BI-IMPULSIVE ORBITAL TRANSFERS BETWEEN NON-COPLANAR ORBITS WITH MINIMUM TIME FOR A PRESCRIBED FUEL CONSUMPTION

Evandro Marconi ROCCO
Antonio Fernando Bertachini de Almeida PRADO
Marcelo Lopes de Oliveira e SOUZA

Instituto Nacional de Pesquisas Espaciais - INPE
Cx. P. 515 – 12201-970 – São José dos Campos, SP, Brasil
E-mail: evandro@dem.inpe.br; prado@dem.inpe.br; marcelo@dem.inpe.br

Abstract. *In this work we consider the problem of two-impulse orbital transfers between non-coplanar elliptical orbits with minimum time for a prescribed fuel consumption. We used the equations presented by Eckel and Vinh, add some new equations to consider cases with different geometries, and solved those equations to develop a software for orbital maneuvers. This software can be used in the next missions developed by INPE. The original method developed by Eckel and Vinh was presented without numerical results in that paper. Thus, the modifications considering cases with different geometries, the implementation and the solutions using this method are contributions of this work. The software was tested in real applications with success.*

1 - INTRODUCTION

The majority of the spacecrafts that have been placed in orbit around the Earth utilize the basic concept of orbital transfers. During the launch, the spacecraft is placed in a parking orbit distinct from the final orbit for which the spacecraft was designed. Therefore, to reach the desired final orbit the spacecraft must perform orbital transfers. Besides that, the spacecraft orbit must be corrected periodically because there are perturbations acting on the spacecraft. Both maneuvers are usually calculated with minimum fuel consumption but without a time constraint. This time constraint imposes a new characteristic to the problem that rules out the majority of the transfer methods available in the literature: [Hohm 25], [Hoel 59], [Gobe 69], [Prad 89], etc. Therefore, the transfer methods must be adapted to this new constraint: [Wang 63], [Lion 68], [Gros 74], [Prus 69], [Prus 70], [Prus 86], [Ivas 81], [Ecke 82], [Ecke 84], [Lawd 93] and [Taur 95]. In Brazil, we have important applications with the launch of the Remote Sensing Satellites RSS1 and RSS2 that belongs to the Complete Brazilian Space Mission and with the launch of the China Brazil Earth Resources Satellites CBERS1 and CBERS2.

In this work we consider the problem of two-impulse orbital transfers between non-coplanar elliptical orbits with minimum time for a prescribed fuel consumption. This problem is very important because most of spacecrafts utilize a propulsion system only capable of providing a fixed value of velocity increment, and the velocity increment is direct related to the fuel consumption. On the other hand, in many missions are important to perform the maneuvers in the minimum time, as for instance in the case of remote sensing satellites because during the maneuver the collected data are of low quality and therefore they are not able to be used. Thus, we used the equations presented by [Ecke 84], add some new equations to consider cases with different geometries, and solved those equations to develop a software for orbital maneuvers. This software can be used in the next missions developed by INPE.

2 - DEFINITION OF THE PROBLEM

The orbital transfer of a spacecraft from an initial orbit to a desired final orbit consists [Mare 79] in a change of state (position, velocity and mass) of the spacecraft, from initial conditions $\vec{r}_0$, $\vec{v}_0$ and m_0 at time t_0 to final conditions $\vec{r}_f$, $\vec{v}_f$ and m_f at time t_f ($t_f \geq t_0$) as shown in Fig. 1.

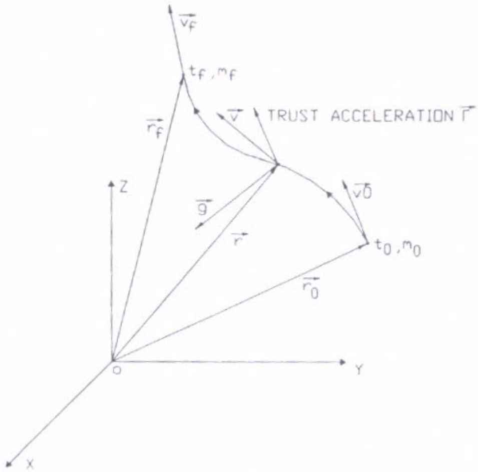

Fig. 1: Orbital Transfer Maneuver, cf. [Mare 79].

