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16. Summary/Notes

This chapter presents a systematic method for obtaining the macroscopic variables in terms of the distribution function. The macroscopic variables are all moments of the distribution function. Expressions, and the physical interpretations, are presented for the macroscopic variables of physical interest, such as the average velocity, flux of a physical quantity, particle current density, momentum flow dyad, pressure dyad, scalar pressure, absolute temperature, heat flow vector, heat flow triad and total energy flux triad.

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AVERAGE VALUES AND MACROSCOPIC VARIABLES

1. AVERAGE VALUE OF A PHYSICAL QUANTITY

A systematic method for obtaining the average values of functions of particle velocities is presented in this chapter. The macroscopic variables, such as number density, average velocity, kinetic pressure, thermal energy flux, and so on, can be considered as average values of physical quantities, involving the collective behavior of a large number of particles. These macroscopic variables are all *moments* of the distribution function.

To each particle in the plasma we can associate some molecular property, $\chi(\underset{\sim}{r}, \underset{\sim}{v}, t)$, which in general may be a function of the position $\underset{\sim}{r}$ of the particle, of its velocity $\underset{\sim}{v}$ and of the time t. This property may be, for example, the mass, the velocity, the momentum, or the energy of the particle.

In order to calculate the average value of $\chi(\underset{\sim}{r}, \underset{\sim}{v}, t)$, recall that $d^6 n_\alpha(\underset{\sim}{r}, \underset{\sim}{v}, t)$ represents the number of particles of type α inside the phase space volume element $d^3r\ d^3v$ about $(\underset{\sim}{r}, \underset{\sim}{v})$ at the instant t. Thus, the total value of $\chi(\underset{\sim}{r}, \underset{\sim}{v}, t)$ for all the particles of type α inside $d^3r\ d^3v$, is given by

$$\chi(\underset{\sim}{r}, \underset{\sim}{v}, t)\, d^6 n_\alpha(\underset{\sim}{r}, \underset{\sim}{v}, t) = \chi(\underset{\sim}{r}, \underset{\sim}{v}, t)\, f_\alpha(\underset{\sim}{r}, \underset{\sim}{v}, t)\, d^3r\, d^3v \qquad (1.1)$$

The total value of $\chi(\underset{\sim}{r}, \underset{\sim}{v}, t)$ for all the particles of type α inside the volume element d^3r of configuration space, irrespective of velocity, is obtained by integrating expression (1.1) over all possible velocities, that is,

$$d^3r \int_V \chi(\underset{\sim}{r}, \underset{\sim}{v}, t)\, f_\alpha(\underset{\sim}{r}, \underset{\sim}{v}, t)\, d^3v \qquad (1.2)$$

The *average value* of $\chi(\underset{\sim}{r}, \underset{\sim}{v}, t)$ can now be obtained by dividing (1.2) by the number of particles of type α inside d^3r about $\underset{\sim}{r}$ at the instant t, $n_\alpha(\underset{\sim}{r}, t)\, d^3r$. We define, therefore, the average value of the property $\chi(\underset{\sim}{r}, \underset{\sim}{v}, t)$, for the particles of type α, by

$$< \chi(\underset{\sim}{r}, \underset{\sim}{v}, t) >_\alpha = \frac{1}{n_\alpha(\underset{\sim}{r}, t)} \int_V \chi(\underset{\sim}{r}, \underset{\sim}{v}, t)\, f_\alpha(\underset{\sim}{r}, \underset{\sim}{v}, t)\, d^3v \qquad (1.3)$$

The symbol $< \;\; >_\alpha$ denotes the average value with respect to velocity space for the particles of type α. Note that the average value is independent of $\underset{\sim}{v}$, being a function of only $\underset{\sim}{r}$ and t.

If we take $\chi = 1$ in Eq. (1.3), the expression for the *number density*, given in Eq. (5.4.2), is obtained.

