some basic transport processes in weakly ionized plasmas using the Boltzmann equation with the relaxation model for the collision term, considering a velocity-dependent collision frequency.

Transport phenomena in plasmas can be promoted by external and internal forces. In a *spatially homogeneous* plasma under the influence of external forces, a drifting of the electrons can occur. This motion induced by external forces is refered to as mobility. Since the electrons have mass, this drifting implies in a transport of mass. Furthermore, since the electrons have electric charge, their motion implies also in conduction of electricity when acted upon by an external electric field. On the other hand, in a *spatially inhomogeneous* plasma, the collisional interactions cause the electrons to drift from the high-pressure to the low-pressure regions. The existence of pressure gradients is associated with the existence of either density gradients or temperature gradients, or both. This motion of the electrons, induced by internal pressure gradients, is called diffusion. Since the electrons also have kinetic energy associated with their randon thermal motion, their drift implies in the transport of thermal energy and therefore in heat conduction. When the plasma is spatially inhomogeneous and is also acted upon by external forces, then the particle flux is due to both diffusion and mobility. The basic transport phenomena which we analyze in this chapter using the Boltzmann equation with the relaxation model are electric conduction, particle diffusion and thermal energy flux.

2. ELECTRIC CONDUCTIVITY IN A NONMAGNETIZED PLASMA

Initially we derive an expression for the AC conductivity of a *weakly ionized plasma*, in which only the collisions between electrons and neutral particles are important. We consider that the spatial inhomogeneity and the anisotropy of the nonequilibrium distribution function of the electrons are very small, so that we can apply the results derived in section 4 of Chapter 21. Thus, according to Eq. (21.4.17) we have

$$\left[\frac{\delta f\ (\underset{\sim}{r},\ \underset{\sim}{v},\ t)}{\delta t}\right]_{coll} = -\ \nu_r(v)\ [\, f(\underset{\sim}{r},\ \underset{\sim}{v},\ t) - f_0(v)\,] \qquad (2.1)$$

where $f_0(v)$ denotes the homogeneous isotropic equilibrium distribution function of the electrons, and $\nu_r(v)$ is a velocity-dependent relaxation collision frequency. Expression (2.1) assumes that the neutral particles are stationary and do not recoil as they collide with electrons, in view of their much larger mass.

2.1 - Solution of Boltzmann equation

We assume that the electron distribution function $f(\underset{\sim}{r}, \underset{\sim}{v}, t)$ deviates only slightly from the equilibrium function $f_0(v)$, so that

$$f(\underset{\sim}{r}, \underset{\sim}{v}, t) = f_0(v) + f_1(\underset{\sim}{r}, \underset{\sim}{v}, t) \quad ; \; | f_1 | \ll f_0 \qquad (2.2)$$

where $f_1(\underset{\sim}{r}, \underset{\sim}{v}, t)$ corresponds to the small anisotropy and spatial inhomogeneity of the electrons in the nonequilibrium state. Using (2.2) and the relaxation model (2.1), the collision term in the Boltzmann equation becomes

$$\left(\frac{\delta f}{\delta t}\right)_{coll} = - \nu_r(v) \; f_1(\underset{\sim}{r}, \underset{\sim}{v}, t) \qquad (2.3)$$

Substituting (2.2) and (2.3) into the Boltzmann equation, and neglecting second order quantities, we obtain

$$\frac{\partial}{\partial t} f_1(\underset{\sim}{r}, \underset{\sim}{v}, t) + (\underset{\sim}{v} \cdot \underset{\sim}{\nabla}) f_1 (\underset{\sim}{r}, \underset{\sim}{v}, t) - \frac{e}{m_e} \underset{\sim}{E}(\underset{\sim}{r}, t) \cdot \underset{\sim}{\nabla}_v f_o(v) =$$

$$= - \nu_r(v) \quad f_1(\underset{\sim}{r}, \underset{\sim}{v}, t) \tag{2.4}$$

where we have considered the electric field $\underset{\sim}{E}(\underset{\sim}{r}, t)$ as the only field externally applied to the plasma. For the purpose of evaluating the conductivity, the perturbation $f_1(\underset{\sim}{r}, \underset{\sim}{v}, t)$ in the velocity distribution function can be assumed to be essentially independent of the position coordinate $\underset{\sim}{r}$, and therefore denoted by $f_1(\underset{\sim}{v}, t)$, since the main effect associated with a spatial variation is the diffusion of particles and, at the moment, we are interested primarily in the charged particle current density induced by an electric field. The electric field is considered to vary harmonically in time at a frequency ω, according to

$$\underset{\sim}{E}(\underset{\sim}{r}, t) = \underset{\sim}{E}(\underset{\sim}{r}) \; e^{-i\omega t} \tag{2.5}$$

Therefore, we assume that $f_1(\underset{\sim}{v}, t)$ has also the same time variation,

$$f_1(\underset{\sim}{v}, t) = f_1(\underset{\sim}{v}) \; e^{-i\omega t} \tag{2.6}$$

Consequently, for the phasor amplitudes, the Boltzmann equation (2.4) simplifies to

$$- i\omega f_1(\underset{\sim}{v}) - \frac{e}{m_e} \underset{\sim}{E}(\underset{\sim}{r}) \cdot \underset{\sim}{\nabla}_v f_o(v) = -\nu_r(v) f_1(\underset{\sim}{v}) \tag{2.7}$$

Using the following identity, given in Eq. (18.3.17),

$$\underset{\sim}{\nabla}_v f_o(v) = \frac{\underset{\sim}{v}}{v} \frac{d f_o(v)}{dv} \tag{2.8}$$

we obtain, from (2.7),

$$f_1 (\underset{\sim}{v}) = \frac{ie}{m_e} \frac{\underset{\sim}{E}(\underset{\sim}{r}) \cdot \underset{\sim}{v}}{v \left[\omega + i\nu_r(v)\right]} \frac{df_o(v)}{dv} \tag{2.9}$$

2.2 - <u>Electric current density and conductivity</u>

The electric current density is given by

$$\underset{\sim}{J}(\underset{\sim}{r}, t) = - e n_e < \underset{\sim}{v} >_e = - e \int_V \underset{\sim}{v} f(\underset{\sim}{r}, \underset{\sim}{v}, t) d^3v \tag{2.10}$$

Using Eqs. (2.2), (2.6) and (2.9) we find that

$$\underset{\sim}{J}(\underset{\sim}{r}, t) = \underset{\sim}{J}(\underset{\sim}{r}) e^{-i\omega t} \tag{2.11}$$

where

$$\underset{\sim}{J}(\underset{\sim}{r}) = -e \int_V \underset{\sim}{v} \; f_1(\underset{\sim}{v}) \; d^3v$$

$$= - \frac{ie^2}{m_e} \int_V \frac{\underset{\sim}{v} \, [\underset{\sim}{E}(\underset{\sim}{r}) \cdot \underset{\sim}{v}]}{v \, [\omega + i\nu_r(v)]} \; \frac{df_o(v)}{dv} \, d^3v \qquad (2.12)$$

In this result we have assumed that the electrons have no average flow velocity in the equilibrium state, that is,

$$\underset{\sim}{u}_o = \frac{1}{n} \int_V \underset{\sim}{v} \; f_o(v) \; d^3v = 0 \qquad (2.13)$$

In spherical coordinates (v, θ, ϕ) in velocity space (Fig. 1), we have $d^3v = v^2 \, dv \sin\theta \; d\theta \; d\phi$, so that Eq. (2.12) can be rewritten as

$$\underset{\sim}{J}(\underset{\sim}{r}) = - \frac{ie^2}{me} \int_0^\infty \frac{v \, dv}{[\omega + i\nu_r(v)]} \; \frac{df_o(v)}{dv} \int_0^\pi d\theta \; \sin\theta \int_0^{2\pi} \underset{\sim}{v}[\underset{\sim}{E}(\underset{\sim}{r}) \cdot \underset{\sim}{v}] \; d\phi \qquad (2.14)$$

Using the following orthogonality relation

$$\int_0^\pi \int_0^{2\pi} v_i \; v_j \; \sin\theta \;\; d\theta \; d\phi = \frac{4\pi}{3} \, v^2 \, \delta_{ij} \qquad (2.15)$$

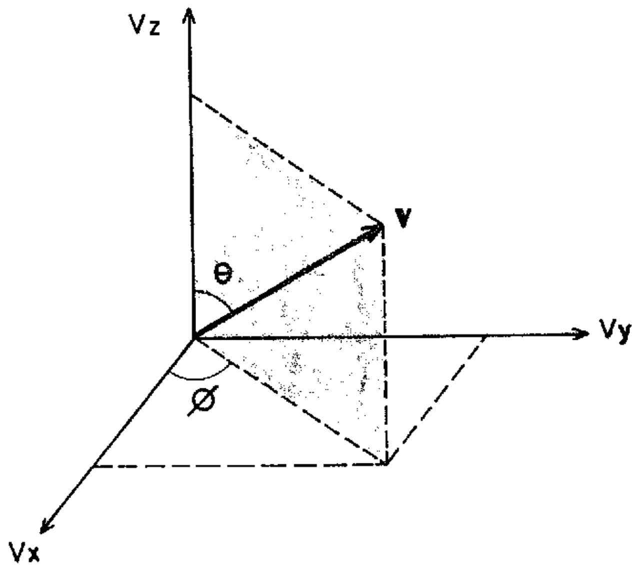

Fig. 1 - Spherical coordinates (v, θ, ϕ) in velocity space.

with i, j = x, y, z, we find that

$$\int_0^{\pi} d\theta \, \sin\theta \int_0^{2\pi} d\phi \quad \underset{\sim}{v}(\underset{\sim}{E} \cdot \underset{\sim}{v}) \;=\; \frac{4\pi}{3} \, v^2 \, \underset{\sim}{E} \tag{2.16}$$

Consequently, Eq. (2.14) becomes

$$\underset{\sim}{J}(\underset{\sim}{r}) \;=\; - \frac{i\, 4\pi\, e^2}{3\, m_e} \, \underset{\sim}{E}(\underset{\sim}{r}) \int_0^{\infty} \frac{v^3}{[\omega + i\nu_r(v)]} \, \frac{df_o(v)}{dv} \, dv \tag{2.17}$$

From the relation $\underset{\sim}{J} = \sigma \underset{\sim}{E}$ we identify the following expression for the electric conductivity

$$\sigma = - \frac{i\,4\pi\,e^2}{3\,m_e} \int_0^\infty \frac{v^3}{[\omega + i\nu_r(v)]} \frac{df_0(v)}{dv}\,dv \tag{2.18}$$

An alternative expression for the electric conductivity can be obtained by integrating (2.18) by parts,

$$\sigma = - \frac{i\,4\pi\,e^2}{3\,m_e} \left\{ \frac{v^3\,f_0(v)}{[\omega + i\nu_r(v)]} \right\} \Bigg|_0^\infty +$$

$$+ \frac{i\,4\pi\,e^2}{3\,m_e} \int_0^\infty f_0(v) \frac{d}{dv} \left\{ \frac{v^3}{[\omega + i\nu_r(v)]} \right\} dv \tag{2.19}$$

The integrated out term on the right-hand side of this expression vanishes, since f_0 goes to zero faster than v^3 goes to infinity, as v approaches infinity. In general the isotropic equilibrium distribution function, $f_0(v)$, decreases exponentially as v goes to infinity.