The maneuvers can be classified in: maneuvers partially free, when one or more parameter is free (for example, the time spent with the maneuver); or maneuvers completely constrained, when all parameters are constrained. In this case the spacecraft perform an orbital transfer maneuver from a specific point in the initial orbit to another specific point in the final orbit (for example, rendezvous maneuvers). In this work we consider the orbital transfer maneuvers partially free, and that the spacecraft propulsion system is able to apply an impulsive thrust. Therefore, we have the instantaneous variation of the spacecraft velocity.

3 - PRESENTATION OF THE METHOD

The bases for this method are the equations presented by [Ecke 84]. These equations furnish the transfer orbit between non-coplanar elliptical orbits with minimum fuel and fixed time transfer, or the transfer orbit with minimum time transfer for a prescribed fuel consumption. But in this work we consider only the problem with minimum time transfer for a prescribed fuel consumption. The problem with minimum fuel and fixed time transfer has already been considered by [Rocc 97] and summarized by [Rocc 99].

The equations were presented in the literature but the method was not implemented neither tested by Eckel and Vinh, and it is only valid for a specific geometry. They used the plane of the transfer orbit as the reference plane but we decide to use the equatorial plane as the reference plane because in this way it is easy to obtain and to apply the results in real applications. Using the transfer orbit as the reference plane almost all the results obtained belongs to the same specific geometry, so we change the reference system, adding the equations 1 to 6 to consider cases with more complex geometry. Therefore, the method was implemented to develop a software for orbital maneuvers. Thus, the modification, the implementation and the solutions using this method are contributions of this work. By varying the total velocity increment necessary to the maneuver the software developed furnishes a set of results that are the solution of the problem of bi-impulsive optimal orbital transfer with minimum time for a prescribed fuel consumption.

Given two non-coplanar terminal orbits we desire to obtain a transfer orbit that performs an orbital maneuver from the initial orbit to the final orbit with minimum time and fixed total velocity increment. The orbits are specified by their orbital elements (subscript 1: initial orbit; subscript 2: final orbit; no subscript: transfer orbit):

Table 1 – Orbital Elements.

a	Semi-major axis
e	Eccentricity
p	Semi-latus rectum
ω	Longitude of the periapsis
i	Inclination
Ω	Longitude of the ascending node
M	Mean anomaly
E	Eccentric anomaly
λ	Angle between the planes of the initial and final orbits
β_1	True anomaly of the point N obtained in the plane of the initial orbit
β_2	True anomaly of the point N obtained in the plane of the final orbit
I_1	Location of the first impulse
I_2	Location of the second impulse
Δ	Transfer angle obtained in the plane of the transfer orbit
γ_1	Plane change angle result of the first impulse
γ_2	Plane change angle result of the second impulse
V_1	Velocity increment generated by the first impulse
V_2	Velocity increment generated by the second impulse
V	Total velocity increment
T	Time spent in the maneuver
α_1	True anomaly of the point I_1 obtained in the plane of the initial orbit
α_2	True anomaly of the point I_2 obtained in the plane of the final orbit
r_1	Distance from point I_1
r_2	Distance from point I_2
f_1	True anomaly of the point I_1 obtained in the plane of the transfer orbit
f_2	True anomaly of the point I_2 obtained in the plane of the transfer orbit
x_1	Radial component of the first impulse
x_2	Radial component of the second impulse
y_1	Transverse component of the first impulse in the plane of the initial orbit
y_2	Transverse component of the second impulse in the plane of the transfer orbit
z_1	Component of the first impulse orthogonal to the initial orbit
z_2	Component of the second impulse orthogonal to the transfer orbit
h_i	Horizontal component of V_i

The geometry of the maneuver is shown in Fig. 2.

