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16.Summary/Notes

This chapter analyses the basic phenomena of electric conductivity and diffusion in plasmas. Expressions are derived for the DC and AC conductivity of an isotropic plasma, as well as of an anisotropic magnetoplasma, using the macroscopic equations pertinent to the cold plasma model. The problems of free diffusion and ambipolar diffusion in plasmas, are analysed in terms of the warm plasma model. Diffusion in a fully ionized plasma is also discussed, using the magnetohydrodynamic equations.

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CHAPTER 10

PLASMA CONDUCTIVITY AND DIFFUSION

1. INTRODUCTION

In the previous chapters we have introduced the fundamental elements of kinetic theory and the macroscopic transport equations necessary for the study of a variety of important phenomena in plasmas. Many plasma phenomena can be analyzed using the macroscopic transport equations, either considering the plasma as a multi-constituent fluid or by treating the whole plasma as a single conducting fluid. In some cases, however, a satisfactory description can only be obtained through the use of the phase space distribution function.

In this and in the following chapter we investigate a number of basic plasma phenomena which illustrate the use of the cold and warm plasma models, and of the phase space distribution function. Phenomena that can be analyzed treating the whole plasma as a single conducting fluid are usually considered under the general title of magnetohydrodynamics (MHD), and will be studied in Chapters 12, 13 and 15.

2. THE LANGEVIN EQUATION

Before we consider the phenomena of plasma conductivity and diffusion, it is convenient to introduce a very simple form of the equation of conservation of momentum for a weakly ionized cold plasma, known as the Langevin equation. In a weakly ionized plasma the number density of the charged particles are much smaller than that of the neutral particles. In this case the interactions between charged and neutral particles are dominant. The electron-electron and electron-ion interactions are considered to be of secondary importance.

The *macroscopic* equation of motion for an *average* electron, under the action of the Lorentz force and the collisional forces, can be written as

$$m_e \frac{D\underset{\sim}{u}_e}{Dt} = - e\ (\underset{\sim}{E} + \underset{\sim}{u}_e \times \underset{\sim}{B}) + (\underset{\sim}{F}_{coll})_e \qquad (2.1)$$

where $\underset{\sim}{u}_e(\underset{\sim}{r}, t)$ is the average electron velocity and $(\underset{\sim}{F}_{coll})_e$ denotes symbolically the rate of change of the average electron momentum due to collisions with neutral particles. The macroscopic collision term $(\underset{\sim}{F}_{coll})_e$ can be expressed in a phenomenological way as the product of the average electron momentum with an *effective* constant collision frequency, ν_c, for momentum transfer between electrons and heavy (neutral) particles,

$$(\underset{\sim}{F}_{coll})_e = - \nu_c\ m_e\ \underset{\sim}{u}_e \qquad (2.2)$$

In doing this we are neglecting the average motion of the neutral particles, as they are much more massive than the electrons. Note that this does not mean that the velocities of the individual neutral particles are zero, but only that they are completely *random* so that their *average* velocity is zero. We obtain, therefore, the following equation, known as the *Langevin equation*,

$$m_e \frac{D\underset{\sim}{u}_e}{Dt} = - e\ (\underset{\sim}{E} + \underset{\sim}{u}_e \times \underset{\sim}{B}) - \nu_c\ m_e\ \underset{\sim}{u}_e \tag{2.3}$$

The effect of the collision term in the Langevin equation can be seen as follows. In the absence of the electric and magnetic fields, Eq. (2.3) reduces to

$$\frac{D\underset{\sim}{u}_e}{Dt} = - \nu_c\ \underset{\sim}{u}_e \tag{2.4}$$

whose solution is

$$\underset{\sim}{u}_e\ (t) = \underset{\sim}{u}_e\ (0)\ \exp\ (-\nu_c t) \tag{2.5}$$

Thus, the collisions between electrons and neutral particles tend to decrease the average electron velocity exponentially in time, at a rate governed by the collision frequency.

An equation analogous to Eq.(2.3) can be written for the ions

$$m_i \frac{D\underset{\sim}{u}_i}{Dt} = e\,(\underset{\sim}{E} + \underset{\sim}{u}_i \times \underset{\sim}{B}) - (\underset{\sim}{F}_{coll})_i \qquad (2.6)$$

where $\underset{\sim}{u}_i$ denotes the average ion velocity. In many cases of interest, as in high-frequency phenomena, we can neglect the motion of the heavy ions and assume that their average velocity is zero, since the mass of the ions is typically about 10^3 or 10^4 times greater than the mass of the electrons. The type of plasma in which only the motion of the electrons is important, is usually called a *Lorentz gas*. When dealing with very low frequencies, however, the motion of the ions must be considered.

Despite the approximations implicit in the Langevin equation, it has been successfully used to describe a variety of phenomena in plasmas, including the propagation of electromagnetic waves in cold magnetoplasmas. Particularly, the analysis of the characteristics of electromagnetic wave propagation in the Earth's ionosphere, using this equation, has been quite sucessful. A great advantage of the Langevin equation is its simplicity.

3. LINEARIZATION OF THE LANGEVIN EQUATION

In the form presented in Eq. (2.3), the Langevin equation contains nonlinear terms which involve the product of two variables. In many situations of interest the difficulty inherent in the nonlinear terms can be eliminated through a linearization approximation.

The total time derivative contains the nonlinear convective term $(\underset{\sim}{u}_e . \underset{\sim}{\nabla}) \underset{\sim}{u}_e$, which is called the inertial term in hydrodynamics. The omission of this inertial term is justified when the average velocity and its space derivatives are small, or when $\underset{\sim}{u}_e$ is normal to its gradient (such as in the case of transverse waves).

For the nonlinear term $\underset{\sim}{u}_e \times \underset{\sim}{B}$, we can separate the magnetic flux density $\underset{\sim}{B}(\underset{\sim}{r}, t)$ into two parts

$$\underset{\sim}{B} (\underset{\sim}{r}, t) = \underset{\sim}{B}_o + \underset{\sim}{B}' (\underset{\sim}{r}, t) \tag{3.1}$$

where $\underset{\sim}{B}_o$ is constant and $\underset{\sim}{B}'(\underset{\sim}{r}, t)$ is the variable component. Without any loss of generality we may write for the Lorentz force term

$$q (\underset{\sim}{E} + \underset{\sim}{u}_e \times \underset{\sim}{B}) = q (\underset{\sim}{E} + \underset{\sim}{u}_e \times \underset{\sim}{B}_o + \underset{\sim}{u}_e \times \underset{\sim}{B}') \tag{3.2}$$

For situations in which

$$|\underset{\sim}{u}_e \times \underset{\sim}{B}'| << |\underset{\sim}{E}| \tag{3.3}$$

the nonlinear term $\underset{\sim}{u}_e \times \underset{\sim}{B}'$ in (3.2) can be neglected.

With the linearization approximations the Langevin equation becomes

$$m_e \frac{\partial \underset{\sim}{u}_e}{\partial t} = - e (\underset{\sim}{E} + \underset{\sim}{u}_e \times \underset{\sim}{B}_o) - m_e \nu_c \underset{\sim}{u}_e \tag{3.4}$$

For steady state problems, and for many problems involving wave propagation, this linearized form of the Langevin equation is applicable.

A case of great practical interest is the one in which the variables $\underset{\sim}{E}$, $\underset{\sim}{B}'$ and $\underset{\sim}{u}_e$ all vary harmonically in space and time. The treatment in terms of plane waves has the advantage of great mathematical simplicity, besides the fact that any complex and physically realizable wave motion can be synthesized in terms of a superposition of plane waves. Let us consider, therefore, plane wave solutions for $\underset{\sim}{E}$, $\underset{\sim}{B}'$ and $\underset{\sim}{u}_e$ in the form

$$\underset{\sim}{E},\ \underset{\sim}{B}',\ \underset{\sim}{u}_e \propto \exp\left[i\ (\underset{\sim}{k}\cdot\underset{\sim}{r} - \omega t)\right] \qquad (3.5)$$

where ω denotes the angular frequency, $\underset{\sim}{k}$ is the propagation vector normal to the wave front, and $\underset{\sim}{r}$ is a position vector drawn from the origin of a coordinate system to the point considered on the wave front (Fig. 1).

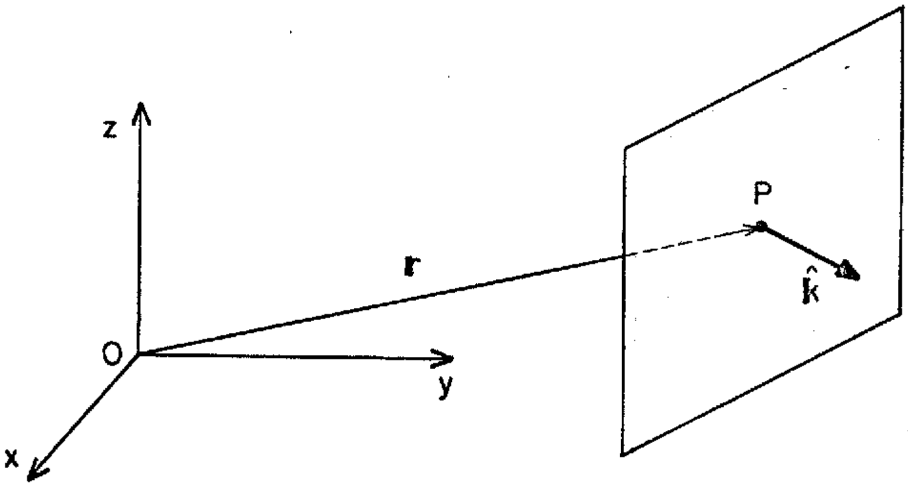

Fig. 1 - Position vector $\underset{\sim}{r}$ drawn from the origin of a coordinate system (x, y, z) to a point P on the wave front, whose normal is given by the propagation vector $\underset{\sim}{k}$.