The integrals which appear in expressions (2.18) and (2.19) can be calculated explicitly only after specifying $f_0(v)$ and $\nu_r(v)$. The functional dependence of ν_r on v is generally determined experimentally from cross section measurements.

If we assume that ν_r is independent of v, then we obtain, from (2.19), for any $f_o(v)$,

$$\sigma = \frac{i\,4\pi\, e^2}{3\, m_e\,(\omega + i\nu_r)} \int_0^\infty f_o(v)\; 3v^2\; dv$$

$$= \frac{i\, n_o\, e^2}{m_e(\omega + i\nu_r)}$$

$$= \frac{\nu_r\, n_o\, e^2}{m_e(\omega^2 + \nu_r^2)} + i\, \frac{\omega\, n_o\, e^2}{m_e(\omega^2 + \nu_r^2)} \tag{2.20}$$

where n_o denotes the electron number density at equilibrium,

$$n_o = 4\pi \int_0^\infty f_o(v)\; v^2\; dv \tag{2.21}$$

The result (2.20) is identical to the one obtained in section 5, of Chapter 10 [see Eq. (10.5.5)], for the longitudinal conductivity.

2.3 - Conductivity for Maxwellian distribution function

Let us consider that $f_o(v)$ is given by the Maxwellian distribution function,

$$f_o(v) = n_o \left(\frac{m_e}{2\pi kT}\right)^{3/2} \exp\left(-\frac{m_e v^2}{2kT}\right) \tag{2.22}$$

Defining a dimensionless variable by

$$\xi = \left(\frac{m_e}{2kT}\right)^{1/2} v \tag{2.23}$$

it can be verified that

$$v^3 \frac{df_o(v)}{dv} dv = - \frac{2 n_o}{\pi^{3/2}} \xi^4 \exp(-\xi^2) d\xi \tag{2.24}$$

Substituting this expression into (2.18) and rationalizing, we find

$$\sigma = \frac{8 n_o e^2}{3 m_e \pi^{1/2}} \left[\int_o^\infty \frac{\nu_r(\xi) \xi^4 \exp(-\xi^2)}{\nu_r^2(\xi) + \omega^2} d\xi + i\omega \int_o^\infty \frac{\xi^4 \exp(-\xi^2)}{\nu_r^2(\xi) + \omega^2} d\xi \right] \tag{2.25}$$

This equation can be used to calculate the electric conductivity of a weakly ionized plasma when the equilibrium distribution function of the electrons is the Maxwell-Boltzmann distribution, for any dependence of the collision frequency ν_r on speed v. In particular, if ν_r is independent of v, then (2.25) reduces directly to the result (2.20).

3. ELECTRIC CONDUCTIVITY IN A MAGNETIZED PLASMA

We consider now a weakly ionized plasma immersed in an externally applied magnetostatic field, $\underset{\sim}{B}_o$. As in the previous section, we assume that the distribution function of the electrons in the nonequilibrium state is only slightly perturbed from the equilibrium value. For purposes of calculating the conductivity, it can also be assumed that the plasma is homogeneous in space. Therefore, we can write

$$f(\underset{\sim}{v},\ t) \;=\; f_o(v) + f_1(\underset{\sim}{v},\ t) \tag{3.1}$$

where $|f_1| \ll f_o$. Suppose that an AC electric field is applied to the plasma, having a harmonic time dependence according to

$$\underset{\sim}{E}(\underset{\sim}{r},\ t) \;=\; \underset{\sim}{E}(\underset{\sim}{r})\ \exp\,(-\,i\omega t) \tag{3.2}$$

Consequently, we also have

$$f_1(\underset{\sim}{v},\ t) \;=\; f_1(\underset{\sim}{v})\ \exp\,(-\,i\omega t) \tag{3.3}$$

The total magnetic field will be denoted by

$$\underset{\sim}{B}_t(\underset{\sim}{r},\ t) \;=\; \underset{\sim}{B}_o + \underset{\sim}{B}(\underset{\sim}{r},\ t) \tag{3.4}$$

where $\underline{B}_o$ is the externally applied field and $\underset{\sim}{B}(\underset{\sim}{r}, t)$ is a first order quantity which has the same harmonic time dependence as the electric field.

3.1 - Solution of Boltzmann equation

The Boltzmann equation satisfied by the homogeneous distribution function of the electrons, and with the relaxation model (2.3) for the collision term, can be written as

$$\frac{\partial f_1(\underset{\sim}{v}, t)}{\partial t} - \frac{e}{m_e}\left[\underset{\sim}{E}(\underset{\sim}{r}, t) + \underset{\sim}{v} \times \underset{\sim}{B}_t(\underset{\sim}{r}, t)\right] \cdot \underset{\sim}{\nabla}_v \left[f_o(v) + f_1(\underset{\sim}{v}, t)\right] =$$

$$= - \nu_r(v) \quad f_1(\underset{\sim}{v}, t) \qquad (3.5)$$

From the identity (2.8) we see that the term $(\underset{\sim}{v} \times \underset{\sim}{B}_t) \cdot \underset{\sim}{\nabla}_v f_o(v)$ vanishes, since it involves the dot product of two mutually orthogonal vector functions. Neglecting second order terms, the Boltzmann equation for the phasor amplitudes becomes

$$\left[\nu_r(v) - i\omega\right] f_1(\underset{\sim}{v}) - \frac{e}{m_e}(\underset{\sim}{v} \times \underset{\sim}{B}_o) \cdot \underset{\sim}{\nabla}_v f_1(\underset{\sim}{v}) =$$

$$= \frac{e}{m_e} \underset{\sim}{E} \cdot \underset{\sim}{\nabla}_v f_o(v) \qquad (3.6)$$

In cylindrical coordinates ($v_\perp$, ϕ, $v_\parallel$) in velocity space (Fig. 2), with the $\underset{\sim}{v}_\parallel$ vector along the magnetostatic field $\underset{\sim}{B}_o$, we have, from Eq. (19.2.10),

$$(\underset{\sim}{v} \times \hat{\underset{\sim}{B}}_o) \cdot \underset{\sim}{\nabla}_v \, f_1(\underset{\sim}{v}) \;=\; - \frac{df_1(\underset{\sim}{v})}{d\phi} \tag{3.7}$$

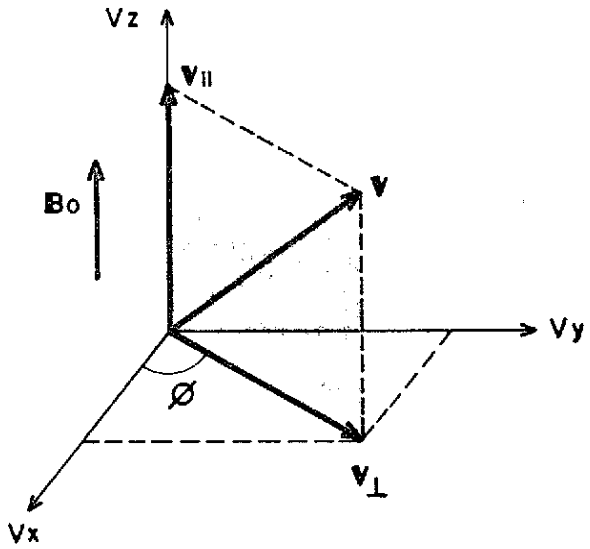

Fig. 2 - Cylindrical coordinates ($v_\perp$, ϕ, $v_\parallel$) in velocity space.

Substituting (3.7) into (3.6) and using the identity (2.8), we obtain

$$\frac{df_1\,(\underset{\sim}{v})}{d\phi} + \frac{\nu_r(v) - i\omega}{\omega_{ce}} f_1\,(\underset{\sim}{v}) = \frac{e}{m_e\,\omega_{ce}}\; \underset{\sim}{E} \cdot \frac{\underset{\sim}{v}}{v}\; \frac{df_o(v)}{dv} \tag{3.8}$$

where we have used the notation $\omega_{ce} = e\,B_o/m_e$, which represents the electron cyclotron frequency. Notice that the speed v does not depend on ϕ, since $v^2 = v_\perp^2 + v_\parallel^2$.

It is convenient to decompose the electric field vector into right circularly polarized (E_+), left circularly polarized (E_-) and longitudinal ($E_\parallel$) components, that is,

$$\underset{\sim}{E} = E_+ \frac{(\hat{\underset{\sim}{x}} + i\hat{\underset{\sim}{y}})}{\sqrt{2}} + E_- \frac{(\hat{\underset{\sim}{x}} - i\hat{\underset{\sim}{y}})}{\sqrt{2}} + E_\parallel \hat{\underset{\sim}{z}} \tag{3.9}$$

where

$$E_\pm = \frac{1}{\sqrt{2}} (E_x \mp iE_y) \tag{3.10}$$

Similarly, we can also decompose the electron velocity as

$$\underset{\sim}{v} = v_+ \frac{(\hat{\underset{\sim}{x}} + i\hat{\underset{\sim}{y}})}{\sqrt{2}} + v_- \frac{(\hat{\underset{\sim}{x}} - i\hat{\underset{\sim}{y}})}{\sqrt{2}} + v_\parallel \hat{\underset{\sim}{z}} \tag{3.11}$$

where

$$v_\pm = \frac{1}{\sqrt{2}} (v_x \mp iv_y)$$

$$= \frac{1}{\sqrt{2}} v_\perp \exp(\mp i\phi) \tag{3.12}$$

since $v_x = v_\perp \cos\phi$, $v_y = v_\perp \sin\phi$ and $\exp(\pm i\phi) = \cos\phi \pm i \sin\phi$. Thus, using (3.9) and (3.11),

$$\underset{\sim}{E} \cdot \underset{\sim}{v} = E_+ v_- + E_- v_+ + E_\parallel v_\parallel$$

$$= \frac{v_\perp}{\sqrt{2}} (E_+ e^{i\phi} + E_- e^{-i\phi}) + E_\parallel v_\parallel \tag{3.13}$$

Substituting this expression into the Boltzmann equation (3.8), we obtain

$$\frac{df_1(\underset{\sim}{v})}{d\phi} + \frac{\nu_r(v) - i\omega}{\omega_{ce}} f_1(\underset{\sim}{v}) = \frac{e}{m_e \omega_{ce}} \left[\frac{v_\perp}{\sqrt{2}} (E_+ e^{i\phi} + \right.$$

$$\left. + E_- e^{-i\phi}) + E_\parallel v_\parallel \right] \frac{1}{v} \frac{df_o(v)}{dv} \tag{3.14}$$

As in subsection 2.2, of Chapter 19, we now introduce the notation

$$F_+(\underset{\sim}{v}) = \frac{e}{m_e \omega_{ce}} E_+ \frac{v_\perp}{v} \frac{e^{i\phi}}{\sqrt{2}} \frac{df_o(v)}{dv} \tag{3.15}$$

$$F_-(\underset{\sim}{v}) = \frac{e}{m_e \omega_{ce}} E_- \frac{v_\perp}{v} \frac{e^{-i\phi}}{\sqrt{2}} \frac{df_o(v)}{dv} \tag{3.16}$$

$$F_{\parallel}(\underset{\sim}{v}) = \frac{e}{m_e\,\omega_{ce}}\,E_{\parallel}\,\frac{v_{\parallel}}{v}\,\frac{df_o(v)}{dv} \tag{3.17}$$

which allows (3.14) to be written as

$$\frac{df_1(\underset{\sim}{v})}{d\phi} + \frac{\nu_r(v) - i\omega}{\omega_{ce}}\,f_1(\underset{\sim}{v}) = F_+(\underset{\sim}{v}) + F_-(\underset{\sim}{v}) + F_{\parallel}(\underset{\sim}{v}) \tag{3.18}$$