2. AVERAGE VELOCITY AND PECULIAR VELOCITY

Consider now $\chi(\underset{\sim}{r}, \underset{\sim}{v}, t)$ as being the velocity $\underset{\sim}{v}$ of the type α particles in the vicinity of the position $\underset{\sim}{r}$ at the instant of time t. Eq. (1.4) then gives the macroscopic *average velocity* $\underset{\sim}{u}_\alpha(\underset{\sim}{r}, t)$ for the particles of type α,

$$\underset{\sim}{u}_\alpha(\underset{\sim}{r}, t) = < \underset{\sim}{v} >_\alpha = \frac{1}{n_\alpha(\underset{\sim}{r}, t)} \int_v \underset{\sim}{v}\, f_\alpha(\underset{\sim}{r}, \underset{\sim}{v}, t)\, d^3v \tag{2.1}$$

which is the same expression given in Eq. (5.4.4).

Note that $\underset{\sim}{r}$, $\underset{\sim}{v}$ and t are taken as independent variables, whereas the average velocity $\underset{\sim}{u}_\alpha$ depends on the position $\underset{\sim}{r}$ and the time t. For the cases in which χ is independent of the velocity, we have

$$< \chi(\underset{\sim}{r}, t) >_\alpha = \chi_\alpha(\underset{\sim}{r}, t) \tag{2.2}$$

so that, for example, $< \underset{\sim}{u}_\alpha > = \underset{\sim}{u}_\alpha$. In what follows, the index α after the average value symbol will be omitted whenever it is redundant, that is, $< \underset{\sim}{u}_\alpha >_\alpha = < \underset{\sim}{u}_\alpha >$.

The *peculiar* or *random valocity* $\underset{\sim}{c}_\alpha$ is defined as the velocity of a type α particle relative to the average velocity $\underset{\sim}{u}_\alpha$,

$$\underset{\sim}{c}_\alpha = \underset{\sim}{v} - \underset{\sim}{u}_\alpha \tag{2.3}$$

Consequently we have $< \underset{\sim}{c}_\alpha > = 0$, since $< \underset{\sim}{v} >_\alpha = \underset{\sim}{u}_\alpha$. The peculiar velocity $\underset{\sim}{c}_\alpha$ is the one associated with the random thermal kinetic energy of the particles of type α. When $\underset{\sim}{u}_\alpha$ vanishes we have $\underset{\sim}{c}_\alpha = \underset{\sim}{v}$.

3. FLUX

From the concept of distribution function many other macroscopic variables can be defined in terms of average values. Macroscopic variables such as the particle current density (particle flux), the pressure dyad or tensor, and the heat-flow vector (thermal energy flux), involve the flux of some molecular property χ ($\underset{\sim}{r}$, $\underset{\sim}{v}$, t). The *flux* of χ ($\underset{\sim}{r}$, $\underset{\sim}{v}$, t) is defined as the amount of the quantity χ ($\underset{\sim}{r}$, $\underset{\sim}{v}$, t) transported across some given surface, per unit area, and per unit time.

Consider a surface element dS, inside the plasma. If the distribution of velocities is isotropic the flux will be independent of the relative orientation in space of the surface element dS. However, more generally, when the velocity distribution is anisotropic the flux will depend on the relative orientation in space of dS. Suppose, therefore, that the surface element of magnitude dS is oriented along some direction specified by the unit vector $\hat{\underset{\sim}{n}}$, that is,

$$d\underset{\sim}{S} = \hat{\underset{\sim}{n}}\, dS \tag{3.1}$$

$\hat{\underset{\sim}{n}}$ being normal to the surface element. In the case of an open surface there are two possible directions for the normal $\hat{\underset{\sim}{n}}$, one opposite to the other. The direction which is taken as positive is related to the positive sense of traversing the perimeter (bounding curve) of the open surface, according to the following convention: if the positive sense of traversal of the perimeter of a horizontal open surface is taken as counterclockwise, then the positive normal to the open surface is up; if the positive sense of traversal of the perimeter is clockwise, then the positive normal to the open surface is down, as shown in Fig. 1. For a closed surface the normal unit vector is conventionally chosen to point outward .

The particles inside the plasma, due to their velocities, will move across the surface element $d\underset{\sim}{S}$ carrying the property χ ($\underset{\sim}{r}$, $\underset{\sim}{v}$, t) with them. We want to calculate the number of particles of type α that move across $d\underset{\sim}{S}$ during the time interval dt.