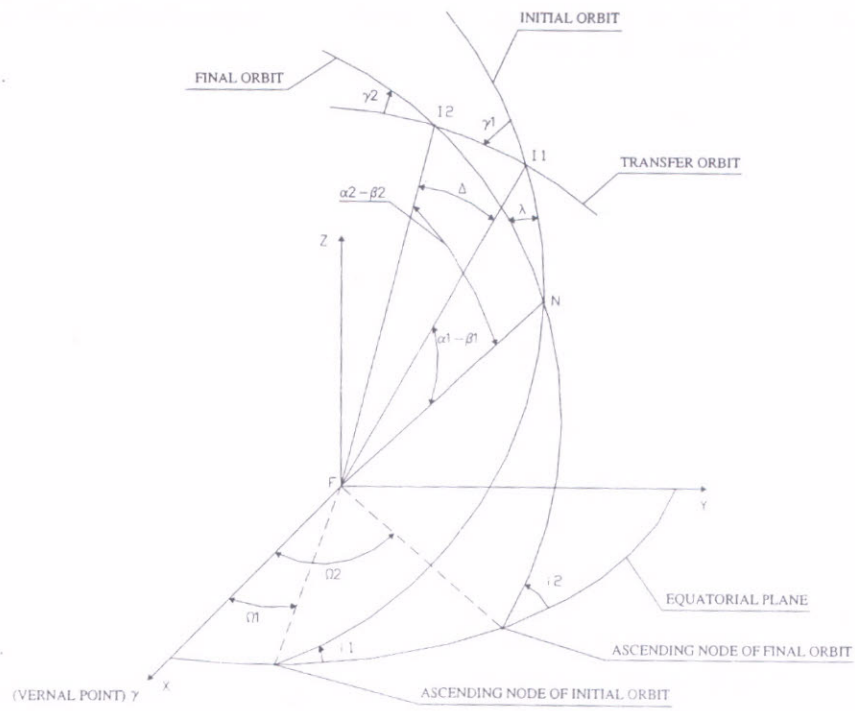

Fig. 2: Geometry of the Maneuver.

From the geometry of the maneuver we obtain β_1, β_2, λ and the transfer angle Δ:

$$\beta_1 = \arctan\left[\frac{\sin(\Omega_2 - \Omega_1)\tan(180^\circ - i_2)}{\sin i_1 + \tan(180^\circ - i_2)\cos i_1 \cos(\Omega_2 - \Omega_1)}\right] - \omega_1 \tag{1}$$

$$\beta_2 = \arctan\left[\frac{\sin(\Omega_2 - \Omega_1)\tan i_1}{\sin i_2 + \tan i_1 \cos(180^\circ - i_2)\cos(\Omega_2 - \Omega_1)}\right] - \omega_2 \tag{2}$$

$$\lambda = \arcsin\left[\frac{\sin(\Omega_2 - \Omega_1)\sin i_1}{\sin(\omega_2 + \beta 2)}\right] = \arcsin\left[\frac{\sin(\Omega_2 - \Omega_1)\sin i_2}{\sin(\omega_1 + \beta_1)}\right] \tag{3}$$

$$\cos\Delta = \cos(\beta_1 - \alpha_1)\cos(\alpha_2 - \beta_2) + \sin(\beta_1 - \alpha_1)\sin(\alpha_2 - \beta_2)\cos(180^\circ - \lambda) \tag{4}$$

$$\sin\Delta = \frac{\sin(\alpha_2 - \beta_2)\sin(180^\circ - \lambda)}{\sin B} \tag{5}$$

$$B = \arctan\left[\frac{\sin(180^\circ - \lambda)}{\sin(\beta_1 - \alpha_1)\cot(\alpha_2 - \beta_2) - \cos(\beta_1 - \alpha_1)\cos(180^\circ - \lambda)}\right] \tag{6}$$