This differential equation is similar to Eq. (19.2.26), replacing the term $-kv_{\parallel}$ by $i\nu_r(v)$. Therefore, its solution can be obtained by inspection of the corresponding results contained in subsection 2.2, of Chapter 19. Hence, using Eqs. (19.2.27) to (19.2.34), we obtain

$$f_1(\underset{\sim}{v}) = \frac{i\,\omega_{ce}}{\omega + i\nu_r(v) - \omega_{ce}}\,F_+(\underset{\sim}{v}) + \frac{i\,\omega_{ce}}{\omega + i\nu_r(v) + \omega_{ce}}\,F_-(\underset{\sim}{v}) +$$

$$+ \frac{i\,\omega_{ce}}{\omega + i\nu_r(v)}\,F_{\parallel}(\underset{\sim}{v}) \tag{3.19}$$

or, substituting (3.15), (3.16) and (3.17), into (3.19),

$$f_1(\underset{\sim}{v}) = \frac{i\,e}{m_e}\,\frac{1}{v}\,\frac{df_o(v)}{dv}\left\{\frac{v_{\perp}}{\sqrt{2}}\left[\frac{E_+\,e^{i\phi}}{\omega + i\nu_r(v) - \omega_{ce}} + \right.\right.$$

$$+ \frac{E_- \, e^{-i\phi}}{\omega + i\nu_r(v) + \omega_{ce}} \Bigg] + \frac{v_{\parallel} \, E_{\parallel}}{\omega + i\nu_r(v)} \Bigg\} \tag{3.20}$$

3.2 - Electric current density and conductivity

Assuming that the electron gas has no average flow velocity in the equilibrium state ($\underset{\sim}{u}_0 = 0$), we can write for the electric current density,

$$\underset{\sim}{J} = - e \int_V \underset{\sim}{v} \, f_1(\underset{\sim}{v}) \; d^3v \tag{3.21}$$

As in Eqs. (3.9) to (3.12), we can also decompose the current density into three components, according to

$$J_+ = - e \int_V v_+ \, f_1(\underset{\sim}{v}) \; d^3v \tag{3.22}$$

$$J_- = - e \int_V v_- f_1(\underset{\sim}{v}) \; d^3v \tag{3.23}$$

$$J_{\parallel} = - e \int_V v_{\parallel} \, f_1(\underset{\sim}{v}) \; d^3v \tag{3.24}$$

For purposes of calculating the conductivity, it is convenient to use spherical coordinates (v, θ, ϕ) in velocity space (Fig. 1), so that $v_\perp = v \sin\theta$, $v_\parallel = v \cos\theta$ and $d^3v = v^2 dv \sin\theta \; d\theta \; d\phi$. Plugging $f_1(\underset{\sim}{v})$, from (3.20), into the expressions for J_+, J_- and $J_\parallel$, given in (3.22), (3.23) and (3.24), respectively, transforming to spherical coordinates, and performing the integrals over ϕ [making use of Eq. (19.2.51)], we find

$$J_\pm = - \frac{i \pi e^2}{m_e} E_\pm \int_0^\pi \sin^3\theta \; d\theta \int_0^\infty \frac{v^3}{\omega + i\nu_r(v) \mp \omega_{ce}} \frac{df_0(v)}{dv} dv \tag{3.25}$$

$$J_\parallel = - \frac{i 2 \pi e^2}{m_e} E_\parallel \int_0^\pi \cos^2\theta \; \sin\theta \, d\theta \int_0^\infty \frac{v^3}{\omega + i\nu_r(v)} \frac{df_0(v)}{dv} \tag{3.26}$$

Note that, in (3.25), either upper signs or lower signs are to be used. Carring out the integrations over θ in these last two equations, yields

$$J_\pm = - \frac{i 4\pi e^2}{3m_e} E_\pm \int_0^\infty \frac{v^3}{\omega + i\nu_r(v) \mp \omega_{ce}} \frac{df_0(v)}{dv} dv \tag{3.27}$$

$$J_{\parallel} = - \frac{i\,4\pi\,e^2}{3\,m_e}\,E_{\parallel} \int_0^{\infty} \frac{v^3}{\omega + i\nu_r(v)}\,\frac{df_0(v)}{dv}\,dv \qquad (3.28)$$

The advantage of using the right and left circularly polarized components in the plane normal to $\underset{\sim}{B}_0$ is that the corresponding equations for J_+ and J_- are uncoupled, so that J_+ depends only on E_+, whereas J_- depends only on E_-. Therefore, writing $\underset{\sim}{J} = \underset{\approx}{\sigma} \cdot \underset{\sim}{E}$, where $\underset{\approx}{\sigma}$ is the conductivity tensor, we obtain, from (3.27) and (3.28),

$$\begin{pmatrix} J_+ \\ J_- \\ J_{\parallel} \end{pmatrix} = \begin{pmatrix} \sigma_+ & 0 & 0 \\ 0 & \sigma_- & 0 \\ 0 & 0 & \sigma_{\parallel} \end{pmatrix} \begin{pmatrix} E_+ \\ E_- \\ E_{\parallel} \end{pmatrix} \qquad (3.29)$$

with the following expressions for the elements of the conductivity tensor

$$\sigma_{\pm} = - \frac{i\,4\pi\,e^2}{3\,m_e} \int_0^{\infty} \frac{v^3}{\omega + i\nu_r(v) \mp \omega_{ce}}\,\frac{df_0(v)}{dv}\,dv \qquad (3.30)$$

$$\sigma_{\parallel} = - \frac{i\,4\pi\,e^2}{3\,m_e} \int_0^{\infty} \frac{v^3}{\omega + i\nu_r(v)}\,\frac{df_0(v)}{dv}\,dv \qquad (3.31)$$

Note that the longitudinal conductivity $\sigma_{\parallel}$ is the same as that for the case of a nonmagnetized plasma, deduced in the previous section.

The elements of the conductivity tensor, in *Cartesian coordinates*, can be obtained as follows. From Eqs. (3.9) and (3.10) we can write in matrix form

$$\begin{pmatrix} E_+ \\ E_- \\ E_{\parallel} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \tag{3.32}$$

Using a relation analogous to (3.32) for the current density $\underset{\sim}{J}$, and inverting it, we obtain

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} J_+ \\ J_- \\ J_{\parallel} \end{pmatrix} \tag{3.33}$$

Substituting (3.29) into (3.33), and combining the resulting expression with (3.32), we find that

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} \sigma_\perp & -\sigma_H & 0 \\ \sigma_H & \sigma_\perp & 0 \\ 0 & 0 & \sigma_\parallel \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \tag{3.34}$$

where

$$\sigma_\perp = \frac{1}{2} (\sigma_+ + \sigma_-) \tag{3.35}$$

$$\sigma_H = \frac{i}{2} (\sigma_+ - \sigma_-) \tag{3.36}$$

with σ_+, σ_- and $\sigma_\parallel$ as given in Eqs. (3.30) and (3.31).

The integrals over v can only be evaluated after specifying the dependence of ν_r on v. In general, when ν_r is an arbitrary function of v, the elements of the conductivity tensor have to be determined by a numerical procedure. In cases when ν_r can be expressed as a polynominal in v, it is possible to obtain simple expressions for the conductivities in the limiting cases of very high and very low collision frequencies. In particular, for the special case when ν_r is independent of v, the integrals over v in Eqs. (3.30) and (3.31) can be explicitly evaluated, yielding

$$\sigma_{\pm} = \frac{i\, n_0\, e^2}{m_e\, (\omega + i\nu_r \mp \omega_{ce})} \tag{3.37}$$

$$\sigma_{\parallel} = \frac{i\, n_0\, e^2}{m_e\, (\omega + i\nu_r)} \tag{3.38}$$

If these expressions are substituted into (3.35) and (3.36), we obtain the following results for the Cartesian components $\sigma_\perp$ and σ_H of the conductivity tensor:

$$\sigma_\perp = \frac{i\, n_0\, e^2\, (\omega + i\nu_r)}{m_e\, [(\omega + i\nu_r)^2 - \omega_{ce}^2]} \tag{3.39}$$

$$\sigma_H = - \frac{n_0\, e^2\, \omega_{ce}}{m_e\, [(\omega + i\nu_r)^2 - \omega_{ce}^2]} \tag{3.40}$$

These are the same results deduced in section 5, of Chapter 10, which were calculated using the macroscopic transport equations with a constant collision frequency.

4. FREE DIFFUSION

In this section we derive an expression for the free diffusion coefficient of a weakly ionized plasma, considering that the relaxation collision frequency is a function of the speed of the electrons. For the analysis of diffusion phenomena we must consider specifically a spatial inhomogeneity in the electron density. Hence, we assume that the equilibrium velocity distribution function of the electrons has a spatial inhomogeneity, but is isotropic in velocity space, and will be denoted as $f_0(\underset{\sim}{r}, v)$. Since we are interested in calculating the electron flux due to diffusion only, we also assume that there are no external electromagnetic fields applied to the plasma. Furthermore, we study the free diffusion problem only under steady state conditions, in which all physical parameters are time-independent.

4.1 - Perturbation distribution function

We assume that, under diffusion, the actual distribution function of the electrons, $f(\underset{\sim}{r}, \underset{\sim}{v})$, deviates only slightly from the equilibrium value $f_0(\underset{\sim}{r}, v)$, so that we can write

$$f(\underset{\sim}{r}, \underset{\sim}{v}) = f_0(\underset{\sim}{r}, v) + f_1(\underset{\sim}{r}, \underset{\sim}{v}) \tag{4.1}$$

where $f_1(\underset{\sim}{r}, \underset{\sim}{v})$ is a first order quantity, $|f_1| \ll f_0$.