For the space and time dependence given in (3.5), the differential operators $\underset{\sim}{\nabla}$ and $\partial / \partial t$ are transformed into simple algebraic operators, according to $\underset{\sim}{\nabla} \rightarrow i \underset{\sim}{k}$ and $\partial / \partial t \rightarrow -i\omega$. Substituting (3.1) into Maxwell equation $\underset{\sim}{\nabla} \times \underset{\sim}{E} = - \partial \underset{\sim}{B} / \partial t$, we obtain

$$i\underset{\sim}{k} \times \underset{\sim}{E} = i\omega \, \underset{\sim}{B}' \tag{3.6}$$

where $\partial \underset{\sim}{B}_o / \partial t = 0$, since $\underset{\sim}{B}_o$ is constant. Therefore,

$$\underset{\sim}{B}' = \frac{\underset{\sim}{k} \times \underset{\sim}{E}}{\omega} \tag{3.7}$$

and plugging this result back into (3.3) yields the condition

$$|\underset{\sim}{u}_e \times (\underset{\sim}{k} \times \underset{\sim}{E}) / \omega \, | << | \underset{\sim}{E} | \tag{3.8}$$

The magnitude of the nonlinear term $\underset{\sim}{u}_e \times \underset{\sim}{B}'$ may be equal to or smaller than $|u_e \; k \; E/\omega|$. Hence, the nonlinear term can be neglected if

$$|u_e \; k \; /\omega \; | << 1 \tag{3.9}$$

or, equivalently, if

$$| \; u_e | << |\omega \; / k| \tag{3.10}$$

The term (ω/k) represents the phase velocity of the plane wave. Since this term is usually of the order of the velocity of light, c, whereas the magnitude of the mean velocity of the electrons, u_e, is much less than c, the nonlinear term can generally be neglected. However, in cases of resonance ω/k is very small, whereas u_e becomes large. Under conditions of resonance, therefore, the nonlinear terms is important and a nonlinear analysis must be used.

4. DC CONDUCTIVITY AND ELECTRON MOBILITY

In this section we apply the steady state Langevin equation to derive an expression for the DC (direct current) conductivity of a weakly ionized plasma, for which the Lorentz model (electron gas) is applicable. The applied electric field is constant and uniform. Two situations will be considered: (1) *isotropic* plasma without a magnetic field, and (2) *anisotropic* plasma immersed in a uniform and constant magnetic field.

4.1 - Isotropic plasma

In the absence of a magnetic field, the steady state Langevin equation for the electrons becomes

$$0 = - e \underset{\sim}{E} - m_e \nu_c \underset{\sim}{u}_e \qquad (4.1)$$

In this case the action of the applied electric field is balanced dynamically by the collisions between electrons and neutral particles. The electric current density associated with the motion of the electrons is

$$\underset{\sim}{J} = - e n_e \underset{\sim}{u}_e \qquad (4.2)$$

Combining Eqs. (4.1) and (4.2), gives

$$\underset{\sim}{J} = \left(\frac{n_e e^2}{m_e \nu_c}\right) \underset{\sim}{E} \qquad (4.3)$$

From Ohm's law, $\underset{\sim}{J} = \sigma_0 \underset{\sim}{E}$, we identify the following expression for the *DC conductivity* of an isotropic electron gas

$$\sigma_0 = \frac{n_e e^2}{m_e \nu_c} \qquad (4.4)$$

The *electron mobility*, μ_e, is defined as the ratio of the mean velocity of the electrons to the applied electric field,

$$\mu_e = \frac{u_e}{E} \tag{4.5}$$

Therefore, from Eq. (4.1) we obtain

$$\mu_e = - \frac{e}{m_e \nu_c} = - \frac{\sigma_o}{n_e e} \tag{4.6}$$

4.2 - Anisotropic magnetoplasma

In the presence of a magnetic field the plasma becomes anisotropic. The steady state Langevin equation can be written as

$$0 = - e (\underset{\sim}{E} + \underset{\sim}{u}_e \times \underset{\sim}{B}_o) - m_e \nu_c \underset{\sim}{u}_e \tag{4.7}$$

where $\underset{\sim}{B}_o$ is a constant and uniform magnetic field. Combining (4.7) with $\underset{\sim}{J} = - e n_e \underset{\sim}{u}_e$, yields

$$\left(\frac{m_e \nu_c}{n_e e}\right) \underset{\sim}{J} = e (\underset{\sim}{E} + \underset{\sim}{u}_e \times \underset{\sim}{B}_o) \tag{4.8}$$

which may be written in the form

$$\underset{\sim}{J} = \sigma_o (\underset{\sim}{E} + \underset{\sim}{u}_e \times \underset{\sim}{B}_o) \tag{4.9}$$

where σ_o is given in (4.4). This last equation is a simplified form of the generalized Ohm's law.

At this point it is worth to consider a useful result which arises when the collisional effects are negligible. When $\nu_c \to 0$ the DC conductivity becomes very large ($\sigma_o \to \infty$) so that we must have, from (4.9),

$$\underset{\sim}{E} + \underset{\sim}{u}_e \times \underset{\sim}{B}_o = 0 \tag{4.10}$$

This expression represents, therefore, the simplified form of the generalized Ohm's law for a plasma with a very large conductivity. In this case, taking the cross product of Eq. (4.10) with $\underset{\sim}{B}_o$, and noting that

$$(\underset{\sim}{u}_e \times \underset{\sim}{B}_o) \times \underset{\sim}{B}_o = - \underset{\sim}{u}_{e\perp} B_o^2 \tag{4.11}$$

we obtain

$$\underset{\sim}{u}_{e\perp} = \frac{\underset{\sim}{E} \times \underset{\sim}{B}_o}{B_o^2} \tag{4.12}$$

This result shows that, in the absence of collisions, the electrons have a *drift velocity*, $\underset{\sim}{u}_{e\perp}$, perpendicular to both the electric and magnetic fields. Since this result is independent of the mass and charge of the particles, the same result will be obtained for the ions if their motion is taken into account. This can be easily shown considering the Langevin equation for the ions. Thus, in the absence of collisions, both electrons and ions move together with the drift velocity (4.12), and there is no electric current ($\underset{\sim}{J} = 0$). When the collisional effects are not negligible, the motion of the ions suffers a larger retardation than that of the electrons, as a result of collisions. In this case, there is an electric current (assuming $n_e = n_i$)

$$\underset{\sim}{J}_{\perp} = e n_e \, (\underset{\sim}{u}_{i_\perp} - \underset{\sim}{u}_{e_\perp}) \tag{4.13}$$

perpendicular to both $\underset{\sim}{E}$ and $\underset{\sim}{B}_0$, known as the *Hall current*. Note that, since $u_{e_\perp} > u_{i_\perp}$, this current is in the direction of $-(\underset{\sim}{E} \times \underset{\sim}{B}_0)$, that is, opposite to the drift velocity of both types of particles.

Returning now to the generalized Ohm's law in the simplified form (4.9), let us rewrite it in a way which relates the current density directly to the applied electric field. We define, therefore, a *DC conductivity dyad* (or *tensor*), $\underset{\approx}{\sigma}$, by the equation

$$\underset{\sim}{J} = \underset{\approx}{\sigma} \cdot \underset{\sim}{E} \tag{4.14}$$

In order to obtain an expression for $\underset{\approx}{\sigma}$, consider a Cartesian coordinate system with the z axis parallel to the magnetic field, that is, $\underset{\sim}{B}_0 = B_0 \hat{z}$. Replacing $\underset{\sim}{u}_e$ in Eq. (4.9) by $-\underset{\sim}{J}/(en_e)$, we get

$$\underset{\sim}{J} = \sigma_0 \underset{\sim}{E} - \frac{\sigma_0 B_0}{e n_e} (\underset{\sim}{J} \times \hat{z}) \tag{4.15}$$

Noting that

$$\underset{\sim}{J} \times \hat{\underset{\sim}{z}} = \begin{vmatrix} \hat{\underset{\sim}{x}} & \hat{\underset{\sim}{y}} & \hat{\underset{\sim}{z}} \\ J_x & J_y & J_z \\ 0 & 0 & 1 \end{vmatrix} = J_y \hat{\underset{\sim}{x}} - J_x \hat{\underset{\sim}{y}} \tag{4.16}$$

we obtain the following set of equations for the $\hat{\underset{\sim}{x}}$, $\hat{\underset{\sim}{y}}$ and $\hat{\underset{\sim}{z}}$ components of Eq. (4.15)

$$\hat{\underset{\sim}{x}}: \quad J_x = \sigma_o \; E_x - \frac{\omega_{ce}}{\nu_c} J_y \tag{4.17}$$

$$\hat{\underset{\sim}{y}}: \quad J_y = \sigma_o \; E_y + \frac{\omega_{ce}}{\nu_c} J_x \tag{4.18}$$

$$\hat{\underset{\sim}{z}}: \quad J_z = \sigma_o \; E_z \tag{4.19}$$

where ω_{ce} denotes the electron cyclotron frequency. We can combine Eqs. (4.17) and (4.18) to eliminate J_y from the first one and J_x from the second one, obtaining

$$J_x = \frac{\nu_c^2}{(\nu_c^2 + \omega_{ce}^2)} \sigma_o \; E_x - \frac{\nu_c \; \omega_{ce}}{(\nu_c^2 + \omega_{ce}^2)} \sigma_o \; E_y \tag{4.20}$$

$$J_y = \frac{\nu_c \ \omega_{ce}}{(\nu_c^2 + \omega_{ce}^2)} \sigma_o \ E_x + \frac{\nu_c^2}{(\nu_c^2 + \omega_{ce}^2)} \sigma_o \ E_y \tag{4.21}$$