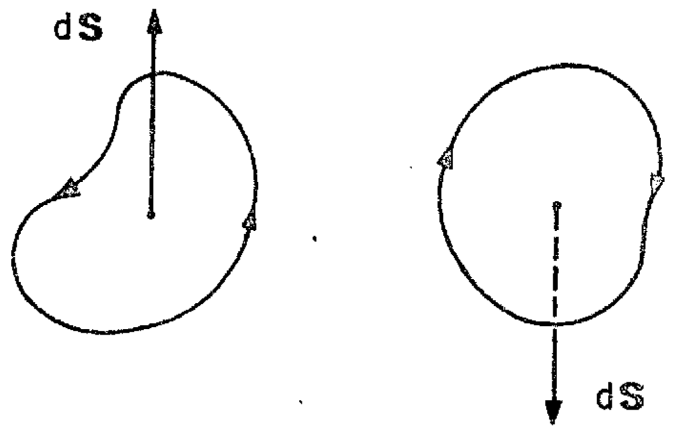

Fig. 1 - Direction of positive normal to the surface element d S as related to the sense of traversing the perimeter of d S.

The particles with velocity between $\underset{\sim}{v}$ and $\underset{\sim}{v} + d\underset{\sim}{v}$ that will cross $d\underset{\sim}{S}$ in the time interval between t and t + dt, must lie initially in the volume of prism of base d S and side v dt, as indicated in Fig. 2. The volume of this prism is

$$d^3r = d\underset{\sim}{S} \cdot \underset{\sim}{v}\, dt = \hat{\underset{\sim}{n}} \cdot \underset{\sim}{v}\, dS\, dt \tag{3.2}$$

From the definition of $f_\alpha (\underset{\sim}{r}, \underset{\sim}{v}, t)$, the number of particles of type α in the volume of this prism, that have velocities between $\underset{\sim}{v}$ and $\underset{\sim}{v} + d\underset{\sim}{v}$ is

$$f_\alpha (\underset{\sim}{r}, \underset{\sim}{v}, t)\, d^3r\, d^3v = f_\alpha (\underset{\sim}{r}, \underset{\sim}{v}, t)\, \hat{\underset{\sim}{n}} \cdot \underset{\sim}{v}\, dS\, dt\, d^3v \tag{3.3}$$

so that the total amount of $\chi(\underset{\sim}{r}, \underset{\sim}{v}, t)$ transported across $d\underset{\sim}{S}$, in the time interval dt, is obtained by multiplying this number of particles by $\chi(\underset{\sim}{r}, \underset{\sim}{v}, t)$ and integrating the result over all possible velocities, that is,

$$\int_V \chi(\underset{\sim}{r}, \underset{\sim}{v}, t) \; f_\alpha(\underset{\sim}{r}, \underset{\sim}{v}, t) \, \hat{\underset{\sim}{n}} \cdot \underset{\sim}{v} \; d^3v \; dS \; dt \tag{3.4}$$

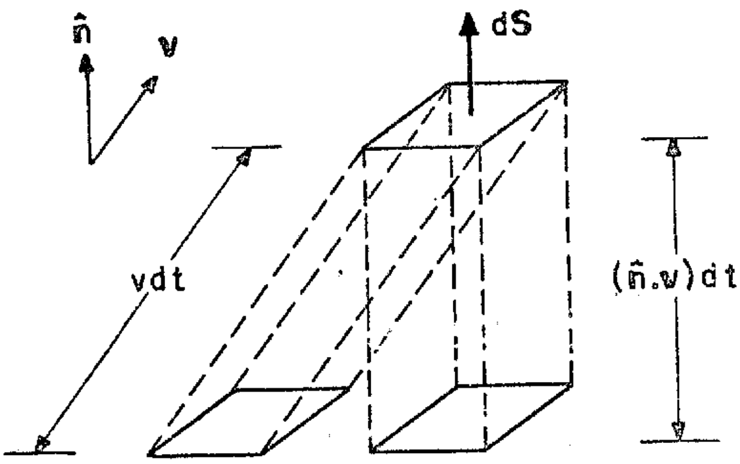

Fig. 2 - Prism of volume $d^3r = d\underset{\sim}{S} \cdot \underset{\sim}{v} \; dt = \hat{\underset{\sim}{n}} \cdot \underset{\sim}{v} \; dS \; dt$ containing the particles of type α with velocities between $\underset{\sim}{v}$ and $\underset{\sim}{v} + d\underset{\sim}{v}$, and which will cross $d\underset{\sim}{S}$ in the time interval dt.