Under steady state conditions, in the absence of external forces, and using the relaxation model for the collision term, the Boltzmann equation simplifies to

$$\underset{\sim}{v} \cdot \underset{\sim}{\nabla} f_0(\underset{\sim}{r}, v) = - \nu_r(v)\, f_1(\underset{\sim}{r}, \underset{\sim}{v}) \tag{4.2}$$

where only the first order terms have been retained. Thus, we obtain directly for the perturbation distribution function,

$$f_1(\underset{\sim}{r}, \underset{\sim}{v}) = - \frac{1}{\nu_r(v)} \underset{\sim}{v} \cdot \underset{\sim}{\nabla} f_0(\underset{\sim}{r}, v) \tag{4.3}$$

4.2 - Particle flux

The expression for the particle current density (or flux) for the electrons, considering $\underset{\sim}{u}_0 = 0$, is

$$\underset{\sim}{\Gamma}_e = n_e \langle \underset{\sim}{v} \rangle_e = \int_V f_1(\underset{\sim}{r}, \underset{\sim}{v}) \, \underset{\sim}{v} \, d^3v \tag{4.4}$$

Substituting (4.3) into (4.4), gives

$$\underset{\sim}{\Gamma}_e = - \int_V \frac{1}{\nu_r(v)} \underset{\sim}{v} \, [\underset{\sim}{v} \cdot \underset{\sim}{\nabla} f_0(\underset{\sim}{r}, v)] \, d^3v \tag{4.5}$$

In spherical coordinates in velocity space (v, θ, ϕ) we have $d^3v = v^2 \sin\theta \, dv \, d\theta \, d\phi$, and using the result contained in Eq. (2.16) we obtain

$$\int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi \; \underset{\sim}{v} \, [\underset{\sim}{v} \cdot \underset{\sim}{\nabla} f_0(\underset{\sim}{r}, v)] = \frac{4\pi}{3} v^2 \, \underset{\sim}{\nabla} f_0(\underset{\sim}{r}, v) \tag{4.6}$$

Therefore, the electron flux vector (4.5) can be written as

$$\underset{\sim}{\Gamma}_e = - \frac{4\pi}{3} \int_0^\infty \frac{v^4}{\nu_r(v)} \underset{\sim}{\nabla} f_0(\underset{\sim}{r}, v)\, dv \tag{4.7}$$

4.3 - Free diffusion coefficient

The distribution function $f_0(\underset{\sim}{r}, v)$ is in general a function of the electron number density n_e, the electron speed v, and the electron temperature T_e, so that it can generally be written in the form

$$f_0(\underset{\sim}{r}, v) = n_e\, F(v, T_e) \tag{4.8}$$

since the number density appears only as a result of normalization of the distribution function. The function $f_0(\underset{\sim}{r}, v)$ could be, for example, a local Maxwellian distribution.

For the purpose of calculating the free electron diffusion coefficient, we assume that the electron temperature has no spatial variation, so that

$$\underset{\sim}{\nabla} f_0(\underset{\sim}{r}, v) = \underset{\sim}{\nabla} n_e(\underset{\sim}{r})\; F(v, T_e) \tag{4.9}$$

or, using (4.8),

$$\nabla f_0(\underset{\sim}{r}, v) = \nabla n_e(\underset{\sim}{r}) \frac{f_0(\underset{\sim}{r}, v)}{n_e(\underset{\sim}{r})} \tag{4.10}$$

Substituting (4.10) into (4.7), we obtain

$$\underset{\sim}{\Gamma}_e = -\frac{4\pi}{3} \frac{\nabla n_e(\underset{\sim}{r})}{n_e(\underset{\sim}{r})} \int_0^\infty \frac{v^4}{\nu_r(v)} f_0(\underset{\sim}{r}, v)\, dv \tag{4.11}$$

Defining the free electron diffusion coefficient, D_e, by the relation

$$\underset{\sim}{\Gamma}_e = - D_e \nabla n_e(\underset{\sim}{r}) \tag{4.12}$$

we deduce the following expression for D_e, by inspection of (4.11),

$$D_e = \frac{4\pi}{3\, n_e(\underset{\sim}{r})} \int_0^\infty \frac{v^4}{\nu_r(v)} f_0(\underset{\sim}{r}, v)\, dv \tag{4.13}$$

Note that this expression for D_e is constant, independent of $\underset{\sim}{r}$ and v, in view of Eqs. (4.8) and (4.9).

If we consider $f_0(\underset{\sim}{r}, v)$ as being a modified (or local) Maxwellian distribution function given by

$$f_0(\underset{\sim}{r}, v) = n_e(\underset{\sim}{r}) \left(\frac{m_e}{2\pi k T_e}\right)^{3/2} \exp\left(-\frac{m_e v^2}{2kT_e}\right) \tag{4.14}$$

then (4.13) becomes

$$D_e = \frac{4\pi}{3} \left(\frac{m_e}{2\pi k T_e}\right)^{3/2} \int_0^\infty \frac{v^4}{\nu_r(v)} \exp\left(-\frac{m_e v^2}{2kT_e}\right) dv \qquad (4.15)$$

Furthermore, if the relaxation collision frequency ν_r is taken to be constant, independent of v, then the integral in (4.15) can be explicitly evaluated [see Eq. (7.4.22)], which gives

$$D_e = \frac{k\, T_e}{m_e\, \nu_r} \qquad (4.16)$$

This is the same result obtained in section 8, of Chapter 10 [see Eq. (10.8.9)], which was deduced using the macroscopic transport equations with a constant collision frequency.

5. DIFFUSION IN A MAGNETIC FIELD

In this section we want to include the effects of an externally applied magnetostatic field, $\underset{\sim}{B}_0$, on the problem of electron diffusion in a weakly ionized plasma. We consider the same assumptions made in the previous section, except for the inclusion of the external magnetic field.

5.1 - Solution of Boltzmann equation

Retaining only the first order terms, the linearized Boltzmann equation is now

$$\underset{\sim}{v} \cdot \underset{\sim}{\nabla} f_0(\underset{\sim}{r}, v) - \frac{e}{m_e} (\underset{\sim}{v} \times \underset{\sim}{B}_0) \cdot \underset{\sim}{\nabla}_v f_1(\underset{\sim}{r}, \underset{\sim}{v}) = - \nu_r(v)\, f_1(\underset{\sim}{r}, \underset{\sim}{v}) \qquad (5.1)$$

Note that in view of the isotropy of $f_0(\underset{\sim}{r}, v)$ we can use the identity (2.8), so that

$$(\underset{\sim}{v} \times \underset{\sim}{B}_0) \cdot \underset{\sim}{\nabla}_v\, f_0(\underset{\sim}{r}, v) = 0 \qquad (5.2)$$

In cylindrical coordinates $(v_\perp, \phi, v_\parallel)$ in velocity space (Fig. 2) we have, from (3.7),

$$(\underset{\sim}{v} \times \hat{\underset{\sim}{B}}_0) \cdot \underset{\sim}{\nabla}_v f_1(\underset{\sim}{r}, \underset{\sim}{v}) = - \frac{df_1(\underset{\sim}{r}, \underset{\sim}{v})}{d\phi} \qquad (5.3)$$

Choosing the unit vector $\hat{\underset{\sim}{z}}$ along the magnetic field $\underset{\sim}{B}_0$, we can write

$$\underset{\sim}{v} \cdot \underset{\sim}{\nabla} f_0(\underset{\sim}{r}, v) = (v_\perp \cos\phi \frac{\partial}{\partial x} + v_\perp \sin\phi \frac{\partial}{\partial y} + v_\parallel \frac{\partial}{\partial z})\, f_0\, (\underset{\sim}{r}, v) \qquad (5.4)$$

Substituting (5.4) and (5.3) into (5.1), and rearranging, yields

$$\left[\frac{d}{d\phi} + \frac{\nu_r(v)}{\omega_{ce}}\right] f_1(\underset{\sim}{r}, \underset{\sim}{v}) = -\frac{1}{\omega_{ce}}\left(v_\perp \cos\phi \frac{\partial}{\partial x} + \right.$$

$$\left. + \; v_\perp \sin\phi \frac{\partial}{\partial y} + v_\parallel \frac{\partial}{\partial z}\right) f_o(\underset{\sim}{r}, v) \qquad (5.5)$$

In order to solve this linear differential equation let

$$f_1(\underset{\sim}{r}, \underset{\sim}{v}) = F_1(\underset{\sim}{r}, \underset{\sim}{v}) + F_2(\underset{\sim}{r}, \underset{\sim}{v}) + F_3(\underset{\sim}{r}, \underset{\sim}{v}) \qquad (5.6)$$

where F_1, F_2 and F_3 are the solutions of (5.5) corresponding respectively, to only the first, the second and the third terms within parenthesis in the right-hand side of (5.5), that is,

$$\left[\frac{d}{d\phi} + \frac{\nu_r(v)}{\omega_{ce}}\right] F_1(\underset{\sim}{r}, \underset{\sim}{v}) = -\frac{1}{\omega_{ce}} v_\perp \cos\phi \frac{\partial f_o(\underset{\sim}{r}, v)}{\partial x}$$

(5.7)

$$\left[\frac{d}{d\phi} + \frac{\nu_r(v)}{\omega_{ce}}\right] F_2(\underset{\sim}{r}, \underset{\sim}{v}) = -\frac{1}{\omega_{ce}} v_\perp \sin\phi \frac{\partial f_o(\underset{\sim}{r}, v)}{\partial y}$$

(5.8)

$$\left[\frac{d}{d\phi} + \frac{\nu_r(v)}{\omega_{ce}}\right] F_3(\underset{\sim}{r}, \underset{\sim}{v}) = -\frac{1}{\omega_{ce}} v_{\parallel} \frac{\partial f_o(\underset{\sim}{r}, v)}{\partial z} \qquad (5.9)$$

To solve (5.7) let us first rewrite it in the form

$$\left[\frac{d}{d\phi} + \frac{\nu_r(v)}{\omega_{ce}}\right] F_1(\underset{\sim}{r}, \underset{\sim}{v}) \equiv \exp\left[-\frac{\nu_r(v)}{\omega_{ce}}\phi\right] \cdot$$

$$\cdot \frac{d}{d\phi}\left\{F_1(\underset{\sim}{r}, \underset{\sim}{v}) \exp\left[\frac{\nu_r(v)}{\omega_{ce}}\phi\right]\right\} =$$

$$= -\frac{1}{\omega_{ce}} v_{\perp} \cos\phi \frac{\partial f_o(\underset{\sim}{r}, v)}{\partial x} \qquad (5.10)$$

The solution of this differential equation is given by

$$F_1(\underset{\sim}{r}, \underset{\sim}{v}) = -\frac{1}{\omega_{ce}} v_{\perp} \frac{\partial f_o(\underset{\sim}{r}, v)}{\partial x} \exp\left[-\frac{\nu_r(v)}{\omega_{ce}}\phi\right] \cdot$$

$$\cdot \int_{-\infty}^{\phi} \cos\phi' \exp\left[\frac{\nu_r(v)}{\omega_{ce}}\phi'\right] d\phi'$$

$$= -v_{\perp} \frac{\partial f_o(\underset{\sim}{r}, v)}{\partial x} \frac{[\nu_r(v)\cos\phi + \omega_{ce}\sin\phi]}{[\nu_r^2(v) + \omega_{ce}^2]} \qquad (5.11)$$

Notice that $F_1(\underset{\sim}{r}, \underset{\sim}{v})$ is a periodic function of ϕ, with period 2π.

In a similar way, the solutions of (5.8) and (5.9) are given, respectively, by

$$F_2(\underset{\sim}{r}, \underset{\sim}{v}) = - v_{\perp} \frac{\partial f_o(\underset{\sim}{r}, v)}{\partial y} \frac{[\nu_r(v) \sin\phi - \omega_{ce} \cos\phi]}{[\nu_r^2(v) + \omega_{ce}^2]} \tag{5.12}$$

$$F_3(\underset{\sim}{r}, \underset{\sim}{v}) = - v_{\parallel} \frac{\partial f_o(\underset{\sim}{r}, v)}{\partial z} \frac{1}{\nu_r(v)} \tag{5.13}$$

Adding (5.11), (5.12) and (5.13), gives the solution for $f_1(\underset{\sim}{r}, \underset{\sim}{v})$ in terms of $f_o(\underset{\sim}{r}, v)$ and $\nu_r(v)$.