In matrix form we can write, therefore,

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \sigma_o \begin{pmatrix} \frac{\nu_c^2}{(\nu_c^2 + \omega_{ce}^2)} & \frac{\nu_c \ \omega_{ce}}{(\nu_c^2 + \omega_{ce}^2)} & 0 \\ \frac{\nu_c \ \omega_{ce}}{(\nu_c^2 + \omega_{ce}^2)} & \frac{\nu_c^2}{(\nu_c^2 + \omega_{ce}^2)} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \tag{4.22}$$

which is now in the form given in (4.14). The DC condutivity dyad is therefore given by

$$\underset{\approx}{\sigma} = \begin{pmatrix} \sigma_\perp & -\sigma_H & 0 \\ \sigma_H & \sigma_\perp & 0 \\ 0 & 0 & \sigma_\parallel \end{pmatrix} \tag{4.23}$$

where we have used the notation

$$\sigma_\perp \equiv \frac{\nu_c^2}{(\nu_c^2 + \omega_{ce}^2)} \sigma_o \tag{4.24}$$

$$\sigma_H \equiv \frac{\nu_c \, \omega_{ce}}{(\nu_c^2 + \omega_{ce}^2)} \, \sigma_o \qquad (4.25)$$

$$\sigma_{\parallel} \equiv \sigma_o = \frac{n_e \, e^2}{m_e \, \nu_c} \qquad (4.26)$$

To illustrate the physical meaning of the components of $\underset{\approx}{\sigma}$ it is convenient to separate the applied electric field in a component parallel to $\underset{\sim}{B}_o$, $\underset{\sim}{E}_{\parallel}$, and a component in the plane normal to $\underset{\sim}{B}_o$, $\underset{\sim}{E}_{\perp}$, as shown in Fig. 2. The element $\sigma_{\perp}$ is called the *perpendicular* (or *transverse*) *conductivity* (also called Pedersen conductivity) since it governs the electric current in the direction of the component of the electric field normal to the magnetic field ($\parallel \underset{\sim}{E}_{\perp}$, $\perp \underset{\sim}{B}_o$), while σ_H, known as the *Hall conductivity,*

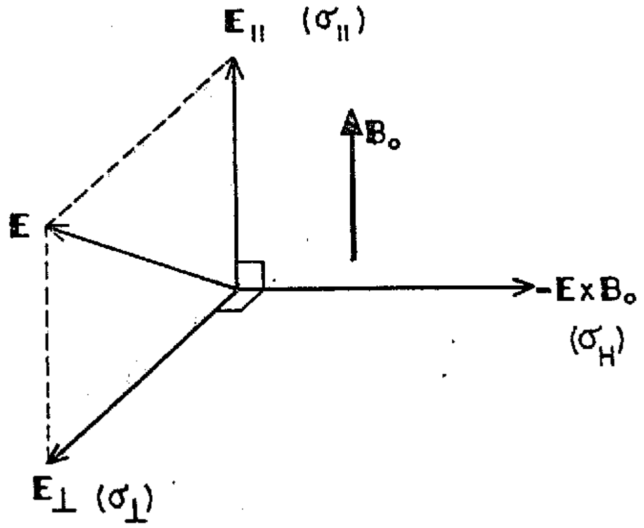

Fig. 2 - Relative orientation of the vector fields $\underset{\sim}{E}_{\parallel}$, $\underset{\sim}{E}_{\perp}$ and $-\underset{\sim}{E} \times B_o$; the conductivities $\sigma_{\parallel}$, $\sigma_{\perp}$, and σ_H govern the magnitude of the electric currents along these directions, respectively.

governs the electric current in the direction perpendicular to both the electric and magnetic fields ($\perp \underset{\sim}{E}$, $\perp \underset{\sim}{B}_0$). The element $\sigma_{\parallel}$ is the *longitudinal conductivity*, since it governs the electric current in the direction of the electric field component along the magnetic field ($\parallel \underset{\sim}{E}_{\parallel}$, $\parallel \underset{\sim}{B}_0$). Note that the electric current along $\underset{\sim}{B}_0$ is governed by the same conductivity (σ_0) as in the case of an isotropic plasma.

The dependence of $\sigma_{\perp}$ and σ_H on the ratio of the cyclotron frequency to the collision frequency is shown in Fig. 3. As the ratio (ω_{ce}/ν_c) increases, $\sigma_{\perp}$ and σ_H decrease rapidly, the efect being more pronouced for $\sigma_{\perp}$. Thus, when (ω_{ce}/ν_c) is relatively large, very little current is produced across the magnetic field lines, as compared with the current produced along the field lines, for the same applied electric field. Note that $\sigma_{\parallel}$ increases as ν_c

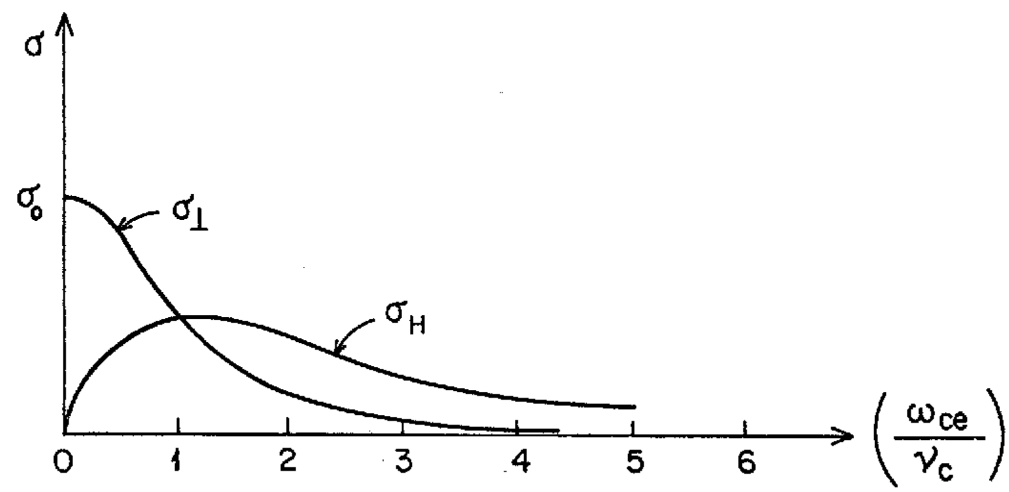

Fig. 3 - Dependence of the Hall conductivity σ_H and the perpendicular conductivity $\sigma_{\perp}$ on the ratio of the cyclotron frequency ω_{ce} to the collision frequency ν_c.

diminishes and is independent of the magnitude of the magnetic field and, therefore, of ω_{ce}. Thus, in a rarefied plasma immersed in a relatively strong magnetic field, the electric current flows essentially along the magnetic field lines.

Note that in the absence of a magnetic field ($\omega_{ce} = 0$) Eqs. (4.24), (4.25) and (4.26) give $\sigma_\perp = \sigma_\parallel = \sigma_0$ and $\sigma_H = 0$; thus, the plasma becomes isotropic.

We deduce next an expression for the electron mobility. Due to the anisotropy introduced by the magnetic field we have, in this case, a *mobility dyad* $\underset{\approx}{\mu}_e$. We define the electron mobility dyad by the equation

$$\underset{\sim}{u}_e = \underset{\approx}{\mu}_e \cdot \underset{\sim}{E} \tag{4.27}$$

Since $\underset{\sim}{J} = - en_e \; \underset{\sim}{u}_e$, we can write

$$\underset{\sim}{J} = - en_e \; \underset{\approx}{\mu}_e \cdot \underset{\sim}{E} \tag{4.28}$$

Comparing this last equation with $\underset{\sim}{J} = \underset{\approx}{\sigma} \cdot \underset{\sim}{E}$, we find that

$$\underset{\approx}{\mu}_e = - \frac{1}{en_e} \underset{\approx}{\sigma} \tag{4.29}$$

Explicit expressions for the components of $\underset{\approx}{\mu}_e$ can be easily written down considering Eqs. (4.23), (4.24), (4.25) and (4.26).

5. AC CONDUCTIVITY AND ELECTRON MOBILITY

Consider now the case when the electric field $\underset{\sim}{E}$ $(\underset{\sim}{r}, t)$ and the mean electron velocity $\underset{\sim}{u}_e$ $(\underset{\sim}{r}, t)$ vary harmonically in time, that is, as exp $(-i\omega t)$. We have seen that for time harmonic disturbances $\partial / \partial t$ is replaced by $- i\omega$. Therefore, the linearized Langevin equation (3.4) becomes

$$- i \omega m_e \underset{\sim}{u}_e = - e (\underset{\sim}{E} + \underset{\sim}{u}_e \times \underset{\sim}{B}_o) - m_e \nu_c \underset{\sim}{u}_e \qquad (5.1)$$

which can be written as

$$0 = - e (\underset{\sim}{E} + \underset{\sim}{u}_e \times \underset{\sim}{B}_o) - m_e (\nu_c - i\omega) \underset{\sim}{u}_e \qquad (5.2)$$

This equation is identical to Eq. (4.7), except for the change in the collision frequency ν_c to $(\nu_c - i\omega)$. We obtain, therefore, solutions similar to the ones obtained for the dc conductivity dyad in the previous section, except that now we must replace ν_c by $(\nu_c - i\omega)$ in each element of the dyad. Therefore, the expressions for the *frequency - dependent* perpendicular conductivity, Hall conductivity and longitudinal conductivity are, respectively,

$$\sigma_\perp = \frac{\tilde{\nu}^2}{(\tilde{\nu}^2 + \omega_{ce}^2)} \sigma_o \qquad (5.3)$$

$$\sigma_H = \frac{\tilde{\nu} \ \omega_{ce}}{(\tilde{\nu}^2 + \omega_{ce}^2)} \sigma_o \qquad (5.4)$$

$$\sigma_{\parallel} = \sigma_o = \frac{n_e\ e^2}{m_e\ \tilde{\nu}} = \frac{n_e\ e^2\ (\nu_c + i\omega)}{m_e\ (\nu_c^2 + \omega^2)} \tag{5.5}$$

where we have used the notation

$$\tilde{\nu} = \nu_c - i\omega \tag{5.6}$$

When the collisions between electrons and neutral particles can be neglected ($\nu_c = 0$) we have $\tilde{\nu} = -i\omega$. For this *collisionless case*, the expressions for the components of the ac conductivity dyad are

$$\sigma_{\perp} = \frac{\omega^2}{(\omega^2 - \omega_{ce}^2)}\ \sigma_o \tag{5.7}$$

$$\sigma_H = \frac{i\omega\ \omega_{ce}}{(\omega^2 - \omega_{ce}^2)}\ \sigma_o \tag{5.8}$$

$$\sigma_{\parallel} = \sigma_o = i\ \frac{n_e\ e^2}{m_e\ \omega} \tag{5.9}$$

The electron mobility, in any of the cases considered in this section, can be easily written down considering the relation (4.29).