Note that the contributions corresponding to a rotation of the segment v dt over all possible directions about $d\underset{\sim}{S}$ are taken into account in the integration over velocity space. Particles that cross $d\underset{\sim}{S}$ in a direction such that $\hat{\underset{\sim}{n}} \cdot \underset{\sim}{v}$ is positive give a positive contribution to the flux in the direction of $\hat{\underset{\sim}{n}}$, while particles that cross $d\underset{\sim}{S}$ in a direction such that $\hat{\underset{\sim}{n}} \cdot \underset{\sim}{v}$ is negative give a negative contribution to the flux in the direction of $\hat{\underset{\sim}{n}}$. This is illustrated in Fig. 3

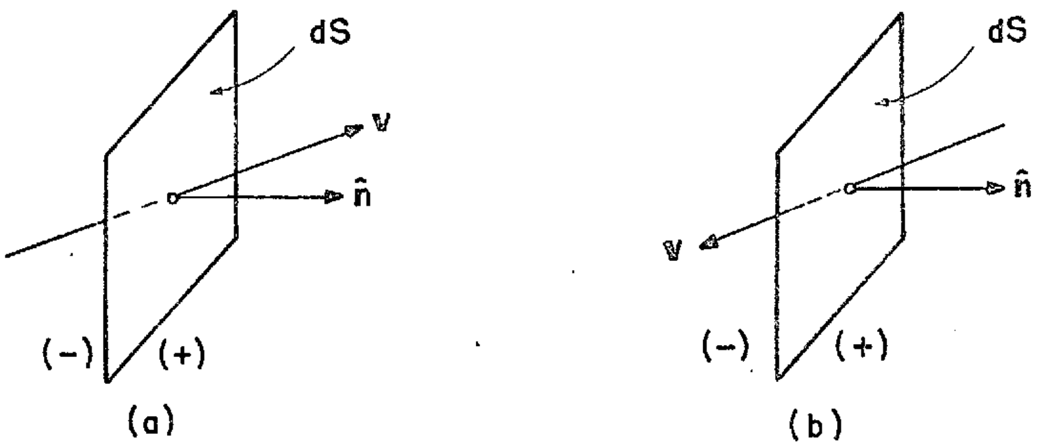

Fig. 3 - (a) Particles that cross $d\underset{\sim}{S}$ from the (-) region to the (+) region contribute positively to the flux in direction $\hat{\underset{\sim}{n}}$, while (b) particles that cross $d\underset{\sim}{S}$ from the (+) region to the (-) region contribute negatively to the flux in direction $\hat{\underset{\sim}{n}}$.

The net amount of the quantity $\chi(\underset{\sim}{r}, \underset{\sim}{v}, t)$ transported by the particles of type α, per unit area, and per unit time, is obtained by dividing expression (3.4) by dS dt. The *flux* in the direction $\hat{\underset{\sim}{n}}$, $\Phi_{\alpha n}(\chi)$, is, therefore, given by

$$\Phi_{\alpha n}(\chi) = \int_v \chi(\underset{\sim}{r}, \underset{\sim}{v}, t) \; \hat{\underset{\sim}{n}} \cdot \underset{\sim}{v} \; f_\alpha(\underset{\sim}{r}, \underset{\sim}{v}, t) \, d^3v \tag{3.5}$$

or, using the average value symbol,

$$\Phi_{\alpha n} = n_\alpha(\underset{\sim}{r}, t) < \chi(\underset{\sim}{r}, \underset{\sim}{v}, t) \; \hat{\underset{\sim}{n}} \cdot \underset{\sim}{v} >_\alpha$$

$$= n_\alpha < \chi \, v_n >_\alpha \tag{3.6}$$

where $v_n = \hat{\underset{\sim}{n}} \cdot \underset{\sim}{v}$ denotes the component of $\underset{\sim}{v}$ along the direction specified by the unit vector $\hat{\underset{\sim}{n}}$.