5.2 - Particle flux and diffusion coefficients

From (4.4), the expression for the x component of the electron flux vector is found to be

$$\Gamma_{ex} = \int_V v_x f_1(\underset{\sim}{r}, \underset{\sim}{v}) d^3v \tag{5.14}$$

In cylindrical coordinates (Fig. 2) we have $d^3v = v_{\perp} dv_{\perp} dv_{\parallel} d\phi$ and $v_x = v_{\perp} \cos\phi$. Therefore,

$$\Gamma_{ex} = \int_0^{\infty} dv_\perp \int_0^{2\pi} d\phi \int_{-\infty}^{+\infty} dv_\parallel \;\; v_\perp^2 \cos\phi \;\; f_1(\underset{\sim}{r}, \underset{\sim}{v}) \tag{5.15}$$

Using Eqs. (5.6), (5.11), (5.12) and (5.13), and performing the integration over ϕ, we obtain

$$\Gamma_{ex} = -\pi \int_0^{\infty} dv_\perp \int_{-\infty}^{+\infty} dv_\parallel \;\; \frac{v_\perp^3 \left[\nu_r(v)\; \partial f_o(\underset{\sim}{r}, v)/\partial x - \omega_{ce}\; \partial f_o(\underset{\sim}{r}, v)/\partial y\right]}{\left[\nu_r^2(v) + \omega_{ce}^2\right]} \tag{5.16}$$

To perform the integrals in (5.16) it is convenient to use spherical coordinates (v, θ, ϕ) in velocity space (Fig. 1). Transforming to spherical coordinates, Eq. (5.16) becomes

$$\Gamma_{ex} = -\pi \int_0^{\infty} dv \int_0^{\pi} d\theta \;\; \cdot$$

$$\cdot \; \frac{v^4 \sin^3\theta \left[\nu_r(v)\; \partial f_o(\underset{\sim}{r}, v)/\partial x \; - \; \omega_{ce}\; \partial f_o(\underset{\sim}{r}, v)/\partial y\right]}{\left[\nu_r^2(v) + \omega_{ce}^2\right]} \tag{5.17}$$

Carrying out the integration over θ, we obtain

$$\Gamma_{ex} = -\frac{4\pi}{3} \int_0^{\infty} \frac{v^4 \nu_r(v)}{\nu_r^2(v) + \omega_{ce}^2} \;\; \frac{\partial f_o(\underset{\sim}{r}, v)}{\partial x} \; dv \; -$$

$$- \frac{4\pi}{3} \int_{o}^{\infty} \frac{v^4 \, \omega_{ce}}{\nu_r^2(v) + \omega_{ce}^2} \; \frac{\partial f_o(\underset{\sim}{r}, v)}{\partial y} \, dv \tag{5.18}$$

This equation can be written in the form

$$\Gamma_{ex} = - \frac{\partial}{\partial x} \left[D_{\perp} \, n_e(\underset{\sim}{r}) \right] - \frac{\partial}{\partial y} \left[- D_H \, n_e(\underset{\sim}{r}) \right] \tag{5.19}$$

where the electron diffusion coefficients $D_{\perp}$ and D_H are given by

$$D_{\perp} = \frac{4\pi}{3 \, n_e \, (\underset{\sim}{r})} \int_{o}^{\infty} \frac{v^4 \, \nu_r(v)}{\nu_r^2(v) + \omega_{ce}^2} \; f_o(\underset{\sim}{r}, v) \, dv \tag{5.20}$$

$$D_H = \frac{4\pi}{3 \, n_e \, (\underset{\sim}{r})} \int_{o}^{\infty} \frac{v^4 \, \omega_{ce}}{\nu_r^2(v) + \omega_{ce}^2} \; f_o(\underset{\sim}{r}, v) \, dv \tag{5.21}$$

Along similar lines, we obtain for the y component of the electron flux vector,

$$\Gamma_{ey} = - \frac{\partial}{\partial x} \left[D_H \, n_e(\underset{\sim}{r}) \right] - \frac{\partial}{\partial y} \left[D_{\perp} \, n_e(\underset{\sim}{r}) \right] \tag{5.22}$$

and for the z-component,

$$\Gamma_{ez} = - \frac{\partial}{\partial z} \left[D_{\parallel} \, n_e(\underset{\sim}{r}) \right] \tag{5.23}$$

where

$$D_{\parallel} = \frac{4\pi}{3\, n_e\,(\underset{\sim}{r})} \int_0^{\infty} \frac{v^4}{\nu_r(v)} \, f_0(\underset{\sim}{r},\, v) \quad dv \tag{5.24}$$

Eqs. (5.19), (5.22) and (5.23) can be written in a succint vector form as

$$\underset{\sim}{\Gamma}_e = - \underset{\sim}{\nabla} \cdot \left[\underset{\approx}{D} \; n_e\,(\underset{\sim}{r}) \right] \tag{5.25}$$

where $\underset{\approx}{D}$ denotes the dyadic coefficient for electron diffusion in a magnetic field, given in matrix form by

$$\underset{\approx}{D} = \begin{pmatrix} D_{\perp} & D_H & 0 \\ -D_H & D_{\perp} & 0 \\ 0 & 0 & D_{\parallel} \end{pmatrix} \tag{5.26}$$

The diffusion coefficient $D_{\parallel}$ is the same as that obtained in the absence of a magnetostatic field ($D_{\parallel} = D_e$). Therefore, the diffusion of particles along the magnetic field is the same as if there were no field present, whereas the diffusion in the perpendicular direction is inhibited by the magnetic field since $D_{\perp} < D_{\parallel}$, as can be verified from Eqs. (5.20) and (5.24).

For the special case in which $f_0(\underset{\sim}{r}, v)$ is given by a local Maxwellian distribution function, as in (4.14), and ν_r is independent of v, the integrals in Eqs. (5.20), (5.21) and (5.24) can be evaluated directly, yielding

$$D_\perp = \frac{\nu_r^2}{\nu_r^2 + \omega_{ce}^2} D_e \tag{5.27}$$

$$D_H = \frac{\nu_r \omega_{ce}}{\nu_r^2 + \omega_{ce}^2} D_e \tag{5.28}$$

$$D_\parallel = D_e = \frac{kT_e}{m_e \nu_r} \tag{5.29}$$

which are the same results obtained in section 9, of Chapter 10, deduced from the macroscopic transport equations [see Eqs. (10.9.4) to (10.9.7)].

6. HEAT FLOW

We shall now derive expressions for the heat flow vector, $\underset{\sim}{q}_e$, and for the thermal conductivity, K_e, due to the random motion of the electrons in a weakly ionized plasma. As in the previous sections, we shall determine the nonequilibrium distribution function $f(\underset{\sim}{r}, \underset{\sim}{v})$, under steady state conditions, by applying a perturbation technique to the Boltzmann equation, using the relaxation model for the collision term. To simplify matters we assume that there are no externally applied electromagnetic fields.

Using (4.1), we find that the Boltzmann equation, for this case, is the same as that given by (4.2). Therefore, as in subsection 4.1,

$$f_1(\underset{\sim}{r}, \underset{\sim}{v}) = - \frac{1}{\nu_r(v)} \underset{\sim}{v} \cdot \underset{\sim}{\nabla} f_0(\underset{\sim}{r}, v) \tag{6.1}$$

6.1 - General expression for the heat flow vector

The expression for the heat flow vector due to the thermal motion of the electrons, and considering $\underset{\sim}{u}_0 = 0$, is

$$\underset{\sim}{q}_e = \frac{1}{2} m_e \int_V v^2 \underset{\sim}{v} f_1(\underset{\sim}{r}, \underset{\sim}{v}) \, d^3v \tag{6.2}$$

Substituting (6.1) into (6.2), yields

$$\underset{\sim}{q}_e = - \frac{1}{2} m_e \int_V \frac{v^2}{\nu_r(v)} \underset{\sim}{v} \left[\underset{\sim}{v} \cdot \underset{\sim}{\nabla} f_0(\underset{\sim}{r}, v) \right] d^3v \tag{6.3}$$

In spherical coordinates in velocity space and using (4.6), we obtain, from (6.3),

$$\underset{\sim}{q}_e = - \frac{2\pi m_e}{3} \int_0^\infty \frac{v^6}{\nu_r(v)} \underset{\sim}{\nabla} f_o(\underset{\sim}{r}, v)\, dv \tag{6.4}$$

This expression gives the electron heat flow vector, $\underset{\sim}{q}_e$, in terms of the distribution function $f_o(\underset{\sim}{r}, v)$ and the relaxation collision frequency $\nu_r(v)$.

6.2 - Thermal conductivity for a constant kinetic pressure

Next, we evaluate (6.4) for the case when $f_o(\underset{\sim}{r}, v)$ is given by a local Maxwellian distribution function,

$$f_o(\underset{\sim}{r}, v) = n_e(\underset{\sim}{r}) \left[\frac{m_e}{2\pi k T_e(\underset{\sim}{r})} \right]^{3/2} \exp \left[- \frac{m_e v^2}{2kT_e(\underset{\sim}{r})} \right] \tag{6.5}$$

in which both n_e and T_e may have a spatial variation, but such that the electron kinetic pressure stays constant, that is,

$$p_e = n_e(\underset{\sim}{r})\, k\, T_e(\underset{\sim}{r}) = \text{constant}. \tag{6.6}$$

From (6.6) we have

$$k\, T_e(\underset{\sim}{r})\, \underset{\sim}{\nabla} n_e(\underset{\sim}{r}) = - n_e(\underset{\sim}{r})\, k\, \underset{\sim}{\nabla} T_e(\underset{\sim}{r}) \tag{6.7}$$

and calculating the gradient of (6.5) we find

$$\underset{\sim}{\nabla} f_o(\underset{\sim}{r}, v) = \left[- \frac{5}{2} + \frac{m_e v^2}{2kT_e(\underset{\sim}{r})} \right] \frac{\underset{\sim}{\nabla} T_e(\underset{\sim}{r})}{T_e(\underset{\sim}{r})} f_o(\underset{\sim}{r}, v) \tag{6.8}$$

Substituting (6.8) into (6.4) gives

$$\underset{\sim}{q}_e = - \frac{2\pi m_e}{3} \frac{\underset{\sim}{\nabla} T_e(\underset{\sim}{r})}{T_e(\underset{\sim}{r})} \int_0^\infty \frac{v^6}{\nu_r(v)} \left[- \frac{5}{2} + \frac{m_e v^2}{2kT_e(\underset{\sim}{r})} \right] f_o(\underset{\sim}{r}, v)\, dv \tag{6.9}$$

This equation can be written in the form

$$\underset{\sim}{q}_e = - K_e \underset{\sim}{\nabla} T_e (\underset{\sim}{r}) \tag{6.10}$$

where K_e is the thermal conductivity coefficient given by

$$K_e = \frac{2\pi m_e}{3T_e(\underset{\sim}{r})} \int_0^\infty \frac{v^6}{\nu_r(v)} \left[- \frac{5}{2} + \frac{m_e v^2}{2kT_e(\underset{\sim}{r})} \right] f_o(\underset{\sim}{r}, v)\, dv \tag{6.11}$$

In the special case when ν_r is independent of v we can write (6.11) as

$$K_e = \frac{2\pi m_e}{3\, T_e(\underset{\sim}{r})\, \nu_r} \left[- \frac{5}{2} \int_0^\infty v^6 f_o(\underset{\sim}{r}, v)\, dv \quad + \right.$$

$$+ \frac{m_e}{2KT_e(\underset{\sim}{r})} \int_0^\infty v^8 \, f_o(\underset{\sim}{r}, v) \, dv \Bigg] \tag{6.12}$$