6. CONDUCTIVITY WITH ION MOTION

The evaluation of the conductivity dyad, when the contribution due to the motion of the ions is included, can be performed in a straightforward way. For this purpose, consider the linearized Langevin equation for the type α species,

$$m_\alpha \frac{\partial \underset{\sim}{u}_\alpha}{\partial t} = q_\alpha \, (\underset{\sim}{E} + \underset{\sim}{u}_\alpha \times \underset{\sim}{B}_o) - m_\alpha \, \nu_{c\alpha} \, \underset{\sim}{u}_\alpha \tag{6.1}$$

where $\nu_{c\alpha}$ is an effective collision frequency or damping term for the type α species resulting from collisions with *neutral* particles. Note that the equations (6.1), for each charged particle species, are uncoupled. Therefore, the net current density is given by

$$\underset{\sim}{J} = \sum_\alpha n_\alpha \, q_\alpha \, \underset{\sim}{u}_\alpha = \sum_\alpha \underset{\sim}{J}_\alpha = (\sum_\alpha \underset{\approx}{\sigma}_\alpha) \cdot \underset{\sim}{E} \tag{6.2}$$

and the total conductivity is simply

$$\underset{\approx}{\sigma} = \sum_\alpha \underset{\approx}{\sigma}_\alpha \tag{6.3}$$

For a plasma with electrons and several types of ions (index j) we obtain, using Eqs. (5.3) (5.4) and (5.5),

$$\sigma_{\perp} = \frac{(\nu_{ce} - i\omega)^2}{\left[(\nu_{ce} - i\omega)^2 + \omega_{ce}^2\right]}\sigma_{oe} + \sum_j \frac{(\nu_{cj} - i\omega)^2}{\left[(\nu_{cj} - i\omega)^2 + \omega_{cj}^2\right]}\sigma_{oj} \qquad (6.4)$$

$$\sigma_H = \frac{(\nu_{ce} - i\omega)\,\omega_{ce}}{\left[(\nu_{ce} - i\omega)^2 + \omega_{ce}^2\right]}\sigma_{oe} - \sum_j \frac{(\nu_{cj} - i\omega)\,\omega_{cj}}{\left[(\nu_{cj} - i\omega)^2 + \omega_{cj}^2\right]}\sigma_{oj} \qquad (6.5)$$

$$\sigma_{\parallel} = \sigma_{oe} + \sum_j \sigma_{oj} = \frac{n_e\, e^2}{m_e}\;\frac{(\nu_{ce} + i\omega)}{(\nu^2_{ce} + \omega^2)} + \sum_j \frac{n_j\, e^2}{m_j}\;\frac{(\nu_{cj} + i\omega)}{(\nu_{cj}^2 + \omega^2)} \qquad (6.6)$$

where $\omega_{c\alpha} = e\, B_o/m_\alpha$. In terms of the plasma frequency

$$\omega_{p\alpha} = \left(\frac{n_o\, e^2}{m_\alpha\, \varepsilon_o}\right)^{1/2} \qquad (6.7)$$

the elements of the conductivity dyad for the multi-component plasma become

$$\sigma_{\perp} = \varepsilon_o \left[\frac{\omega_{pe}^2\,(\nu_{ce} - i\omega)}{(\nu_{ce} - i\omega)^2 + \omega_{ce}^2} + \sum_j \frac{\omega_{pj}^2\,(\nu_{cj} - i\omega)}{(\nu_{cj} - i\omega)^2 + \omega_{cj}^2}\right] \qquad (6.8)$$

$$\sigma_H = \varepsilon_o \left[\frac{\omega_{pe}^2\,\omega_{ce}}{(\nu_{ce} - i\omega)^2 + \omega_{ce}^2} - \sum_j \frac{\omega_{pj}^2\,\omega_{cj}}{(\nu_{cj} - i\omega)^2 + \omega_{cj}^2}\right] \qquad (6.9)$$

$$\sigma_{\parallel} = \varepsilon_o \left[\frac{\omega_{pe}^2}{(\nu_{ce} - i\omega)} + \sum_j \frac{\omega_{pj}^2}{(\nu_{cj} - i\omega)}\right] \qquad (6.10)$$

7. THE PLASMA AS A DIELECTRIC MEDIUM

The plasma can also be treated as a dielectric medium characterized by a dielectric dyad, in which the internal particle behavior is not considered. So far, we have treated the plasma as a collection of charged and neutral particles moving about in their own internal fields. Thus, as far as the constitutive relations are concerned, we have taken

$$\underset{\sim}{D} = \varepsilon_0 \underset{\sim}{E} \tag{7.1}$$

$$\underset{\sim}{B} = \mu_0 \underset{\sim}{H} \tag{7.2}$$

which is the case for vacuum, and the plasma effects show up through the motion and interaction of the charged particles inside the plasma. In this sense, the use of the Langevin equation constitutes a *microscopic description* involving the average motion of the particles in the plasma. A different approach is provided by a *macroscopic description* through the use of a dielectric dyad, in which we are concerned only with the gross macroscopic properties of the plasma and not with the motion of the particles.

Thus, instead of the Langevin equation, let us consider the following Maxwell equation

$$\underset{\sim}{\nabla} \times \underset{\sim}{B} = \mu_0 \, (\underset{\sim}{J} + \varepsilon_0 \, \partial \underset{\sim}{E} \, / \, \partial t) \tag{7.3}$$

and incorporate the effects of the plasma in the conductivity dyad $\underset{\approx}{\sigma}$, defined by the equation

$$\underset{\sim}{J} = \underset{\approx}{\sigma} \cdot \underset{\sim}{E} \tag{7.4}$$

Substituting (7.4) into (7.3), and assuming time-harmonic variations of the form exp (- iωt), we obtain

$$\underset{\sim}{\nabla} \times \underset{\sim}{B} = \mu_0 \underset{\approx}{\sigma} \cdot \underset{\sim}{E} - i\omega \mu_0 \varepsilon_0 \underset{\sim}{E} \tag{7.5}$$

If we let $\underset{\approx}{1}$ be the unit dyad, we can write

$$\underset{\sim}{\nabla} \times \underset{\sim}{B} = - i\omega \mu_0 \varepsilon_0 \left(\underset{\approx}{1} + \frac{i}{\omega\varepsilon_0} \underset{\approx}{\sigma}\right) \cdot \underset{\sim}{E} \tag{7.6}$$

or, equivalently, as

$$\underset{\sim}{\nabla} \times \underset{\sim}{B} = - i\omega \mu_0 \underset{\approx}{\varepsilon} \cdot \underset{\sim}{E} \tag{7.7}$$

where

$$\underset{\approx}{\varepsilon} \equiv \varepsilon_0 \left(\underset{\approx}{1} + \frac{i}{\omega\varepsilon_0} \underset{\approx}{\sigma}\right) \tag{7.8}$$

is called the *dielectric dyad for the plasma*. The use of the dielectric dyad represents, therefore, a different approach for the treatment of a plasma, as compared to the one we have used so far. Adopting this approach, Eq. (7.1) must be replaced by

$$\underset{\sim}{D} = \underset{\approx}{\varepsilon} \cdot \underset{\sim}{E} \tag{7.9}$$

and the plasma is considered as a dielectric medium, without bringing into the picture its internal particle behavior. Note that $\underset{\approx}{\varepsilon}$ depends on the frequency ω.

The dielectric dyad can be written in matrix form as

$$\underset{\approx}{\varepsilon} = \varepsilon_0 \begin{bmatrix} \varepsilon_1 & -\varepsilon_2 & 0 \\ \varepsilon_2 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} \tag{7.10}$$

where the following notation was introduced

$$\varepsilon_1 \equiv 1 + \frac{i}{\omega\varepsilon_0}\sigma_\perp \tag{7.11}$$

$$\varepsilon_2 \equiv \frac{i}{\omega\varepsilon_0}\sigma_H \tag{7.12}$$

$$\varepsilon_3 \equiv 1 + \frac{i}{\omega\varepsilon_0}\sigma_0 \tag{7.13}$$

For the case of a multi-constituent plasma the total conductivity must be used in (7.8), so that the expressions to be substituted for $\sigma_\perp$, σ_H and σ_0 are those given in Eqs. (6.4), (6.5) and (6.6).

8. FREE ELECTRON DIFFUSION

The presence of a pressure gradient term in the momentum transport equation provides a force which tends to smooth out any inhomogeneities in the density of the plasma. The diffusion of particles in a plasma results from this pressure gradient force. To deduce the electron diffusion coefficient for a warm weakly ionized plasma we will use the momentum transport equation for the electrons with a constant collision frequency between electrons and neutral particles. We assume that the deviations from the equilibrium state caused by inhomogeneities in the density are very small, so that they may be considered as small first order quantities. This means that the mean velocity of the electrons $\underset{\sim}{u}_e$ is also a first order quantity, and since the velocity distribuition will be approximately isotropic, we can replace the pressure dyad $\underset{\approx}{p}_e$ by a scalar pressure p_e.