When $\chi(\underset{\sim}{r}, \underset{\sim}{v}, t)$ is a scalar quantity, $\Phi_{\alpha n}(\chi)$ can be considered as the component along $\hat{\underset{\sim}{n}}$, of a *vector flux* $\underset{\sim}{\Phi}_\alpha(\chi)$, that is,

$$\Phi_{\alpha n}(\chi) = \hat{\underset{\sim}{n}} \cdot \underset{\sim}{\Phi}_\alpha(\chi) \tag{3.7}$$

with

$$\underset{\sim}{\Phi}_\alpha(\chi) = n_\alpha < \chi \, \underset{\sim}{v} >_\alpha \tag{3.8}$$

If χ ($\underset{\sim}{r}$, $\underset{\sim}{v}$, t) represents a *vector* quantity, then we will have a *flux dyad* (or *tensor*),

$$\underset{\approx}{\Phi}_{\alpha} (\underset{\sim}{\chi}) = n_{\alpha} < \underset{\sim}{\chi} \, \underset{\sim}{v} >_{\alpha} \tag{3.9}$$

and if it represents a *dyad* quantity we will have a *flux triad,* and so on.

In many situations of practical interest it is important to separately consider the contribution to the flux due to the average velocity $\underset{\sim}{u}_{\alpha}$, and that due to the random velocity $\underset{\sim}{c}_{\alpha}$ of the particles of type α. Substituting $\underset{\sim}{v} = \underset{\sim}{c}_{\alpha} + \underset{\sim}{u}_{\alpha}$ in Eq. (3.6) gives

$$\Phi_{\alpha n} (\chi) = n_{\alpha} < \chi \, c_{\alpha n} > + \, n_{\alpha} < \chi \, u_{\alpha n} > \tag{3.10}$$

where $c_{\alpha n} = \underset{\sim}{\hat{n}} \cdot \underset{\sim}{c}_{\alpha}$ and $u_{\alpha n} = \underset{\sim}{\hat{n}} \cdot \underset{\sim}{u}_{\alpha}$

For the cases in which the flow velocity $\underset{\sim}{u}_{\alpha}$ is zero or, equivalently, if we take $d\underset{\sim}{S}$ to be in a frame of reference moving with the average velocity $\underset{\sim}{u}_{\alpha}$, Eq. (3.10) becomes

$$\Phi_{\alpha n} (\chi) = n_{\alpha} < \chi \, c_{\alpha n} > \tag{3.11}$$

which is the flux of χ ($\underset{\sim}{r}$, $\underset{\sim}{v}$, t) along $\underset{\sim}{\hat{n}}$ due to the *random* motions of the particles of type α.

4. PARTICLE CURRENT DENSITY

The *particle current density* (or *particle flux*) is defined as the number of particles passing through a given surface, per unit area, and per unit time. Taking $\chi = 1$ in Eq.(3.6), we obtain the flux of particles of type α in the direction $\hat{\underset{\sim}{n}}$,

$$\Gamma_{\alpha n}(\underset{\sim}{r}, t) = n_\alpha < v_n >_\alpha$$

$$= n_\alpha u_{\alpha n} \tag{4.1}$$

since $< c_{\alpha n} > = 0$. When $\underset{\sim}{u}_\alpha$ vanishes, it is of interest to consider only the flux in the positive direction instead of the resulting net flux. The number of particles of type α that cross a given surface along the direction $\hat{\underset{\sim}{n}}$ from the same side, per unit area and time, due to their random motions, is given by

$$\Gamma_{\alpha n}^{(+)}(\underset{\sim}{r}, t) = \int_{\substack{v \\ (\hat{\underset{\sim}{n}} \cdot \underset{\sim}{c}_\alpha > 0)}} \hat{\underset{\sim}{n}} \cdot \underset{\sim}{c}_\alpha \, f_\alpha(\underset{\sim}{r}, \underset{\sim}{v}, t) \, d^3v \tag{4.2}$$

where the integral in velocity space is over only the velocities for which $\hat{\underset{\sim}{n}} \cdot \underset{\sim}{c}_\alpha > 0$.

The *random mass flux* in the positive direction of $\hat{\underset{\sim}{n}}$ is, consequently, $m_\alpha \, \Gamma_{\alpha n}^{(+)} \, (\underset{\sim}{r}, t)$, where m_α is the mass of the type α particle.