Considering that the spacecraft propulsion system is able to apply an impulsive thrust, and that maneuver is bi-impulsive, the total velocity increment is:

$$V = V_1 + V_2 = F(X) \tag{7}$$

The time of the transfer maneuver is:

$$T = G(X) \tag{8}$$

Therefore, the problem is the minimization of T for a prescribed V. If the total velocity increment is prescribed, being equal to a value V_0, we have the constrained relation:

$$V - V_0 = 0 \tag{9}$$

Thus, we have the performance index:

$$J = T + k(V - V_0) \tag{10}$$

From [Ecke 84] we know that the solution of the problem depend on three variables: the semi-latus rectum p of the transfer orbit and the true anomaly α_1 and α_2 that define the position of impulses in the initial and final orbits. Therefore, we have the necessary conditions:

$$\frac{\partial V}{\partial p} + k\frac{\partial T}{\partial p} = 0 \quad ; \quad \frac{\partial V}{\partial \alpha_1} + k\frac{\partial T}{\partial \alpha_1} = 0 \quad ; \quad \frac{\partial V}{\partial \alpha_2} + k\frac{\partial T}{\partial \alpha_2} = 0 \tag{11}$$

By eliminating the Lagrange's multiplier k from equations 11 we have the set of two equations:

$$\frac{\partial V}{\partial p}\frac{\partial T}{\partial \alpha_1} - \frac{\partial V}{\partial \alpha_1}\frac{\partial T}{\partial p} = 0 \quad ; \quad \frac{\partial V}{\partial p}\frac{\partial T}{\partial \alpha_2} - \frac{\partial V}{\partial \alpha_2}\frac{\partial T}{\partial p} = 0 \tag{12}$$

Evaluating the partial derivatives in these equations and doing some simplifications we have the final optimal conditions:

$$\begin{aligned}&(X_1 + YZes\inf_2)(S_1 q_1 - T_1 es\inf_1) + S_1 T_1 \\ &+ W_1\left(\frac{W_1 - W_2}{sin\Delta} q_2 - W_1 \tan\frac{\Delta}{2}\right) - \frac{W_1 Zer_1 e_1 sin\alpha_1}{q_1 p_1 s\inf_1 \, sin\gamma_1} = 0\end{aligned} \tag{13}$$

$$\begin{aligned}&(X_2 + YZes\inf_1)(S_2 q_2 - T_2 es\inf_2) + S_2 T_2 \\ &- W_2\left(\frac{W_2 - W_1}{sin\Delta} q_1 - W_2 \tan\frac{\Delta}{2}\right) + \frac{W_2 Zer_2 e_2 sina_2}{q_2 p_2 s\inf_2 \, sin\gamma_2} = 0\end{aligned} \tag{14}$$

which utilize the relations shown in appendix A.

Thus, we have an equation system composed by equations 9, 13 and 14. Solving this equation system by Newton Raphson Method (cf. [Pres 92]), we obtain the transfer orbit that performs the maneuver between two terminal non-coplanar elliptical orbits spending a minimum time but with a specific fuel consumption.

4 - RESULTS

Figures 3 to 8 present some results obtained in [Rocc 97] and [Rocc 99] with the software developed. They not only show the tendency of the parameters, but they quantify the evolution of the variables studied. The graphs were obtained through the variation of the total velocity increment necessary to perform the maneuver. Thus, each point was obtained executing the software to the specific total velocity increment. The points were joined by a line that shows the behavior of that orbital element.

We utilized as an example the maneuver between an initial orbit with semi-major axis of 12030 km, eccentricity 0.02, inclination 0.00873 rad, longitude of the periapse 3.17649 rad, longitude of the ascending node zero and a final orbit with semi-major axis of 11994.7 km, eccentricity 0.016, inclination 0.00602 rad, longitude of the periapse 3.05171 rad, longitude of the ascending node 0.15568. We utilized as initial values l = 17511.16764407 km, α_1 = 3.19194826 rad, and α_2 = 6.18292080 rad. The graphs were obtained through the variation of the total necessary velocity increment from 0.063 to 4.42 km/s.