Consider the case in which $\underset{\sim}{E}$ and $\underset{\sim}{B}$ are zero and the electron temperature T_e is constant. For a *slightly nonuniform* electron number density, we have

$$n_e\ (\underset{\sim}{r},\ t) = n_o + n_e'\ (\underset{\sim}{r},\ t) \tag{8.1}$$

$$p_e\ (\underset{\sim}{r},\ t) = n_e\ (\underset{\sim}{r},\ t)\ k\ T_e = (n_o + n_e')\ k\ T_e \tag{8.2}$$

where $|n_e'| << n_o$ is a first order quantity and n_o is constant. Since $\underset{\sim}{u}_e$ is also a first order quantity, the continuity equation for the electron gas becomes

$$\frac{\partial n_e'}{\partial t} + n_o \underset{\sim}{\nabla} \cdot \underset{\sim}{u}_e = 0 \tag{8.3}$$

where the second order term $n_e' \underset{\sim}{u}_e$ has been neglected. Similarly,for the momentum transport equation,

$$n_e \, m_e \left[\frac{\partial \underset{\sim}{u}_e}{\partial t} + (\underset{\sim}{u}_e \cdot \underset{\sim}{\nabla}) \, \underset{\sim}{u}_e \right] = - \underset{\sim}{\nabla} p_e - n_e \, m_e \, \nu_c \, \underset{\sim}{u}_e \tag{8.4}$$

we obtain, after linearization,

$$n_o \frac{\partial \underset{\sim}{u}_e}{\partial t} = - \frac{k\, T_e}{m_e} \underset{\sim}{\nabla} n_e' - n_o \, \nu_c \, \underset{\sim}{u}_e \tag{8.5}$$

Taking the divergence of this equation, we obtain

$$n_o \frac{\partial}{\partial t} (\underset{\sim}{\nabla} \cdot \underset{\sim}{u}_e) = - \frac{k\, T_e}{m_e} \nabla^2 n_e' - n_o \, \nu_c \, \underset{\sim}{\nabla} \cdot \underset{\sim}{u}_e \tag{8.6}$$

Using Eq. (8.3) to substitute for $n_o \underset{\sim}{\nabla} \cdot \underset{\sim}{u}_e$, yields

$$\frac{\partial^2 n_e'}{\partial t^2} = \frac{k T_e}{m_e} \nabla^2 n_e' - \nu_c \frac{\partial n_e'}{\partial t} \tag{8.7}$$

This equation may also be writtenin the form

$$\frac{\partial n_e'}{\partial t} = D_e \nabla^2 n_e' - \frac{1}{\nu_c} \frac{\partial^2 n_e'}{\partial t^2} \tag{8.8}$$

where we have defined

$$D_e \equiv \frac{k T_e}{m_e \nu_c} \tag{8.9}$$

which is called the *electron free-diffusion coefficient.*

To obtain a rough estimate of the order of magnitude of the various terms in Eq. (8.8), let τ and L represent, respectively, a characteristic time and a characteristic length over which n'_e varies significantly. Thus, any spatial derivative is of the order of L^{-1} and any time derivative is of the order τ^{-1},so that the order of magnitude of the terms in Eq. (8.8) are

$$\frac{\partial n_e'}{\partial t} \simeq \frac{1}{\tau} n_e' \tag{8.10}$$

$$D_e \nabla^2 n_e' \simeq \frac{D_e}{L^2} n_e' \tag{8.11}$$

$$\frac{1}{\nu_c} \frac{\partial^2 n_e'}{\partial t^2} \simeq \frac{1}{\nu_c \tau^2} n_e' \tag{8.12}$$

Comparing Eqs. (8.10) and (8.12) we see that if $\nu_c \tau \gg 1$, that is, if the average number of collisions between the electrons and the heavy neutral particles is large during the time interval τ, then the last term in Eq. (8.8) can be neglected and it reduces to the following *diffusion equation*

$$\frac{\partial n_e'}{\partial t} = D_e \nabla^2 n_e' \tag{8.13}$$

Therefore, when the rate of change in the number density is slow compared to the collision frequency, the number density is governed by a diffusion equation with a free electron diffusion coefficient as given by Eq. (8.9).

The condition $\nu_c \tau \gg 1$ implies in the omission of the acceleration term in the momentum transport equation, that is, $\partial \underset{\sim}{u}_e / \partial t$ is neglected. From the linearized Eq. (8.5), when there are no time variations in $\underset{\sim}{u}_e$, we obtain,

$$n_o \nu_c \underset{\sim}{u}_e = - \frac{k T_e}{m_e} \underset{\sim}{\nabla} n_e' \tag{8.14}$$

which can be written as

$$\underset{\sim}{\Gamma}_e = - D_e \underset{\sim}{\nabla} n_e' \tag{8.15}$$

where $\underset{\sim}{\Gamma}_e = n_o \underset{\sim}{u}_e$ denotes the linearized electronflux. Eq. (8.15) is analogous to the simple Ohm's law $\underset{\sim}{J} = \sigma_o \underset{\sim}{E}$, replacing $\underset{\sim}{J}$ by $\underset{\sim}{\Gamma}_e$, σ_o by D_e, and $\underset{\sim}{E}$ by $-\underset{\sim}{\nabla} n_e'$. Thus, we see that the electron flux $\underset{\sim}{\Gamma}_e$ is caused by a density gradient, in a way analogous to the electric current caused by an electric field, under steady state conditions for $\underset{\sim}{u}_e$.

9. ELECTRON DIFFUSION IN A MAGNETIC FIELD

Consider now the problem of electron diffusion in the presence of a constant and uniform magnetic field $\underset{\sim}{B}_o$. We make the same assumptions as in the previous section, and neglect the acceleration term $\partial \underset{\sim}{u}_e / \partial t$ in the equation of motion.

In the linearized momentum transport equation (8.5), with the time derivative set equal to zero, we include now a magnetic force term, which results in

$$\underset{\sim}{\Gamma}_e = - D_e \underset{\sim}{\nabla} n_e' - \frac{e}{m_e \nu_c} (\underset{\sim}{\Gamma}_e \times \underset{\sim}{B}_o) \tag{9.1}$$

Choosing a Cartesian coordinate system with the z-axis pointing in the direction of the constant $\underset{\sim}{B}_o$ field, that is, $\underset{\sim}{B}_o = B_o \hat{\underset{\sim}{z}}$, we have

$$\underset{\sim}{\Gamma}_e = - D_e \underset{\sim}{\nabla} n_e' - \frac{\omega_{ce}}{\nu_c} (\underset{\sim}{\Gamma}_e \times \hat{\underset{\sim}{z}}) \tag{9.2}$$

This equation is analogous to Eq. (4.15) with $\underset{\sim}{\Gamma}_e$ replacing $\underset{\sim}{J}$, D_e replacing σ_o, and $-\underset{\sim}{\nabla} n_e'$ replacing $\underset{\sim}{E}$ (note that $\omega_{ce}/ \nu_c = \sigma_o B_o/en_e$). Therefore, in analogy with the expression $\underset{\sim}{J} = \underset{\approx}{\sigma} \cdot \underset{\sim}{E}$, we can write

$$\underset{\sim}{\Gamma}_e = - \underset{\approx}{D} \cdot \underset{\sim}{\nabla} n_e' \tag{9.3}$$

where $\underset{\approx}{D}$ is the *dyad coefficient for free electron diffusion* given by

$$\underset{\approx}{D} = \begin{pmatrix} D_\perp & D_H & 0 \\ -D_H & D_\perp & 0 \\ 0 & 0 & D_\parallel \end{pmatrix} \tag{9.4}$$

where the following notation is used

$$D_\perp \equiv \frac{\nu_c^2}{(\nu_c^2 + \omega_{ce}^2)} D_e \tag{9.5}$$

$$D_H \equiv \frac{\nu_c \, \omega_{ce}}{(\nu_c^2 + \omega_{ce}^2)} D_e \tag{9.6}$$

$$D_\parallel \equiv D_e = \frac{k \, T_e}{m_e \, \nu_c} \tag{9.7}$$

A diffusion equation for n_e', when there is a constant and uniform magnetic field present, can also be derived in the same way as in the previous section. First, we write the continuity equation (8.3) in the form

$$\frac{\partial n_e'}{\partial t} + \underset{\sim}{\nabla} \cdot \underset{\sim}{\Gamma}_e = 0 \tag{9.8}$$

Substituting Eq. (9.3) for $\underset{\sim}{\Gamma}_e$, yields

$$\frac{\partial n_e'}{\partial t} = \underset{\sim}{\nabla} \cdot (\underset{\approx}{D} \cdot \underset{\sim}{\nabla} n_e') \tag{9.9}$$

Using Eq. (9.4) we find, by direct calculation in Cartesian coordinates,

$$\underset{\approx}{D} \cdot \underset{\sim}{\nabla} n_e' = \hat{\underset{\sim}{x}} \left(D_\perp \frac{\partial n_e'}{\partial x} + D_H \frac{\partial n_e'}{\partial y} \right) + \hat{\underset{\sim}{y}} \left(-D_H \frac{\partial n_e'}{\partial x} + D_\perp \frac{\partial n_e'}{\partial y} \right) + \hat{\underset{\sim}{z}} D_e \frac{\partial n_e'}{\partial z} \tag{9.10}$$

Substituting this result into (9.9), yields

$$\frac{\partial n_e'}{\partial t} = D_\perp \left(\frac{\partial^2 n_e'}{\partial x^2} + \frac{\partial^2 n_e'}{\partial y^2} \right) + D_e \frac{\partial^2 n_e'}{\partial z^2} \tag{9.11}$$

Since $D_\perp < D_e$ and since $D_\perp$ decreases rapidly with increasing values of ω_{ce}/ν_c (similary to $\sigma_\perp$, as shown in Fig. 3) the diffusion of particles in a direction perpendicular to the magnetic field is always less than that in the direction parallel to the magnetic field.
For values of ω_{ce} much larger than ν_c the diffusion of particles across the magnetic field lines is greatly reduced (from Eqs. (9.5) and (9.6) it can be seen that for $\omega_{ce} >> \nu_c$ we have, approximately, $D_\perp \propto 1/B^2$ and $D_H \propto 1/B$).

As a final point in this section we note that the momentum transport equation for a gas of electrons, neglecting the acceleration term but including the electromagnetic force, and when the temperature is constant, can be written in the general form

$$\underset{\sim}{\Gamma}_e = \mu_e \, (n_e \, \underset{\sim}{E} + \underset{\sim}{\Gamma}_e \times \underset{\sim}{B}) - D_e \, \underset{\sim}{\nabla} n_e \tag{9.12}$$

From this equation we can see that the electron flux is produced by either, or both,electromagnetic fields and density gradients. The ratio of the scalar mobility μ_e to the diffusion coefficient is known as the *Einstein relation* and is given by

$$\frac{\mu_e}{D_e} = - \frac{e}{k T_e} \tag{9.13}$$

10. AMBIPOLAR DIFFUSION

We have seen in section 8 that the steady state momentum equation, in the absence of electromagnetic forces and when the temperature is constant, gives the following diffusion equation for the electrons

$$\underset{\sim}{\Gamma}_e = - D_e \underset{\sim}{\nabla} n'_e \qquad (10.1)$$

where the *electron free-diffusion coefficient* is given by

$$D_e = \frac{k\,T_e}{m_e\,\nu_{ce}} \qquad (10.2)$$

The subscript e has been added here to ν_c to indicate that the constant collision frequency ν_{ce} refers to collisions between electrons and neutral particles.