5. MOMENTUM FLOW DYAD OR TENSOR

This quantity is defined as the net momentum transported per unit area and time, through some surface element $\hat{\underset{\sim}{n}}$ dS. If we take, in Eq. (3.6), $\chi \, (\underset{\sim}{r}, \underset{\sim}{v}, t)$ as the component of momentum of the particles of type α along some direction specified by the unit vector $\hat{\underset{\sim}{j}}$, that is,

$$\chi = m_\alpha \, \underset{\sim}{v} \cdot \hat{\underset{\sim}{j}} = m_\alpha \, v_j \tag{5.1}$$

we obtain the element $P_{\alpha jn} \, (\underset{\sim}{r}, t)$ of the momentum flow tensor for the particles of type α,

$$P_{\alpha jn} \, (\underset{\sim}{r}, t) = n_\alpha < (m_\alpha \, \underset{\sim}{v} \cdot \hat{\underset{\sim}{j}})(\underset{\sim}{v} \cdot \hat{\underset{\sim}{n}}) >_\alpha$$

$$= \rho_\alpha < v_j \, v_n >_\alpha \tag{5.2}$$

where $\rho_\alpha = n_\alpha \, m_\alpha$ denotes the density of the particles of type α. Thus, the momentum flow element $P_{\alpha jn} \, (\underset{\sim}{r}, t)$ represents the flux of the $j^{\underline{th}}$

component of the momentum of the type α particles through a surface element whose normal is oriented along $\hat{\underset{\sim}{n}}$. Since $\underset{\sim}{v} = \underset{\sim}{c}_\alpha + \underset{\sim}{u}_\alpha$, we obtain

$$P_{\alpha jn} (\underset{\sim}{r}, t) = \rho_\alpha < c_{\alpha j} c_{\alpha n} > + \rho_\alpha u_{\alpha j} u_{\alpha n} \tag{5.3}$$

or, in dyadic form,

$$\underset{\approx}{P}_\alpha (\underset{\sim}{r}, t) = \rho_\alpha < \underset{\sim}{c}_\alpha \underset{\sim}{c}_\alpha > + \rho_\alpha \underset{\sim}{u}_\alpha \underset{\sim}{u}_\alpha \tag{5.4}$$

where we have used the fact that $< \underset{\sim}{u}_\alpha \underset{\sim}{c}_\alpha > = \underset{\sim}{u}_\alpha < \underset{\sim}{c}_\alpha > = 0$.

In a Cartesian coordinate system (x, y, z) the momentum flow dyad has the following form, in terms of its components,

$$\begin{aligned} \underset{\approx}{P}_\alpha = {} & \hat{\underset{\sim}{x}}\hat{\underset{\sim}{x}} P_{\alpha xx} + \hat{\underset{\sim}{x}}\hat{\underset{\sim}{y}} P_{\alpha xy} + \hat{\underset{\sim}{x}}\hat{\underset{\sim}{z}} P_{\alpha xz} \\ & + \hat{\underset{\sim}{y}}\hat{\underset{\sim}{x}} P_{\alpha yx} + \hat{\underset{\sim}{y}}\hat{\underset{\sim}{y}} P_{\alpha yy} + \hat{\underset{\sim}{y}}\hat{\underset{\sim}{z}} P_{\alpha yz} \\ & + \hat{\underset{\sim}{z}}\hat{\underset{\sim}{x}} P_{\alpha zx} + \hat{\underset{\sim}{z}}\hat{\underset{\sim}{y}} P_{\alpha zy} + \hat{\underset{\sim}{z}}\hat{\underset{\sim}{z}} P_{\alpha zz} \end{aligned} \tag{5.5}$$

From the rules of matrix multiplication, the dyad $\underset{\approx}{P}_\alpha$ can be expressed as

$$\underset{\approx}{P}_{\alpha} = (\hat{\underset{\sim}{x}}\ \hat{\underset{\sim}{y}}\ \hat{\underset{\sim}{z}}) \begin{pmatrix} P_{\alpha xx} & P_{\alpha xy} & P_{\alpha xz} \\ P_{\alpha yx} & P_{\alpha yy} & P_{\alpha yz} \\ P_{\alpha zx} & P_{\alpha zy} & P_{\alpha zz} \end{pmatrix} \begin{pmatrix} \hat{\underset{\sim}{x}} \\ \hat{\underset{\sim}{y}} \\ \hat{\underset{\sim}{z}} \end{pmatrix} \tag{5.6}$$