Now,

$$\int_0^\infty v^6 \, f_o(\underset{\sim}{r}, v) \, dv = \frac{15 \, k \, T_e(\underset{\sim}{r}) \, p_e}{4\pi \, m_e^2} \tag{6.13}$$

$$\int_0^\infty v^8 \, f_o(\underset{\sim}{r}, v) \, dv = \frac{105 \, k^2 \, T_e^2(\underset{\sim}{r}) \, p_e}{4\pi \, m_e^3} \tag{6.14}$$

so that, substituting (6.13) and (6.14) into (6.12) and simplifying, we obtain the following expression for the thermal conductivity, when ν_r = constant,

$$K_e = \frac{5}{2} \, \frac{k \, p_e}{m_e \, \nu_r} \tag{6.15}$$

6.3 - Thermal conductivity for the adiabatic case

We consider now the case when the electron kinetic pressure is not constant, but follows the adiabatic law

$$p_e(\underset{\sim}{r}) \;\; \rho_e^{-\gamma}(\underset{\sim}{r}) = \text{constant} \tag{6.16}$$

where γ is the adiabatic constant, defined as the ratio of the specific heats at constant pressure and at constant volume, which may be expressed as

$$\gamma = \frac{2 + N}{N} \tag{6.17}$$

where N denotes the number of degrees of freedom. Eq. (6.16) can also be written as

$$n_e(\underset{\sim}{r}) \; T_e(\underset{\sim}{r})^{1/(1 - \gamma)} = \text{constant} \tag{6.18}$$

Taking the gradient of the local Maxwellian distribution function (6.5), and making use of (6.18), we obtain

$$\underset{\sim}{\nabla} f_o(\underset{\sim}{r}, v) = \left[\frac{1}{\gamma - 1} - \frac{3}{2} + \frac{m_e v^2}{2k\, T_e(\underset{\sim}{r})} \right] \frac{\underset{\sim}{\nabla} T_e(\underset{\sim}{r})}{T_e(\underset{\sim}{r})} \; f_o(\underset{\sim}{r}, v) \tag{6.19}$$

Now we substitute (6.19) into (6.4), which gives for the heat flow vector

$$\underset{\sim}{q}_e = - \frac{2\pi\, m_e}{3} \; \frac{\underset{\sim}{\nabla} T_e(\underset{\sim}{r})}{T_e(\underset{\sim}{r})} \int_o^\infty \frac{v^6}{\nu_r(v)} \left[\frac{1}{\gamma - 1} - \frac{3}{2} + \right.$$

$$\left. + \frac{m_e\, v^2}{2kT_e(\underset{\sim}{r})} \right] f_o(\underset{\sim}{r}, v)\; dv \tag{6.20}$$

With reference to Eq. (6.10), we identify the following expression for the thermal conductivity

$$K_e = \frac{2\pi m_e}{3 T_e(\underset{\sim}{r})} \int_0^\infty \frac{v^6}{\nu_r(v)} \left[\frac{1}{\gamma - 1} - \frac{3}{2} + \frac{m_e v^2}{2kT_e(\underset{\sim}{r})} \right] f_0(\underset{\sim}{r}, v) \; dv \tag{6.21}$$

For the special case in which ν_r does not depend on v, we can use the results given in (6.13) and (6.14), so that (6.21) simplifies to

$$K_e = \frac{5}{2} \frac{k p_e}{m_e \nu_r} \left(2 + \frac{1}{\gamma - 1}\right) \tag{6.22}$$

If three degrees of freedom corresponding to the three-dimensional translational motion are considered, we have $\gamma = 5/3$, so that

$$K_e = \frac{35}{4} \frac{k p_e}{m_e \nu_r} \tag{6.23}$$

When the plasma is immersed in an externally applied magnetostatic field $\underset{\sim}{B}_0$, an anisotropy is introduced in the thermal energy flux, so that the thermal conductivity coefficient is replaced by a thermal conductivity dyad $\underset{\approx}{K}$. Expressions for the components of the thermal conductivity dyad can be deduced along lines similar to the calculations presented in section 5 for the diffusion coefficient dyad.

The derivation of explicit expressions for the components of $\underset{\approx}{K}$ in a magnetized plasma will be left as an exercise for the reader.

PROBLEMS

22.1 - In Cartesian coordinates in velocity space (refer to Fig. 1), with the components expressed in spherical coordinates (v,θ,ϕ), we have

$$\underset{\sim}{v} = v\,\hat{\underset{\sim}{v}} = v(\sin\theta\ \cos\phi\ \hat{\underset{\sim}{v}}_x + \sin\theta\ \sin\phi\ \hat{\underset{\sim}{v}}_y + \cos\theta\ \hat{\underset{\sim}{v}}_z)$$

(a) Show that the dyad $\underset{\sim}{v}\,\underset{\sim}{v}$ can be written in matrix form as

$$\underset{\sim}{v}\,\underset{\sim}{v} = v^2 \begin{pmatrix} (\sin^2\theta\ \cos^2\phi) & (\sin^2\theta\ \sin\phi\ \cos\phi) & (\sin\theta\ \cos\theta\ \cos\phi) \\ (\sin^2\theta\ \sin\phi\ \cos\phi) & (\sin^2\theta\ \sin^2\phi) & (\sin\theta\ \cos\theta\ \sin\phi) \\ (\sin\theta\ \cos\theta\ \cos\phi) & (\sin\theta\ \cos\theta\ \sin\phi) & (\cos^2\theta) \end{pmatrix}$$

(b) Prove the following orthogonality relations

$$\int_0^{\pi}\int_0^{2\pi} \sin\theta\ d\theta\ d\phi = 4\pi$$

$$\int_0^{\pi}\int_0^{2\pi} v_i\ \sin\theta\ d\theta\ d\phi = 0$$

$$\int_0^{\pi}\int_0^{2\pi} v_i\ v_j\ \sin\theta\ d\theta\ d\phi = \frac{4\pi}{3}\ v^2\ \delta_{ij}$$

$$\int_0^{\pi}\int_0^{2\pi} v_i\ v_j\ v_k\ \sin\theta\ d\theta\ d\phi = 0$$

with i, j, k = x, y, z, and where δ_{ij} is the Kronecker delta.

22.2 - Using the Maxwell-Boltzmann distribution function (2.22) and the definition (2.23), verify Eq. (2.24)

22.3 - Show that, when ν_r is independent of v, (2.25) reduces to (2.20).

22.4 - Consider Eq. (2.25), which gives the AC electric conductivity of a weakly ionized plasma for a velocity-dependent collision frequency $\nu_r(v)$.

(a) Show that in the high-frequency limit, $\omega^2 \gg \nu_r^2$, we have

$$\sigma = \frac{n_o \; e^2}{m_e \; \omega^2} \; (\nu_c + i\omega)$$

where

$$\nu_c = \frac{8}{3\pi^{1/2}} \int_o^\infty \nu_r(\xi) \; \xi^4 \; \exp(-\xi^2) \; d\xi$$

(b) Show that in the low-frequency limit, $\omega^2 \ll \nu_r^2$, we have

$$\sigma = \frac{n_o\ e^2}{m_e} \left[\frac{1}{\nu_c'} + i\ \frac{\omega}{(\nu_c'')^2} \right]$$

where

$$\frac{1}{\nu_c'} = \frac{8}{3\pi^{1/2}} \int_o^\infty \frac{1}{\nu_r(\xi)}\ \xi^4\ \exp(-\xi^2)\ d\xi$$

$$\frac{1}{(\nu_c'')^2} = \frac{8}{3\pi^{1/2}} \int_o^\infty \frac{1}{\nu_r(\xi)^2}\ \xi^4\ \exp(-\xi^2)\ d\xi$$

(c) For intermediate frequencies, show that

$$\sigma = \frac{n_o\ e^2}{m_e} (\nu_c\ K_1 + i\,\omega\, K_2)$$

where

$$K_1 = \frac{8}{3\pi^{1/2}} \quad \frac{1}{\nu_c\ \omega^2} \int_o^\infty \left[1 - \frac{\nu_r(\xi)^2}{\omega^2} + \frac{\nu_r(\xi)^4}{\omega^4} + \right.$$

$$\left. + \ \ldots \right] \nu_r\ (\xi)\ \xi^4\ \exp(-\xi^2)\ d\xi$$

$$K_2 = \frac{8}{3\pi^{1/2}}\ \frac{1}{\omega^2} \int_o^\infty \left[1 - \frac{\nu_r(\xi)^2}{\omega^2} + \frac{\nu_r(\xi)^4}{\omega^4} + \ \ldots \right] \xi^4 \exp(-\xi^2)\ d\xi$$

and where ν_c is the same quantity defined in part (a) for the high frequency limit.

22.5 - If we define an *effective collision frequency,* $\nu_{eff}(\omega)$, such that the longitudinal electric conductivity is given by

$$\sigma = \frac{i\, n_o\, e^2}{m_e\,[\omega + i\,\nu_{eff}(\omega)]}$$

then, by comparison with Eq. (2.18), we find that

$$[\omega + i\,\nu_{eff}(\omega)]^{-1} = -\frac{4\pi}{3n_o}\int_o^\infty \frac{v^3}{[\omega + i\,\nu_r(v)]}\,\frac{d f_o(v)}{dv}\,dv$$

(a) Show that in the low-frequency limit, $\omega \ll \nu_{eff}(\omega)$, we have

$$\frac{1}{\nu_{eff}} = -\frac{4\pi}{3n_o}\int_o^\infty \frac{v^3}{\nu_r(v)}\,\frac{d f_o(v)}{dv}\,dv$$

(b) Show that in the high-frequency limit, $\omega \gg \nu_{eff}(\omega)$, we have

$$\nu_{eff} = -\frac{4\pi}{3n_o}\int_o^\infty v^3\,\nu_r(v)\,\frac{d f_o(v)}{dv}\,dv$$

Thus, in both limits ν_{eff} is independent of ω.

22.6 - In the expression deduced for ν_{eff} in part (b) of the previous problem (high-frequency limit), consider that f_o is the Maxwell-Boltzmann distribution function and that $\nu_r(v) = \nu_o\, v^n$, where ν_o is a constant and n is an integer.

(a) Show that in this case we have

$$\nu_{eff} = \frac{4\nu_o}{3\pi^{1/2}} \left(\frac{2kT}{m} \right)^{n/2} \Gamma\left(\frac{n+5}{2} \right)$$

where $\Gamma(z)$ is the gamma function defined by

$$\Gamma(z) = \int_0^\infty t^{(z-1)}\, e^{-t}\, dt$$

(b) Calculate the average value of the collision frequency, $< \nu_r(v) >_o$, using the Maxwell-Boltzmann distribution function, and show that

$$< \nu_r(v) >_o = \frac{\nu_{eff}}{(1 + n/3)}$$

22.7 - Derive Eq. (3.34), from Eqs. (3.29) to (3.33).

22.8 - Show that Eqs. (3.30) and (3.31) yield, respectively, Eqs. (3.37) and (3.38), when ν_r is independent of v for any $f_o(v)$.