If we consider similar equations for the ions in a weakly ionized plasma, under the same assumptions, we obtain the following diffusion equation for the ions

$$\underset{\sim}{\Gamma}_i = - D_i \underset{\sim}{\nabla} n'_i \qquad (10.3)$$

where

$$D_i = \frac{k\,T_i}{m_i\,\nu_{ci}} \tag{10.4}$$

denotes the *ion free-diffusion coefficient*, and ν_{ci} is the constant collision frequency between ions and neutral particles.

In deriving the results given by Eqs. (10.1) and (10.3), the mutual interaction between the electrons and the ions were not taken into account. Since the diffusion coefficient is inversely proportional to the mass of the particles, the electrons diffuse faster than the ions leaving an excess of positive charge behind them. This gives rise to a space charge electric field in the same direction as the diffusion of the particles, and which accelerates the diffusion of the ions and slow down that of the electrons. The diffusion in which the effect of the space charge electric field is *not* included is known as *free diffusion*.

For most problems of plasma diffusion, however, the space charge electric field *cannot* be neglected. According to Maxwell equation

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{E} = \rho_c / \varepsilon_0 = \frac{e}{\varepsilon_0}\,(n_i - n_e) \tag{10.5}$$

it is clear that an electric field is present whenever the electron density differs from the ion density. To estimate the importance of the space charge electric field in diffusion problems, we may use dimensional analysis and let L represent a characteristic length over which the number density changes significantly. Thus, from Eq. (10.5) we may write

$$E \approx \frac{e\,n\,L}{\varepsilon_o} \tag{10.6}$$

so that the electric force per unit mass, f_E, is of the order

$$f_E = \frac{e\,E}{m} \simeq \frac{e^2\,n\,L}{m\,\varepsilon_o} \tag{10.7}$$

The diffusion force per unit mass, f_D, obtained from Eq. (10.1), is of the order

$$f_D = \frac{k\,T}{m\,n_o}|\underset{\sim}{\nabla}\,n| \approx \frac{k\,T\,n}{m n_o\,L} \tag{10.8}$$

Therefore, the space charge electric field can be neglected only if $f_E << f_D$, or equivalently, if

$$L^2 << \frac{k\,T\,\varepsilon_o}{n_o\,e^2} = \lambda_D^2 \tag{10.9}$$

where λ_D is the *Debye length*. Since the Debye length is generally very small (see Fig. 2 of Chapter 1), the condition $L << \lambda_D$ is rarely satisfied and for most plasma diffusion problems we cannot neglect the space charge electric field. In what follows we will reexamine, therefore, the problem of plasma diffusion taking into account the motion of both ions and electrons and including the space charge electric field $\underset{\sim}{E}$. The combined diffusion of the electrons and the ions, forced by the space charge $\underset{\sim}{E}$ field, is known as *ambipolar diffusion*. Since the electric field retards the electrons and accelerates the ions, the two kinds of charged particles diffuse at a rate which is intermediate in value to their *free* diffusion rates.

To investigate the characteristics of *ambipolar diffusion* we assume that the disturbance for both electrons and ions are small first order quantities, so that (for α = e, i)

$$n_\alpha (\underset{\sim}{r}, t) = n_0 + n'_\alpha (\underset{\sim}{r}, t) \tag{10.10}$$

with $|n'_\alpha| << n_0$, and that the mean velocities $\underset{\sim}{u}_e$ and $\underset{\sim}{u}_i$ are of very small amplitude. We obtain, under these assumptions, the following linearized mass conservation equations (α = e, i)

$$\frac{\partial n'_\alpha}{\partial t} + n_o \underset{\sim}{\nabla} \cdot \underset{\sim}{u}_\alpha = 0 \qquad (10.11)$$

The linearized momentum equations, assuming that the temperatures are constant, and without a magnetic field, become (α = e, i)

$$\frac{\partial \underset{\sim}{u}_\alpha}{\partial t} = \frac{q_\alpha \underset{\sim}{E}}{m_\alpha} - \frac{k\,T_\alpha}{m_\alpha\, n_o} \underset{\sim}{\nabla}\, n'_\alpha - \nu_{c\alpha}\, \underset{\sim}{u}_\alpha \qquad (10.12)$$

where the space charge $\underset{\sim}{E}$ field satisfies Maxwell equation (10.5). We are assuming that the neutral mean velocity $\underset{\sim}{u}_n$ is zero, and we are neglecting electron-ion collisions, since the plasma is weakly ionized. Taking the divergence of Eq. (10.12) and using Eq. (10.11), we obtain,

$$\frac{\partial^2 n'_\alpha}{\partial t^2} = - \frac{q_\alpha\, n_o}{m_\alpha} (\underset{\sim}{\nabla} \cdot \underset{\sim}{E}) + \frac{k\,T_\alpha}{m_\alpha} \nabla^2 n'_\alpha - \nu_{c\alpha} \frac{\partial n'_\alpha}{\partial t} \qquad (10.13)$$

If we replace $\nabla \cdot E$ from Eq. (10.5), we obtain the following set of coupled equations for the two variables n'_e and n'_i,

$$\frac{\partial^2 n'_e}{\partial t^2} = \frac{e^2 n_o}{m_e \varepsilon_o} (n'_i - n'_e) + \frac{k T_e}{m_e} \nabla^2 n'_e - \nu_{ce} \frac{\partial n'_e}{\partial t} \quad (10.14)$$

$$\frac{\partial^2 n'_i}{\partial t^2} = - \frac{e^2 n_o}{m_i \varepsilon_o} (n'_i - n'_e) + \frac{k T_i}{m_i} \nabla^2 n'_i - \nu_{ci} \frac{\partial n'_i}{\partial t} \quad (10.15)$$

These equations, however, are still too complicated for a detailed analytical treatment and to go further we will make some additional simplifying assumptions. Recall that, if $\nu_c \tau >> 1$, that is, if the average electron or ion has many collisions with neutral particles during the characteristic time for diffusion τ, the term $\partial^2 n' / \partial t^2$ (originated from the acceleration term in the momentum equation) can be neglected. With this assumption we neglect the term in the left-hand side of Eqs. (10.14) and (10.15).Combining these equations we obtain

$$0 = k T_e \nabla^2 n'_e + k T_i \nabla^2 n'_i - m_e \nu_{ce} \frac{\partial n'_e}{\partial t} - m_i \nu_{ci} \frac{\partial n'_i}{\partial t} \quad (10.16)$$

As a second approximation we will set $n'_i = n'_e = n'$ in Eq. (10.16) to obtain the following diffusion equation

$$0 = k (T_e + T_i) \nabla^2 n' - (m_e \nu_{ce} + m_i \nu_{ci}) \frac{\partial n'}{\partial t} \quad (10.17)$$

which can be written in the form

$$\frac{\partial n'}{\partial t} = D_a \nabla^2 n' \qquad (10.18)$$

where

$$D_a = k\,(T_e + T_i) / (m_e \nu_{ce} + m_i \nu_{ci}) \qquad (10.19)$$

is the *ambipolar diffusion coefficient*. Note that the coupling of the two Eqs. (10.14) and (10.15) is a consequence of the electric field term, and that the simplifying approximation $n'_i = n'_e$ was introduced only after the two equations were combined into Eq. (10.16). This approximation implies that the space charge electric field becomes a negligible perturbation with the result that both ions and electrons diffuse together. This situation is known as *perfect ambipolar diffusion*, since the coupling between the two types of charged particles is complete.

Instead of taking $n'_i = n'_e$, a less restrictive simplifying approximation would be to assume

$$n'_i = C\, n'_e \qquad (10.20)$$

where C is a constant. Using this approximation in Eq. (10.16) we obtain

$$0 = k\,(T_e + CT_i)\,\nabla^2 n'_e \quad - (m_e\,\nu_{ce} + C\,m_i\,\nu_{ci})\,\frac{\partial n'_e}{\partial t} \tag{10.21}$$

or

$$\frac{\partial n'_e}{\partial t} = D_a\,\nabla^2 n'_e \tag{10.22}$$

where the ambipolar diffusion coefficient is now given by

$$D_a = \frac{k\,(T_e + C\,T_i)}{(m_e\,\nu_{ce} + C\,m_i\,\nu_{ci})} \tag{10.23}$$

The space charge density is now

$$\rho_c = e\,(n'_i - n'_e) = e\,n'_e\,(C - 1) \tag{10.24}$$

and the electric field can be obtained from Maxwell equation

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{E} = e\,n'_e\,(C - 1)/\varepsilon_o \tag{10.25}$$

The effect of the electric field is to accelerate the diffusion of the ions and to retard the diffusion of the electrons, as compared to their individual free diffusion rates, so that, to a good approximation, both species diffuse together. Whenever there is a significant deviation from charge neutrality ($C \neq 1$), the electric field force becomes very strong as can be seen from the following dimensional argument.

A comparison of the magnitude of the electric force per unit mass, $\underset{\sim}{f}_E = q_\alpha \underset{\sim}{E}/m_\alpha$, and the diffusion force per unit mass, $\underset{\sim}{f}_D = -(k T_\alpha / m_\alpha n_o) \underset{\sim}{\nabla} n'_\alpha$, which are of the order

$$f_E \simeq \frac{L e^2 (n'_i - n'_e)}{m \varepsilon_o} = \frac{L e^2 n'_e (C - 1)}{m \varepsilon_o} \qquad (10.26)$$

$$f_D \simeq \frac{k T n'_e}{m n_o L} \qquad (10.27)$$

shows that

$$f_E / f_D \simeq \frac{L^2}{\lambda_D^2} (C - 1) \qquad (10.28)$$

where $\lambda_D = (k T \varepsilon_o / n_o e^2)^{1/2}$ is the Debye length. Since in most cases L^2 is much larger than λ_D^2, we see that if n'_i differs significantly from n'_e, the electric field force (which tends to equalize n'_i and n'_e) becomes very strong.