It is usual, however, to omit the pre- and post-multiplicative dyadic signs, such as $\hat{\underset{\sim}{x}}\ \hat{\underset{\sim}{x}}$, etc., and denote the dyad only by the 3 x 3 matrix containing the elements $P_{\alpha ij}$. Thus, $P_{\alpha ij}$ corresponds to the element of the $i^{\underline{th}}$ row and the $j^{\underline{th}}$ column. From Eq. (5.3) it is clear that $P_{\alpha ij} = P_{\alpha ji}$ and, consequently, the 3 x 3 matrix in (5.6) is *symmetric*. Therefore, only six of the components of the momentum flow dyad (or tensor) are independent.

6. PRESSURE DYAD

6.1 - Concept of pressure

The pressure of a gas is usually defined as the force per unit area exerted by the molecules of the gas through collisions with the walls of the containing vessel. This force is equal to the rate of transfer of molecular momentum to the walls of the container. This definition applies also to any surface immersed in the gas as, for example, the surface of a material body.

We may generalize this definition of pressure so that it can be applied to any point inside the gas. To this end, we will define pressure in terms of an imaginary surface element $d\,\underset{\sim}{S} = \underset{\sim}{\bar{n}}\, dS$, inside the gas, moving with its average velocity $\underset{\sim}{u}\,(\underset{\sim}{r}, t)$. The pressure on dS is then defined as the rate of transport of molecular momentum per unit area, that is, the *flux of momentum* across $d\underset{\sim}{S}$ due to the random particle motions. When different species of particles are present, as in a plasma, it is useful to define a (partial) pressure due to the particles of type α, as the flux of momentum transported by the particles of type α as they move back and forth across the surface element $\underset{\sim}{\bar{n}}\, d\underset{\sim}{S}$, moving with the average velocity $\underset{\sim}{u}_\alpha\,(\underset{\sim}{r}, t)$.

In the frame of reference of $d\underset{\sim}{S}$ Eq. (3.11) applies, and taking $\chi\,(\underset{\sim}{r}, \underset{\sim}{v}, t)$ as the $j^{\underline{th}}$ component of momentum of the type α particles, $m_\alpha\, c_{\alpha j}$, we obtain the element $p_{\alpha jn}$ of the pressure tensor,

$$p_{\alpha jn} = \rho_\alpha < c_{\alpha j}\, c_{\alpha n} > \qquad (6.1)$$

The *pressure dyad* is, therefore, given by

$$\underset{\sim}{p}_\alpha = \rho_\alpha < \underset{\sim}{c}_\alpha\, \underset{\sim}{c}_\alpha > \qquad (6.2)$$

From Eq. (5.4) we find the following relation between the pressure dyad $\underset{\approx}{p}_\alpha$ and the momentum flow dyad $\underset{\approx}{P}_\alpha$,

$$\underset{\approx}{p}_\alpha = \underset{\approx}{P}_\alpha - \rho_\alpha\, \underset{\sim}{u}_\alpha\, \underset{\sim}{u}_\alpha \qquad (6.3)$$

They are equal only when the flow velocity $\underset{\sim}{u}_\alpha\,(\underset{\sim}{r}, t)$ vanishes.

6.2 - Force per unit area

Consider now a small element of volume inside the plasma, bounded by the closed surface S, and let $d\underset{\sim}{S} = \hat{\underset{\sim}{n}}\, dS$ be an element of area belonging to S, with the unit vector $\hat{\underset{\sim}{n}}$ normal to the surface element and pointing *outward* (see Fig. 4). The *force per unit area* acting on the element of area $\hat{\underset{\sim}{n}}\, dS$, as the result of the random motion of the particles of type α, is given by

$$-\underset{\approx}{p}_\alpha \cdot \hat{\underset{\sim}{n}} = - \rho_\alpha < \underset{\sim}{c}_\alpha (\underset{\sim}{c}_\alpha \cdot \hat{\underset{\sim}{n}})> \tag{6.4}$$

The reason for the minus sign can be seen as follows. Suppose, for the moment, that all type α particles have the same velocity $\underset{\sim}{c}_\alpha$.