22.9 - Deduce Eqs. (5.22) and (5.23) starting from the definition of the electron flux vector, and the expression for $f_1(\underset{\sim}{r}, \underset{\sim}{v})$ given by Eqs. (5.6), (5.11), (5.12) and (5.13).

22.10 - Analyze the problem of heat flow in a weakly ionized plasma immersed in an externally applied magnetostatic field, $\underset{\sim}{B}_o$, and derive expressions for the heat flow vector, $\underset{\sim}{q}_e$, and for the components of the thermal conductivity dyad, $\underset{\approx}{K}$, considering a velocity-dependent collision frequency, $\nu_r(v)$. Analyze the problem for the adiabatic case and for the case of a constant kinetic pressure.

APPENDIX I

USEFUL VECTOR RELATIONS

(1) $\mathbf{A}.\mathbf{B} = \mathbf{B}.\mathbf{A} = A_x B_x + A_y B_y + A_z B_z$

(2) $\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$

(3) $\mathbf{A}.\ (\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\ .\ \mathbf{C} = (\mathbf{C}\times\mathbf{A})\ .\ \mathbf{B}$

(4) $\mathbf{A}\times\ (\mathbf{B}\times\mathbf{C}) = (\mathbf{A}.\mathbf{C})\ \mathbf{B} - (\mathbf{A}.\mathbf{B})\ \mathbf{C}$

(5) $(\mathbf{A}\times\mathbf{B}) \times \mathbf{C} = (\mathbf{A}.\mathbf{C})\ \mathbf{B} - (\mathbf{B}.\mathbf{C})\ \mathbf{A}$

(6) $(\mathbf{A}\times\mathbf{B})\ .\ (\mathbf{C}\times\mathbf{D}) = (\mathbf{A}.\mathbf{C})\ (\mathbf{B}.\mathbf{D}) - (\mathbf{A}.\mathbf{D})(\mathbf{B}.\mathbf{C})$

(7) $(\mathbf{A}\times\mathbf{B}) \times (\mathbf{C}\times\mathbf{D}) = [\mathbf{A}.(\mathbf{B}\times\mathbf{D})]\ \mathbf{C} - [\mathbf{A}.(\mathbf{B}\times\mathbf{C})]\ \mathbf{D}$

(8) $\boldsymbol{\nabla}(\phi\psi) = \phi\boldsymbol{\nabla}\psi + \psi\boldsymbol{\nabla}\phi$

(9) $\boldsymbol{\nabla}.(\phi\mathbf{A}) = \phi\boldsymbol{\nabla}.\mathbf{A} + \mathbf{A}.\boldsymbol{\nabla}\phi$

(10) $\boldsymbol{\nabla}\times(\phi\mathbf{A}) = \phi\boldsymbol{\nabla}\times\mathbf{A} + (\boldsymbol{\nabla}\phi) \times \mathbf{A}$

(11) $\boldsymbol{\nabla}.\ (\mathbf{A}\times\mathbf{B}) = \mathbf{B}\ (\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}.\ (\boldsymbol{\nabla}\times\mathbf{B})$

(12) $\boldsymbol{\nabla}\ (\mathbf{A}.\mathbf{B}) = (\mathbf{A}.\boldsymbol{\nabla})\ \mathbf{B} + (\mathbf{B}.\boldsymbol{\nabla})\ \mathbf{A} + \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A})$

(13) $\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\boldsymbol{\nabla}.\mathbf{B}) + (\mathbf{B}.\boldsymbol{\nabla})\ \mathbf{A} - \mathbf{B}(\boldsymbol{\nabla}.\mathbf{A}) - (\mathbf{A}.\boldsymbol{\nabla})\ \mathbf{B}$

(14) $\underset{\sim}{\nabla} \times (\underset{\sim}{\nabla} \times \underset{\sim}{A}) = \underset{\sim}{\nabla}(\underset{\sim}{\nabla} . \underset{\sim}{A}) - (\underset{\sim}{\nabla} . \underset{\sim}{\nabla}) \underset{\sim}{A}$

(15) $\underset{\sim}{\nabla} . (\underset{\sim}{\nabla} \times \underset{\sim}{A}) = 0$

(16) $\underset{\sim}{\nabla} \times (\underset{\sim}{\nabla} \phi) = 0$

(17) $(\underset{\sim}{\nabla} . \underset{\sim}{\nabla}) \phi = \nabla^2 \phi$

If $\underset{\sim}{r}$ is the radius vector, of magnitude r, drawn from the origin to a general point x,y,z, then

(18) $\underset{\sim}{\nabla} . \underset{\sim}{r} = 3$

(19) $\underset{\sim}{\nabla} \times \underset{\sim}{r} = 0$

(20) $\underset{\sim}{\nabla} r = \underset{\sim}{r}/r$

(21) $\underset{\sim}{\nabla}(1/r) = - \underset{\sim}{r}/r^3$

(22) $\underset{\sim}{\nabla} . (\underset{\sim}{r}/r^3) = -\nabla^2(1/r) = 4\pi \, \delta(\underset{\sim}{r})$

In the following integral relations, V is the volume bounded by the closed surface S and $\hat{\underset{\sim}{n}}$ is a unit normal vector drawn outwardly to the closed surface S:

(23) $\oint_S \phi \, \hat{\underset{\sim}{n}} \, dS = \int (\underset{\sim}{\nabla} \phi) \, dV$

(24) $\oint_S \underset{\sim}{A} . \hat{\underset{\sim}{n}} \, dS = \int_V (\underset{\sim}{\nabla} . \underset{\sim}{A}) \, dV$ (Gauss' theorem)

(25) $\oint_S (\hat{\underset{\sim}{n}} \times \underset{\sim}{A}) \, dS = \int_V (\underset{\sim}{\nabla} \times \underset{\sim}{A}) \, dV$

(26) $\oint_S \phi \; (\underset{\sim}{\nabla}\psi) \cdot \underset{\sim}{\hat{n}} \, dS = \int_V \left[\phi\nabla^2\psi + (\underset{\sim}{\nabla}\phi) \cdot (\underset{\sim}{\nabla}\psi)\right] dV$ (Green's first identity)

(27) $\oint_S (\phi\underset{\sim}{\nabla}\psi - \psi\underset{\sim}{\nabla}\phi) \cdot \underset{\sim}{\hat{n}} \, dS = \int (\phi\nabla^2\psi - \psi\nabla^2\phi) \, dV$ (Green's second identity or Green's theorem)

(28) $\oint_S \left[\underset{\sim}{B}\times(\underset{\sim}{\nabla}\times\underset{\sim}{A}) - \underset{\sim}{A}\times(\underset{\sim}{\nabla}\times\underset{\sim}{B}) \right] \cdot \underset{\sim}{\hat{n}} \, dS =$

$= \int \{ \underset{\sim}{A} \cdot \left[\underset{\sim}{\nabla}\times(\underset{\sim}{\nabla}\times\underset{\sim}{B})\right] - \underset{\sim}{B} \cdot \left[\underset{\sim}{\nabla}\times(\underset{\sim}{\nabla}\times\underset{\sim}{A})\right] \} \, dV$ (Vector version of Green's theorem)

If S is an open surface bounded by the contour C, of which the line element is $d\underset{\sim}{\ell}$, then

(29) $\oint_C \phi \, d\underset{\sim}{\ell} = \int_S \underset{\sim}{\hat{n}} \times (\underset{\sim}{\nabla}\phi) \, dS$

(30) $\oint_C \underset{\sim}{A} \cdot d\underset{\sim}{\ell} = \int_S (\underset{\sim}{\nabla}\times\underset{\sim}{A}) \cdot \underset{\sim}{\hat{n}} \, dS$ (Stoke's theorem)

If $\underset{\approx}{T}$ is a tensor, then

(31) $\underset{\sim}{\nabla} \cdot (\phi\underset{\approx}{T}) = \phi(\underset{\sim}{\nabla}\cdot\underset{\approx}{T}) + (\underset{\sim}{\nabla}\phi) \cdot \underset{\approx}{T}$

(32) $\oint_S \underset{\approx}{T} \cdot \underset{\sim}{\hat{n}} \, dS = \int_V (\underset{\sim}{\nabla}\cdot\underset{\approx}{T}) \, dV$

APENDIX II

USEFUL RELATIONS IN CARTESIAN AND IN CURVILINEAR COORDINATES

1. CARTESIAN COORDINATES (x,y,z)

Orthogonal unit vectors:

$\hat{\underset{\sim}{x}}, \hat{\underset{\sim}{y}}, \hat{\underset{\sim}{z}}$

Orthogonal line elements:

dx, dy, dz

Components of gradient:

$$(\underset{\sim}{\nabla}\psi)_x = \frac{\partial\psi}{\partial x}$$

$$(\underset{\sim}{\nabla}\psi)_y = \frac{\partial\psi}{\partial y}$$

$$(\underset{\sim}{\nabla}\psi)_z = \frac{\partial\psi}{\partial z}$$

Divergence:

$$\underset{\sim}{\nabla}.\underset{\sim}{A} = \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z$$

Components of curl:

$$(\underset{\sim}{\nabla}x\underset{\sim}{A})_x = (\frac{\partial}{\partial y} A_z - \frac{\partial}{\partial z} A_y)$$

$$(\underset{\sim}{\nabla}x\underset{\sim}{A})_y = (\frac{\partial}{\partial z} A_x - \frac{\partial}{\partial x} A_z)$$

$$(\underset{\sim}{\nabla}x\underset{\sim}{A})_z = (\frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x)$$

Laplacian:

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$$

Components of divergence of a tensor:

$$(\underset{\sim}{\nabla}\cdot\underset{\approx}{T})_x = \frac{\partial}{\partial x} T_{xx} + \frac{\partial}{\partial y} T_{yx} + \frac{\partial}{\partial z} T_{zx}$$

$$(\underset{\sim}{\nabla}\cdot\underset{\approx}{T})_y = \frac{\partial}{\partial x} T_{xy} + \frac{\partial}{\partial y} T_{yy} + \frac{\partial}{\partial z} T_{zy}$$

$$(\underset{\sim}{\nabla}\cdot\underset{\approx}{T})_z = \frac{\partial}{\partial x} T_{xz} + \frac{\partial}{\partial y} T_{yz} + \frac{\partial}{\partial z} T_{zz}$$

2. <u>CYLINDRICAL COORDINATES</u> (ρ,ϕ,z)

Orthogonal unit vectors:

$\underset{\sim}{\hat{\rho}}, \underset{\sim}{\hat{\phi}}, \underset{\sim}{\hat{z}}$

Orthogonal line elements:

$d\rho,\ \rho d\phi,\ dz$

Components of gradient:

$$(\underset{\sim}{\nabla}\psi)_\rho = \frac{\partial\psi}{\partial\rho}$$

$$(\underset{\sim}{\nabla}\psi)_\phi = \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}$$

$$(\underset{\sim}{\nabla}\psi)_z = \frac{\partial\psi}{\partial z}$$

Divergence:

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial}{\partial\phi} A_\phi + \frac{\partial}{\partial z} A_z$$

Components of curl:

$$(\underset{\sim}{\nabla} x \underset{\sim}{A})_\rho = \frac{1}{\rho} \frac{\partial}{\partial \phi} A_z - \frac{\partial}{\partial z} A_\phi$$

$$(\underset{\sim}{\nabla} x \underset{\sim}{A})_\phi = \frac{\partial}{\partial z} A_\rho - \frac{\partial}{\partial \rho} A_z$$

$$(\underset{\sim}{\nabla} x \underset{\sim}{A})_z = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\phi) - \frac{1}{\rho} \frac{\partial}{\partial \phi} A_\rho$$