11. DIFFUSION IN A FULLY IONIZED PLASMA

Consider now the problem of diffusion in a fully ionized plasma. For simplicity, we shall describe the plasma as a single conducting fluid for which the equation of motion under steady state conditions, in the presence of magnetic and pressure-gradient forces, is

$$\underset{\sim}{J} \times \underset{\sim}{B} = \underset{\sim}{\nabla} p \tag{11.1}$$

where $\underset{\sim}{J}$ denotes the total electric current density, $\underset{\sim}{B}$ is the magnetic induction, and p represents the total pressure of the conducting fluid. Note that the electric force is zero since the plasma, as a whole, is macroscopically neutral ($\rho_c = 0$). This equation is complemented by the generalized Ohm's law in the following simplified form,

$$\underset{\sim}{J} = \sigma_0 (\underset{\sim}{E} + \underset{\sim}{u} \times \underset{\sim}{B}) \tag{11.2}$$

where σ_0 is the longitudinal electric conductivity and $\underset{\sim}{u}$ is the total fluid velocity.

Taking the cross-product of Eq. (11.2) with $\underset{\sim}{B}$, yields

$$\underset{\sim}{J} \times \underset{\sim}{B} = \sigma_0 (\underset{\sim}{E} \times \underset{\sim}{B} - B^2 \underset{\sim}{u}_{\perp}) \tag{11.3}$$

where $\underset{\sim}{u}_\perp$ is the component of $\underset{\sim}{u}$ in a direction normal to the external field $\underset{\sim}{B}$. Using (11.1) and rearranging, (11.3) gives

$$\underset{\sim}{u}_\perp = \frac{\underset{\sim}{E} \times \underset{\sim}{B}}{B^2} - \frac{1}{\sigma_o B^2} \underset{\sim}{\nabla} p \tag{11.4}$$

This result shows that the total fluid velocity across the magnetic field is given by the $\underset{\sim}{E} \times \underset{\sim}{B}$ drift of the whole plasma plus a diffusion velocity in the direction of $- \underset{\sim}{\nabla} p$.

The flux associated with the diffusion velocity only, is given by

$$\underset{\sim}{\Gamma}_\perp = n \, \underset{\sim}{u}_\perp = - \frac{n}{\sigma_o B^2} \underset{\sim}{\nabla} p \tag{11.5}$$

where n denotes the electron (or total ion) number density. Considering a two-fluid plasma (electrons and one type of ions), we have

$$\begin{aligned} p &= p_e + p_i \\ &= n k (T_e + T_i) \end{aligned} \tag{11.6}$$

so that Eq. (11.5) becomes, assuming the temperatures to be constant,

$$\underset{\sim}{\Gamma}_\perp = - \frac{n k (T_e + T_i)}{\sigma_o B^2} \underset{\sim}{\nabla} n$$

$$= - D_{\perp} \nabla n \tag{11.7}$$

The quantity

$$D_{\perp} = \frac{n\,k\,(T_e + T_i)}{\sigma_o\,B^2} \tag{11.8}$$

is known as the *classical diffusion coefficient* for a fully ionized plasma.

This diffusion coefficient is proportional to $1/B^2$, just as in the case of a weakly ionized plasma. Nevertheless, there are some fundamental differences between $D_{\perp}$, as given by (11.8), and the corresponding coefficient for a partially ionized plasma. Initially note that in a fully ionized plasma $D_{\perp}$ is not constant, but depends on the number density n. Further, since it can be shown that σ_o is proportional to $T^{3/2}$ for a Maxwellian distribution of velocities, $D_{\perp}$ decreases with increasing temperature in a fully ionized plasma, while the opposite is true for a weakly ionized plasma. Finally, the diffusion coefficient $D_{\perp}$ in (11.8) was derived for the whole plasma as a conducting fluid and, since both ions and electrons diffuse together, there is no ambipolar electric field.

In some experiments it has been observed a dependence of $D_{\perp}$ on the magnetic field as B^{-1}, rather than B^{-2}, and the decay of the plasma was found to be exponential, rather than reciprocal with time. Furthermore, the absolute value of $D_{\perp}$ was found to be much larger than that given in (11.8). This anomalously poor magnetic confinement was first noted in laboratory by Bohm, in 1946, who obtained the following semiempirical formula

$$D_{\perp} \equiv D_B = \frac{1}{16} \frac{k T_e}{e B} \tag{11.9}$$

Since this diffusion coefficient does not depend on the density, the decay of the plasma density is exponential with time. This type of diffusion in plasmas is known as *Bohm diffusion*.

PROBLEMS

10.1 - Consider a solid state plasma with the same number of electrons (e) and holes (h). Using the linearized Langevin equation (with α = e, h)

$$m_\alpha \frac{\partial \underset{\sim}{u}_\alpha}{\partial t} = q_\alpha \, (\underset{\sim}{E} + \underset{\sim}{u}_\alpha \times \underset{\sim}{B}_0) - \nu_{c\alpha} \, m_\alpha \, \underset{\sim}{u}_\alpha$$

taking $m_e = m_h$, $\nu_{ce} = \nu_{ch}$, assuming a time dependence for both $\underset{\sim}{E}$ and $\underset{\sim}{u}_\alpha$ of the form exp $(-i\,\omega t)$, and choosing a Cartesian coordinate system with the z axis pointing along the constant and uniform magnetic field $\underset{\sim}{B}_0$, show that the conductivity dyad is given by

$$\underset{\approx}{\sigma} = 2 \begin{bmatrix} \sigma_\perp & 0 & 0 \\ 0 & \sigma_\perp & 0 \\ 0 & 0 & \sigma_0 \end{bmatrix}$$

where $\sigma_\perp = \tilde{\nu}^2 \, \sigma_0/(\nu^2 + \omega_c^2)$, $\sigma_0 = ne^2/\,m\tilde{\nu}$ and $\tilde{\nu} = \nu_c - i\omega$. Explain, in physical terms, why $\sigma_H = 0$ in this case.

10.2 - Assume that the average velocities of the electrons and ions in a completely ionized plasma, in the presence of constant and uniform electric ($\underset{\sim}{E}$) and magnetic ($\underset{\sim}{B}_0$) fields, satisfy, respectively, the following equations of motion

$$m_e \frac{\partial \underset{\sim}{u}_e}{\partial t} = - e(\underset{\sim}{E} + \underset{\sim}{u}_e \times \underset{\sim}{B}_o) - m_e \nu(\underset{\sim}{u}_e - \underset{\sim}{u}_i) \qquad \text{(electrons)}$$

$$m_i \frac{\partial \underset{\sim}{u}_i}{\partial t} = e(\underset{\sim}{E} + \underset{\sim}{u}_i \times \underset{\sim}{B}_o) - m_e \nu(\underset{\sim}{u}_i - \underset{\sim}{u}_e) \qquad \text{(ions)}$$

(a) Determine expressions for the steady state DC conductivities σ_H, $\sigma_\perp$, σ_o.

(b) For $\underset{\sim}{u}_e$, $\underset{\sim}{u}_i$ and $\underset{\sim}{E}$ all proportional to exp $(-i\,\omega\, t)$ and $\underset{\sim}{B}_o$ constant, calculate the AC conductivity dyad for the plasma.

10.3 - Consider the equation $\underset{\sim}{J} = \underset{\approx}{\sigma} \cdot \underset{\sim}{E}$, with $\underset{\approx}{\sigma}$ as given in Eq. (4.23). If we choose a Cartesian coordinate system such that $E_x = E_\perp$, $E_y = 0$, $E_z = E_\parallel$ and $\underset{\sim}{B}_o = B_o \hat{\underset{\sim}{z}}$ (refer to Fig. 2), verify that in this coordinate system we have

$$J_x = \sigma_\perp \ E_\perp$$

$$J_y = \sigma_H \ E_\perp$$

$$J_z = \sigma_\parallel \ E_\parallel$$

Interpret physically this result with reference to Fig. 2.

10.4 - What is the physical meaning of a complex conductivity, as given in Eqs. (5.8) and (5.9)? Consider, for example, that $\underset{\sim}{E}(\underset{\sim}{r}, t) = \underset{\sim}{E}(\underset{\sim}{r}) \exp(-i\,\omega\, t)$, and calculate the real parts of

$\underset{\sim}{E}(\underset{\sim}{r}, t)$ and of $\underset{\sim}{J}(\underset{\sim}{r}, t) = \underset{\approx}{\sigma} \cdot \underset{\sim}{E}(\underset{\sim}{r}, t)$. Interpret physically the results considering the phase differences between $\underset{\sim}{J}$ and $\underset{\sim}{E}$.

10.5 - Write expressions for the components of the dielectric dyad, $\underset{\approx}{\varepsilon}$, of a multi-constituent magnetized plasma.

10.6 - Consider the electrons in a plasma acted upon by a small, constant and uniform external electric field $\underset{\sim}{E}$. Under steady state conditions with no spatial gradients, obtain an expression for the nonequilibrium distribution function f for the electrons, by applying a perturbation technique to the Boltzmann equation (take $f = f_0 + f_1$ with $|f_1| << f_0$ and neglect all second order terms) using the relaxation model for the collision term

$$\left(\frac{\delta f}{\delta t}\right)_{coll} = - \nu(f-f_0)$$

where ν is a relaxation collision frequency and f_0 is the equilibrium Maxwellian distribution function. Assuming that ν is independent of velocity, obtain an expression for the electric conductivity σ_0 of the plasma, by taking $\underset{\sim}{J} = \sigma_0 \underset{\sim}{E}$.

10.7 - Same as Problem 10.6, but including also a constant and uniform magnetic field $\underset{\sim}{B}_0$.