5 - CONCLUSIONS

In Figs. 3 to 6 we can verify that when the total velocity increment increases the semi-major axis and the eccentricity of the transfer orbit also increase, however, the transfer angle and the time spent in the maneuver decrease. These behaviors occur because when the maneuver is performed with a high value of the velocity increment the transfer orbit approaches a parabolic orbit, so the eccentricity approaches one. Then we have a high value of the semi-major axis and a small value of the transfer angle. In Fig. 7 we have the behavior of the plane change angle. We can verify that when the necessary velocity increment increases the absolute value of the plane change angle also increases. This is expected because changes in inclination, in general, spend more fuel. From this figure we conclude that the sum of the plane change angles almost remain constant because the second impulse undo part of the plane change angle that results of the first impulse. In Fig. 8 we can see that when the maneuver spends more time the velocity increment is smaller than when the maneuver spends less time. This is expected because when the maneuver spends more time the impulse directions approach the movement directions. However, we are studying the non-coplanar case, therefore the impulse directions never will be in the movement directions because we always have a component orthogonal to the orbital plane. In these graphs we can see that it was possible to obtain results when we fixed a small value of the velocity increment, but there is a lower limit, which occur when we reach the solution for time free. Besides that, we should advise that the developed program can not supply the solution for all combinations of the input parameters. For certain values of the total velocity increment it can be impossible to obtain one solution because for a very small or very large values of the total velocity increment the solution can not exist, or the numerical algorithms used in the program do not converge for the solution, because the initial values used can be too far from the solution. So, it is recommended a physical analysis of the problem, that takes into account the geometry of the maneuver, to find the range of values for the total velocity increment which is possible to accomplish the maneuver. Another question to be solved is if the solution is a local or global minimum. Up to where we verified, the solution obtained seems to be a global minimum because for the same input parameters, but using different initial values, it was not possible until the moment, to obtain better results. It is important to notice that the software tests automatically all the results, verifying if the maneuver obtained is just a mathematical solution or if it can really be implemented. When we use numerical methods there are some solutions, which satisfy the equations, however, in practice, they are impossible. Concluding, we can verify that these results are very similar to the results obtained by [Rocc 97] for the case of minimum fuel consumption and fixed time transfer. Thus it is clear that the case of minimum time transfer and fixed fuel consumption is almost the converse of the case of minimum fuel consumption and fixed time transfer. Therefore, both cases were studied, implemented and tested with success. The simulations showed that the software developed can be used in real applications and it is capable to generate reliable results.

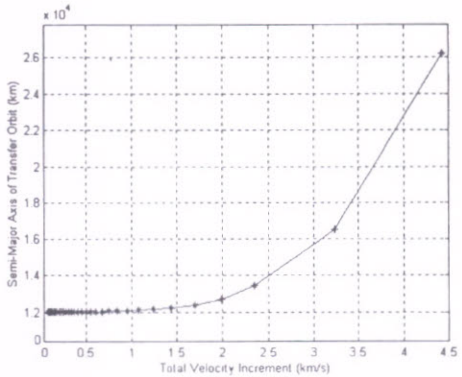

Fig. 3: Semi-Major Axis vs. Total Velocity Increment.

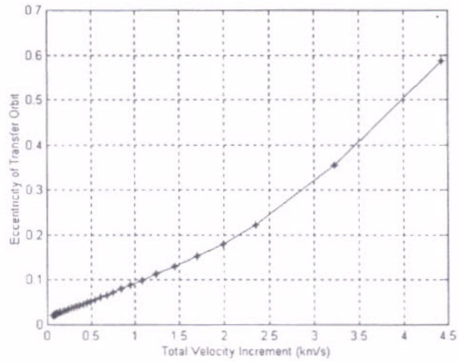

Fig.4: Eccentricity vs. Total Velocity Increment.