Laplacian:

$$\nabla^2 \psi = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \frac{\partial \psi}{\partial \rho}) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2}$$

Components of divergence of a tensor:

$$(\underset{\sim}{\nabla} \cdot \underset{\approx}{T})_\rho = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho T_{\rho\rho}) + \frac{1}{\rho} \frac{\partial}{\partial \phi} (T_{\phi\rho}) + \frac{\partial}{\partial z} T_{z\rho} - \frac{1}{\rho} T_{\phi\phi}$$

$$(\underset{\sim}{\nabla} \cdot \underset{\approx}{T})_\phi = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho T_{\rho\phi}) + \frac{1}{\rho} \frac{\partial}{\partial \phi} T_{\phi\phi} + \frac{\partial}{\partial z} T_{z\phi} + \frac{1}{\rho} T_{\phi\rho}$$

$$(\underset{\sim}{\nabla} \cdot \underset{\approx}{T})_z = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho T_{\rho z}) + \frac{1}{\rho} \frac{\partial}{\partial \phi} T_{\phi z} + \frac{\partial}{\partial z} T_{zz}$$

3. SPHERICAL COORDINATES (r,θ,ϕ)

Orthogonal unit vectors:

$\underset{\sim}{\hat{r}}, \underset{\sim}{\hat{\theta}}, \underset{\sim}{\hat{\phi}}$

Orthogonal line elements:

dr, $r d\theta$, $r \sin\theta \, d\phi$

Components of gradient:

$$(\underset{\sim}{\nabla}\psi)_r = \frac{\partial\psi}{\partial r}$$

$$(\underset{\sim}{\nabla}\psi)_\theta = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

$$(\underset{\sim}{\nabla}\psi)_\phi = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

Divergence:

$$\underset{\sim}{\nabla}.\underset{\sim}{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) +$$

$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}A_\phi$$

Components of curl:

$$(\underset{\sim}{\nabla}\mathrm{x}\underset{\sim}{A})_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}A_\theta$$

$$(\underset{\sim}{\nabla}\mathrm{x}\underset{\sim}{A})_\theta = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}A_r - \frac{1}{r}\frac{\partial}{\partial r}(r\,A_\phi)$$

$$(\underset{\sim}{\nabla}\mathrm{x}\underset{\sim}{A})_\phi = \frac{1}{r}\frac{\partial}{\partial r}(r\,A_\theta) - \frac{1}{r}\frac{\partial}{\partial\theta}A_r$$

Laplacian:

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) +$$

$$+ \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

Components of divergence of a tensor:

$$(\underset{\sim}{\nabla}\cdot\underset{\approx}{T})_r = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{rr}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(T_{\theta r}\sin\theta) + $$

$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}T_{\phi r} - \frac{1}{r}(T_{\theta\theta} + T_{\phi\phi})$$

$$(\underset{\sim}{\nabla}\cdot\underset{\approx}{T})_\theta = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(T_{\theta\theta}\sin\theta) + $$

$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}T_{\phi\theta} + \frac{1}{r}(T_{\theta r} - \cot\theta\, T_{\phi\phi})$$

$$(\underset{\sim}{\nabla}\cdot\underset{\approx}{T})_\phi = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\phi}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(T_{\theta\phi}\sin\theta) + $$

$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}T_{\phi\phi} + \frac{1}{r}(T_{\phi r} + \cot\theta\, T_{\phi\theta})$$

APPENDIX III

PHYSICAL CONSTANTS (MKSA)

c	Speed of light in vacuum	2.998×10^{8} m/sec
ε_o	Permittivity of vacuum	8.854×10^{-12} farad/m
μ_o	Permeability of vacuum	$4\pi \times 10^{-7}$ henry/m
h	Planck's constant	6.626×10^{-34} joule . sec
k	Boltzmann's constant	1.381×10^{-23} joule/K
G	Gravitational constant	6.671×10^{-11} $N m^2 / kg^2$
e	Charge of proton	1.602×10^{-19} coul
m_e	Rest mass of proton	1.673×10^{-27} kg
m_p	Rest mass of electron	9.109×10^{-31} kg
m_n	Rest mass of neutron	1.675×10^{-27} kg
m_p/m_e	Proton/electron mass ratio	1.836×10^{3}
amu	Unified atomic mass unit	1.661×10^{-27} kg
a_o	Bohr radius	5.292×10^{-11} m
r_e	Classical electron radius	2.818×10^{-15} m
N_A	Avogadro's number	6.022×10^{23} mol^{-1}
N_L	Loschmidt's number	2.687×10^{25} m^{-3}
V_o	Molar volume at STP	22.4×10^{-3} m^3/mol
R	Gas constant ($N_A k$)	8.314 joule/(K mol)
g	Standard acceleration of gravity	9.807 m/sec^2

APPENDIX IV

CONVERSION FACTORS FOR UNITS

Charge:	1 coulomb = 2.998×10^9 statcoulomb
Current:	1 ampere $\equiv$ 1 coul/sec = 2.998×10^9 statampere
Potential	1 volt = $(2.998 \times 10^2)^{-1}$ statvolt
Electric field:	1 volt/m = $(2.998 \times 10^4)^{-1}$ statvolt/cm
Magnetic induction:	1 weber/m^2 $\equiv$ 1 tesla = 10^4 gauss
Resistance :	1 ohm = $(2.998)^{-2} \times 10^{-11}$ sec/cm
Conductivity:	1 mho/m = $(2.998)^2 \times 10^9$ sec^{-1}
Capacitance:	1 farad = $(2.998)^2 \times 10^{11}$ cm
Magnetic flux:	1 weber = 10^8 gauss . cm^2 (or maxwells)
Magnetic field:	1 ampere-turn/m = $4\pi \times 10^{-3}$ oersted
Force:	1 newton = 10^5 dyne
Energy:	1 joule = 10^7 erg
	1 electron volt (ev) = 1.602×10^{-19} joule
	1 ev = kT, where k is Boltzmann's constant, for T = 1.160×10^4 K
	1 Rydberg = 13.61 ev
Power:	1 watt $\equiv$ 1 joule/sec = 10^2 erg/sec
Pressure:	1 newton/m^2 = 10 dyne/cm^2
	1 atm = 760 mm Hg = 1.013×10^5 newton/m^2
	1 torr = 1 mm Hg

APPENDIX V

SOME IMPORTANT PLASMA PARAMETERS

1. Electron plasma frequency:

$$\omega_{pe} = \left(\frac{n_e \, e^2}{m_e \, \varepsilon_o} \right)^{1/2} = 56.5 \; n_e^{1/2} \text{ rad/sec } (n_e \text{ in } m^{-3})$$

2. Ion plasma frequency:

$$\omega_{pi} = \left(\frac{n_i \, Z^2 \, e^2}{m_i \, \varepsilon_o} \right)^{1/2}$$

3. Debye length:

$$\lambda_D = \left(\frac{\varepsilon_o \, k \, T}{n_e \, e^2} \right)^{1/2} = 69.0 \left(\frac{T}{n_e} \right)^{1/2} \text{ m}$$

(T in degrees K and n_e in m^{-3})

4. Electron cyclotron frequency:

$$\omega_{ce} = \frac{eB}{m_e} = 1.76 \times 10^{11} B \text{ rad/sec (B in tesla)}$$

5. Ion cyclotron frequency:

$$\omega_{ci} = \frac{Z\,e\,B}{m_i}$$

6. Particle magnetic moment:

$$\underset{\sim}{m} = -\frac{W_\perp}{B^2}\,\underset{\sim}{B} = -\frac{m\,v_\perp^2/2}{B^2}\,\underset{\sim}{B}$$

7. Electron cyclotron radius:

$$r_{ce} = \frac{v_{\perp e}}{\omega_{ce}} = \frac{m_e\,v_{\perp e}}{e\,B}$$

8. Ion cyclotron radius:

$$r_{ci} = \frac{v_{\perp i}}{\omega_{ci}} = \frac{m_i\,v_{\perp i}}{Z\,e\,B}$$

9. Number of electrons in Debye sphere:

$$N_D = \frac{4}{3}\,\pi\,\lambda_D^3\,n_e = 1.37 \times 10^6\,\frac{T^{3/2}}{n_e^{1/2}}$$

(T in degrees K and n_e in m^{-3})

10 - Alfvén velocity:

$$\underset{\sim}{V}_A = \frac{\underset{\sim}{B}}{(\mu_o \ \rho)^{1/2}}$$

11. DC conductivity :

$$\sigma_o = \frac{n_e \ e^2}{m_e \ \nu_e}$$

12. Free electron diffusion coefficient:

$$D_e = \frac{k \, T_e}{m_e \ \nu_e}$$

13. Ambipolar diffusion coefficient:

$$D_a = \frac{k(T_e + T_i)}{m_e \ \nu_{en} + m_i \ \nu_{in}}$$

14. Magnetic pressure:

$$p_m = \frac{B^2}{2\mu_o}$$

15. Magnetic viscosity:

$$\eta_m = \frac{1}{\mu_o \ \sigma_o}$$

16. Magnetic Reynolds number:

$$R_m = \frac{uL}{\eta_m}$$

17. Coulomb cutoff parameter:

$$\Lambda = 12 \pi n_e \lambda_D^3 = 9 N_D$$

$$= 1.23 \times 10^7 \frac{T^{3/2}}{n_e^{1/2}}$$

(T in degrees K and n_e in m^{-3})

18. Electron collision frequencies for momentum transfer:

$$\nu_{ei} = 3.62 \times 10^{-6} n_i T_e^{-3/2} \ln \Lambda \ \sec^{-1}$$

$$\nu_{en} = 2.60 \times 10^{4} \sigma^2 n_n T_e^{1/2} \ \sec^{-1}$$

(T in degrees K, $n_{i,n}$ in m^{-3}; σ is the sum of the radii of the colliding particles and is of the order of 10^{-10} m, and $\ln \Lambda$ is typically about 10).

APPENDIX VI

APPROXIMATE MAGNITUDES IN SOME TYPICAL PLASMAS

PLASMA TYPE	n_o (m^{-3})	T (ev)	ω_{pe} (sec^{-1})	λ_D (m)	$n_o \lambda_D^3$
Interstellar gas	10^6	10^{-1}	6×10^4	1	10^6
Interplanetary gas	10^8	1	6×10^5	1	10^8
Solar corona	10^{12}	10^2	6×10^7	10^{-1}	10^9
Solar atmosphere	10^{20}	1	6×10^{11}	10^{-6}	10^2
Ionosphere	10^{12}	10^{-1}	6×10^7	10^{-3}	10^4
Gas discharge	10^{20}	1	6×10^{11}	10^{-6}	10^2
Hot plasma	10^{20}	10^2	6×10^{11}	10^{-5}	10^5
Diffuse hot plasma	10^{18}	10^2	6×10^{10}	10^{-4}	10^6
Dense hot plasma	10^{22}	10^2	6×10^{12}	10^{-6}	10^4
Thermonuclear plasma	10^{22}	10^4	6×10^{12}	10^{-5}	10^7