10.8 - Imagine a horizontally stratified ionosphere in the absence of a magnetic field, constituted only of electrons (density n, temperature T, charge -e, mass m_e) and one type of ions (density n, temperature T, charge +e, mass m_i), subjected to the gravitational field ($\underset{\sim}{g}$), vertical pressure gradient ($\underset{\sim}{\nabla} p$), and the internal electric field ($\underset{\sim}{E}$) due to the charge separation associated with ambipolar diffusion. Neglect the gravitational force for the electrons and consider the system in equilibrium. Using the collisionless equations of motion for the electrons and the ions, show that the internal electric field acts downward on the electrons with a force $m_i g/2$, and upward on the ions with the same force. Consequently, the net effect is the same as if both ions and electrons had mass $m_i/2$.

10.9 - (a) In order to solve the *diffusion equation*

$$\frac{\partial n(\underset{\sim}{r},t)}{\partial t} = D \nabla^2 n(\underset{\sim}{r},t)$$

by the method of separation of variables, let

$$n(\underset{\sim}{r},t) = S(\underset{\sim}{r})\, T(t)$$

and show that

$$T_k(t) = (\text{constant}) \exp(-D\, k^2 t)$$

$$(\nabla^2 + k^2)\, S(\underset{\sim}{r}) = 0$$

where k^2 is the separation constant.

(b) Assuming that S depends only on the x-coordinate show that

$$S(x) = c(k) \exp(ikx)$$

where k can be either positive or negative, and that

$$n(x,t) = \int_{-\infty}^{+\infty} c(k) \exp(ikx - D\, k^2 t)\, dk$$

$$n_o(x) = \int_{-\infty}^{+\infty} c(k) \exp(ikx)\, dk$$

where $n_o(x) = n(x,o)$ is the known *initial* density distribution.

(c) Using Fourier transform theory, show that

$$c(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} n_o(x) \exp(-ikx)\, dx$$

and, consequently, that

$$n(x,t) = \frac{1}{2(\pi D t)^{1/2}} \int_{-\infty}^{+\infty} n_o(x') \exp\left[-(x-x')^2/4Dt\right] \, dx'$$

(d) Taking as initial condition

$$n_o(x) = \exp(-x^2/x_o^2)$$

prove that

$$n(x,t) = \left(\frac{\tau_D}{\tau_D + 4t}\right)^{1/2} \exp\left[- \frac{x^2}{x_o^2} \left(\frac{\tau_D}{\tau_D + 4t}\right)\right]$$

where $\tau_D = x_o^2/D$ is a characteristic time for diffusion to smooth out the density n.

(e) Generalize the problem for the three-dimensional case in Cartesian coordinates, when $S = S(\underset{\sim}{r})$.

10.10 - Consider the solution of the diffusion equation by separation of variables in the linear geometry of the plasma slab indicated in Fig. P 10.1. Show that the solution of the equation

$$\frac{d^2 S(x)}{dx^2} + k^2 S(x) = 0$$

which satisfies the boundary condition $S = 0$ at $x = \pm L$, is

$$S(x) = \sum_m a_m \cos\left[\frac{(m+1/2)\pi x}{L}\right]$$

and

$$S(x) = \sum_m b_m \sin\left(\frac{m\pi x}{L}\right)$$

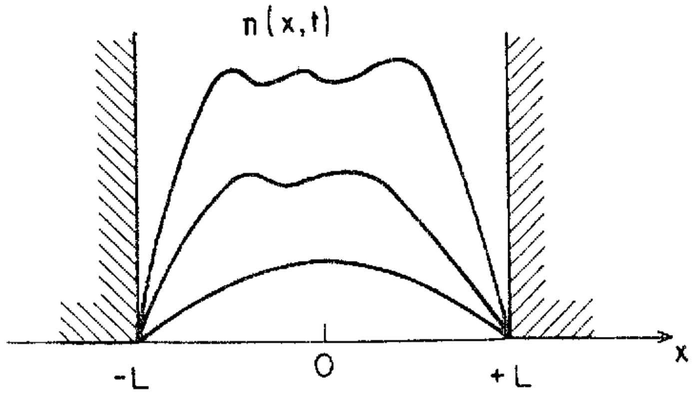

Fig. P10.1

Explain why the solution as a sine series is not a physically acceptable solution for the diffusion problem. Consequently, from $n(x,t) = S(x)\,T(t)$, show that the number density can be expressed as

$$n(x,t) = \sum_m a_m \exp\left\{- D \left[\frac{\pi(m+1/2)}{L}\right]^2 t\right\} \cos\left[\frac{\pi(m+1/2)x}{L}\right]$$

Therefore, the decay time constant for the m^{th} mode is

$$\tau_m = \left[\frac{L}{\pi(m+1/2)}\right]^2 \frac{1}{D}$$

This result shows that the higher modes decay faster than the lower ones. How are the coefficients a_m determined in terms of $n_o(x)$?

10.11 - Show that the solution of the diffusion equation in the case of cylindrical geometry (see Fig. P 10.2),

$$\frac{d^2S(r)}{dr^2} + \frac{1}{r} \frac{dS(r)}{dr} + k^2 S(r) = 0$$

is given in terms of Bessel's functions $J_m(kr)$.

Explain how k must be determined in order that $n(\underline{r},t)$ satisfies the boundary condition $n = 0$ at $r = R_o$.

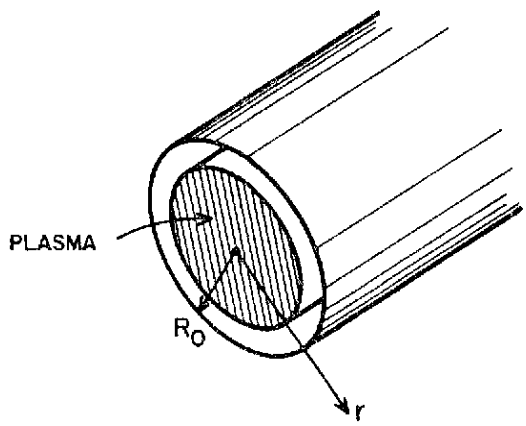

Fig. P 10.2

10.12 - Verify that plane wave solutions to the diffusion equation

$$\frac{\partial n(\underset{\sim}{r},t)}{\partial t} = D \nabla^2 n(\underset{\sim}{r},t)$$

yields the following *dispersion relation* between k and ω

$$k^2 D = i\omega$$

Then, show that for *free electron diffusion* we obtain

$$k^2 V_{se}^2 = i\omega\nu_{ce}$$

where $V_{se} = (k_B T_e/m_e)^{1/2} = (p_e/\rho_e)^{1/2}$ is the isothermal speed of sound in the electron gas and k_B is Boltzmann's constant. Next, show that for *ambipolar diffusion* we obtain

$$k^2 V_{sp}^2 = i\omega \nu_{ci}$$

where

$$V_{sp} = [k_B (T_e + T_i)/m_i]^{1/2} \simeq [(p_e + p_i)/(\rho_e + \rho_i)]^{1/2}$$

is the isothermal plasma sound speed. Calculate the phase velocity and the damping factor for these waves and verify if they are longitudinal or transverse.

10.13 - Consider a weakly ionized plasma immersed in a uniform magnetostatic field $\underset{\sim}{B}_0$ oriented along the z-axis of a Cartesian coordinate system.

(a) Show that the diffusion equation for the electrons (with $D \underset{\sim}{u}_e/Dt \simeq 0$) in the presence of the space charge electric field, $\underset{\sim}{E}$, is given by

$$\underset{\sim}{\Gamma}_e = - \underset{\sim}{\nabla} \cdot (\underset{\approx}{D}_e n_e) + n_e \underset{\approx}{\mu}_e \cdot \underset{\sim}{E}$$

where

$$\underset{\approx}{D}_e = \begin{pmatrix} D_{e\perp} & D_{eH} & 0 \\ -D_{eH} & D_{e\perp} & 0 \\ 0 & 0 & D_{e\parallel} \end{pmatrix}$$

with the following notation

$$D_{e\perp} = \frac{\nu_{ce}^2}{(\nu_{ce}^2 + \omega_{ce}^2)} D_e$$

$$D_{eH} = \frac{\nu_{ce} \; \omega_{ce}}{(\nu_{ce}^2 + \omega_{ce}^2)} D_e$$

$$D_{e\parallel} \equiv D_e = \frac{k \, T_e}{m_e \nu_{ce}}$$

and where

$$\underset{\approx}{\mu}_e = - \frac{e}{k \, T_e} \underset{\approx}{D}_e$$

(b) Deduce the corresponding equation for the ions in the presence of the space charge electric field $\underset{\sim}{E}$. Combine the equations for the electrons and the ions to eliminate the space charge electric field. Then, assuming that the electron and ion fluxes are equal, $\underset{\sim}{\Gamma}_e = \underset{\sim}{\Gamma}_i$, and that their number densities are also equal, $n_e = n_i$, determine the ambipolar diffusion coefficient, and notice that it is not affected by the presence of the magnetostatic field.

10.14 - Consider the following heat flow equation, derived in Problem 8.11, for a stationary electron gas immersed in a magnetic field,

$$\underset{\sim}{q}_e + \frac{\omega_{ce}}{\nu} (\underset{\sim}{q}_e \times \hat{\underset{\sim}{B}}) = - K_o \underset{\sim}{\nabla} T_e$$

Show that this equation can be written in the form

$$\underset{\sim}{q}_e = - \underset{\approx}{K} \cdot \underset{\sim}{\nabla} T_e$$

where $\underset{\approx}{K}$ is the dyadic thermal conductivity coefficient, given by

$$\underset{\approx}{K} = \begin{pmatrix} K_\perp & -K_H & 0 \\ K_H & K_\perp & 0 \\ 0 & 0 & K_o \end{pmatrix}$$

where

$$K_{\perp} = \frac{\nu^2}{(\nu^2 + \omega_{ce}^2)} K_o$$

$$K_H = \frac{\nu \, \omega_{ce}}{(\nu^2 + \omega_{ce}^2)} K_o$$

$$K_o = \frac{5 \, k \, p_e}{2 \, m_e \, \nu}$$