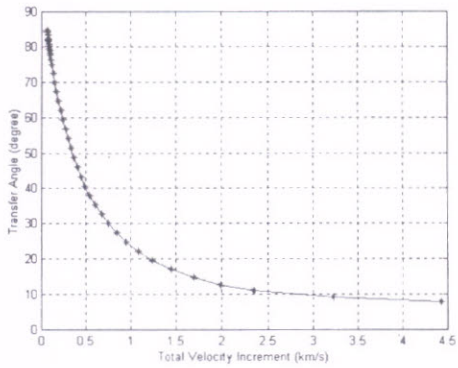

Fig. 5: Transfer Angle vs. Total Velocity Increment.

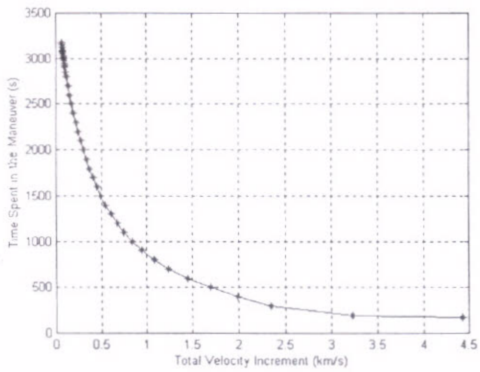

Fig. 6: Time Spent in Maneuver vs. Total Velocity Increment.

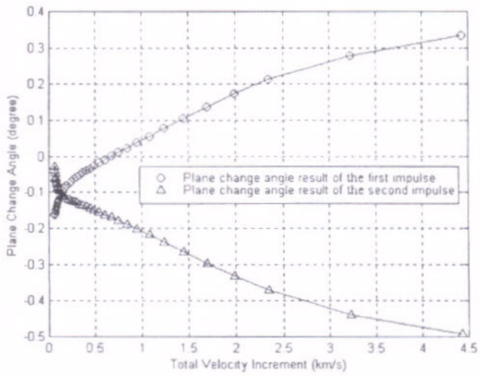

Fig. 7: Plane Change Angle vs. Total Velocity Increment.

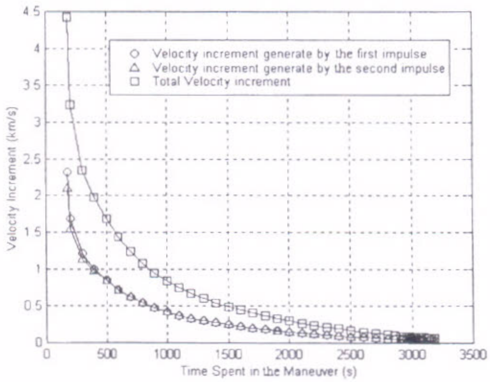

Fig. 8: Velocity Increment vs. Time Spent in Maneuver.

6 - ACKNOWLEDGEMENTS

The authors express their thanks to the National Council for Scientific and Technological Development in Brazil (CNPq) and to the Foundation for Supporting Research in São Paulo State (FAPESP), for supporting this research.

APPENDIX A

$$r_i = \frac{p_i}{1 + e_i \cos\alpha_i} \tag{A.1}$$

$$f_1 = \arctan\left[\cot\Delta - \frac{r_1(p - r_2)}{r_2(p - r_1)\sin\Delta}\right]$$

$$f_2 = \arctan\left[\frac{r_2(p - r_1)}{r_1(p - r_2)\sin\Delta} - \operatorname{cotg}\Delta\right] \tag{A.2}$$

$$p = \frac{r_1 r_2 (\cos f_1 - \cos f_2)}{r_1 \cos f_1 - r_2 \cos f_2} \tag{A.3}$$

$$e = \frac{r_2 - r_1}{r_1 \cos f_1 - r_2 \cos f_2} \tag{A.4}$$

$$a = \frac{p}{1 - e^2} \tag{A.5}$$

$$\gamma_1 = \arcsin\left[-\frac{\sin(\beta_2 - \alpha_2)}{\sin\Delta}\sin\phi\right]$$

$$\gamma_2 = \arcsin\left[-\frac{\sin(\beta_1 - \alpha_1)}{\sin\Delta}\sin\phi\right] \tag{A.6}$$

$$x_1 = \sqrt{\mu}\left(\frac{e}{\sqrt{p}}\sin f_1 - \frac{e_1}{\sqrt{p_1}}\sin\alpha_1\right)$$

$$x_2 = \sqrt{\mu}\left(\frac{e_2}{\sqrt{p_2}}\sin\alpha_2 - \frac{e}{\sqrt{p}}\sin f_2\right) \tag{A.7}$$

$$y_1 = \frac{\sqrt{\mu}}{r_1}\left(\sqrt{p} - \sqrt{p_1}\cos\gamma_1\right)$$

$$y_2 = \frac{\sqrt{\mu}}{r_2}\left(\sqrt{p_2}\cos\gamma_2 - \sqrt{p}\right) \tag{A.8}$$

$$z_i = \frac{\sqrt{\mu p_i}}{r_i} \sin \gamma_i \tag{A.9}$$

$$h_i = \left(y_i^{\,2} + z_i^{\,2} \right)^{1/2} \tag{A.10}$$

$$V_i = \left(x_i^{\,2} + h_i^{\,2} \right)^{1/2} \tag{A.11}$$

$$S_i = \frac{x_i}{V_i} \tag{A.13}$$

$$T_i = \frac{y_i}{V_i} \tag{A.14}$$

$$W_i = \frac{z_i}{V_i} \tag{A.15}$$

$$q_i = \frac{p}{r_i} \tag{A.16}$$

$$E_i = \arccos\left(\frac{e + \cos f_i}{1 + e \cos f_i} \right) \tag{A.17}$$

$$\sin E_i = \frac{\sqrt{1 - e^2} \sin f_i}{1 + e \cos f_i} \tag{A.18}$$

$$M_i = E_i - e \sin E_i \tag{A.19}$$

$$T = \sqrt{\frac{a^3}{\mu}} \left(M_{t_2} - M_{t_1} + 2\pi N \right) \tag{A.20}$$

$$\begin{aligned} X_1 &= \frac{S_1 \cos \Delta - S_2}{\operatorname{sen} \Delta} + T_1 \\ X_2 &= \frac{S_1 - S_2 \cos \Delta}{\operatorname{sen} \Delta} + T_2 \end{aligned} \tag{A.21}$$

$$Y = \frac{1}{\left(1 - e^2\right) \operatorname{sen} \Delta} \left[3e^2 T \sqrt{\frac{\mu}{p^3}} - 2e \left(\frac{1}{q_2 \operatorname{sen} f_2} - \frac{1}{q_1 \operatorname{sen} f_1} \right) + \operatorname{cotg} f_2 - \operatorname{cotg} f_1 \right] \tag{A.22}$$

$$Z = \frac{q_2 X_2 - q_1 X_1 + \left(S_1 + S_2\right) \operatorname{tg} \Delta/2}{\operatorname{cotg} f_1 - \operatorname{cotg} f_2 + Y \left[\left(1 + e^2\right) \operatorname{sen} \Delta + 2e \left(\operatorname{sen} f_2 - \operatorname{sen} f_1 \right) \right]} \tag{A.23}$$

REFERENCES

[Ecke 82] Eckel, K.G., "Optimal Impulsive Transfer with Time Constraint" *Acta Astronautica*, Vol. 9, No. 3, 1982, pp. 139-146.

[Ecke 84] Eckel, K.G. e Vinh, N.X., "Optimal Swtitching Conditions for Minimum Fuel Fixed Time Transfer Between non Coplanar Elliptical Orbits" *Acta Astronautica*, Vol. 11, No. 10/11, 1984, pp. 621-631.

[Gobe 69] Gobetz, F.W.; Doll, J.R., "A Survey of Impulsive Trajectories. AIAA Journal, Vol. 7, No. 5: 801-834.

[Gros 74] Gross, L.R.; Prussing, J.E., 1974, Optimal Multiple-Impulse Direct Ascent Fixed-Time Rendezvous" *AIAA Journal*, Vol. 12, No. 7, 1969, pp. 885-889.

[Hoel 59] Hoelker, R.F.; Silber, R., "The Bi-Elliptical Transfer Between Circular Co-Planar Orbits", *Army Ballistic Missiles Agency*, Redstone Arsenal, Alabama, EUA, 1959.

[Hohm 25] Hohmann, W., "Die Erreichbarkeit Der Himmelskoprper" Oldenbourg, Munique, 1925.

[Ivas 81] Ivashkin, V.V.; Skorokhodov, A.P., "Definition and Analysis of Properties of Optimal Three-Impulse Point-to-Orbit Transfers Under Time Constraint" *Acta Astronautica*, Vol. 8, No. 1, 1981, pp. 11-23.

[Lawd 93] Lawden, D.F., "Time-Closed Optimal Transfer by Two Impulses Between Coplanar Elliptical Orbits" *Journal of Guidance, Control, and Dynamics*, Vol. 16, No. 3, 1993.

[Lion 68] Lion, P.M.; Handelsnman, M., "Primer Vector on Fixed-Time Impulsive Trajectories", *AIAA Journal*, Vol. 6 No. 1, 1968, pp. 127-132.

[Mare 79] Marec, J.P., "Optimal Space Trajectories", New York, NY, Elsevier, 1979.

[Prad 89] Prado, A.F.B.A., "Análise, Seleção e Implementação de Procedimentos que Visem Manobras Ótimas de Satélites Artificiais". Master Thesis, (INPE-5003-TDL/397), 1989.

[Pres 92] Press, W.H.; Teukolsky, S.A.; Vetterling, W.T.; Flannery, B.P., "Numerical Recipes in FORTRAN (The Art of Scientific Computing)", 2nd Ed. Cambridge University Press, 1992.

[Prus 69] Prussing, J.E., "Optimal Four-Impulse Fixed-Time Rendezvous in the Vicinity of a Circular Orbit", *AIAA Journal*, Vol. 7, No. 5, 1969, pp. 928-935.

[Prus 69] Prussing, J.E., 1970, "Optimal Two and Three Impulse Fixed-Time Rendezvous in the Vicinity of a Circular Orbit", *AIAA Journal*, Vol. 8, No. 7, 1969, pp. 1221-1228.

[Prus 86] Prussing, J.E. e Chiu, J.H., "Optimal Multiple-Impulse Time-Fixed Rendezvous Between Circular Orbits", *Journal of Guidance, Control, and Dynamics*, Vol. 9, No. 1, 1986, pp. 17-22.

[Rocc 97] Rocco, E.M., "Transferências Orbitais Bi-Impulsivas com Limite de Tempo" Master Thesis, (INPE-6676-TDI/626), 1997.

[Rocc 99] Rocco, E.M.; Prado, A.F.B.A.; Souza, M.L.O., "Bi-Impulsive Orbital Transfers Between Non-Coplanar Orbits with Time Limit", *Sixth Pan American Congress of Applied Mechanics PACAM VI / 8th International Conference on Dynamic Problems in Mechanics DINAME*. Rio de Janeiro – RJ, 1999.

[Taur 95] Taur, D.R.; Carroll, V.C.; Prussing, J.E., "Optimal Impulsive Time-Fixed Orbital Rendezvous and Interception with Path Constraints", *Journal of Guidance, Control, and Dynamics*, Vol. 18, No. 1, 1995, pp. 54-60.

[Wang 63] Wang, K., "Minimum Time Transfer Between Coplanar Circular Orbits by Two Impulses and Propulsion Requirements", *Astronautica Acta*, Vol. 9 No. 1, 1963, pp. 12-19.