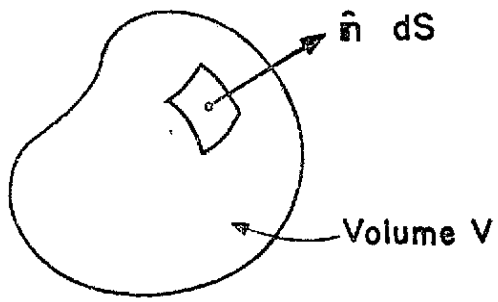

Fig. 4 - Element of volume V bounded by a closed surface S, with the surface element $\hat{\underset{\sim}{n}}\, dS$ pointing outward.

If $\underset{\sim}{c}_\alpha$ forms an angle of less than 90^0 with $\underset{\sim}{\hat{n}}$, then the quantity $n_\alpha\ (\underset{\sim}{c}_\alpha \cdot \underset{\sim}{\hat{n}})$ dS is the number of type α particles *leaving*, per unit time, the volume enclosed by the closed surface S, through $d\underset{\sim}{S}$. The corresponding change (*decrease*) in momentum of the plasma enclosed by the surface S is $-\ n_\alpha\ m_\alpha\ \underset{\sim}{c}_\alpha\ (\underset{\sim}{c}_\alpha \cdot \underset{\sim}{\hat{n}})$ dS,since $(\underset{\sim}{c}_\alpha \cdot \underset{\sim}{\hat{n}})$ is *positive*. On the other hand, if $\underset{\sim}{c}_\alpha$ forms an angle greater than 90^0 with $\underset{\sim}{\hat{n}}$, then $-n_\alpha\ (\underset{\sim}{c}_\alpha \cdot \underset{\sim}{\hat{n}})$ dS represents the number of particles *entering*, per unit time, the bounded volume through $d\underset{\sim}{S}$, and the corresponding change (*increase*) in momentum of the plasma within the closed surface S is again $-n_\alpha\ m_\alpha\ \underset{\sim}{c}_\alpha\ (\underset{\sim}{c}_\alpha \cdot \underset{\sim}{\hat{n}})$ dS, since now $(\underset{\sim}{c}_\alpha \cdot \underset{\sim}{\hat{n}})$ is *negative*.

We conclude, by generalizing this result, that for any arbitrary distribution of individual velocities, the vector quantity

$$-n_\alpha\ m_\alpha < \underset{\sim}{c}_\alpha(\underset{\sim}{c}_\alpha \cdot \underset{\sim}{\hat{n}})> dS = -\underset{\approx}{p}_\alpha \cdot \underset{\sim}{\hat{n}}\ \ dS \tag{6.5}$$

represents the rate of change of momentum of the plasma within the closed surface S, due to the exchange of type α particles through the surface element $\underset{\sim}{\hat{n}}$ dS. Therefore, the force per unit area exerted on an element of area oriented along the unit vector $\underset{\sim}{\hat{n}}$ is $-\underset{\approx}{p}_\alpha \cdot \underset{\sim}{\hat{n}}$. If we take, for example, an element of area along the $\underset{\sim}{\hat{x}}$ direction, that is, $\underset{\sim}{\hat{n}} = \underset{\sim}{\hat{x}}$, we have

$$-\underset{\approx}{p}_\alpha \cdot \underset{\sim}{\hat{x}} = -\underset{\sim}{\hat{x}} p_{\alpha xx}\ \ -\ \ \underset{\sim}{\hat{y}} p_{\alpha yx}\ \ -\ \ \underset{\sim}{\hat{z}} p_{\alpha zx} \tag{6.6}$$

where $p_{\alpha xx}$ is normal to the surface and towards it, just like a hydrostatic pressure, whereas the components $p_{\alpha yx}$ and $p_{\alpha zx}$ are pressures due to shear forces which are tangential to the surface, as indicated in Fig. 5. All other components of $\underset{\sim}{p}_{\alpha}$ are interpreted in an analogous way. Generally, the force per unit area $p_{\alpha jn}$ acts along the negative direction of the axis denoted by the first subscript (j), on a surface whose outward normal is parallel to the axis indicated by the second subscript (n). Alternatively, if the outward normal to the surface is in the negative direction of the axis indicated by the second subscript (n), then the force acts in the same direction as the axis denoted by the first subscript (j).

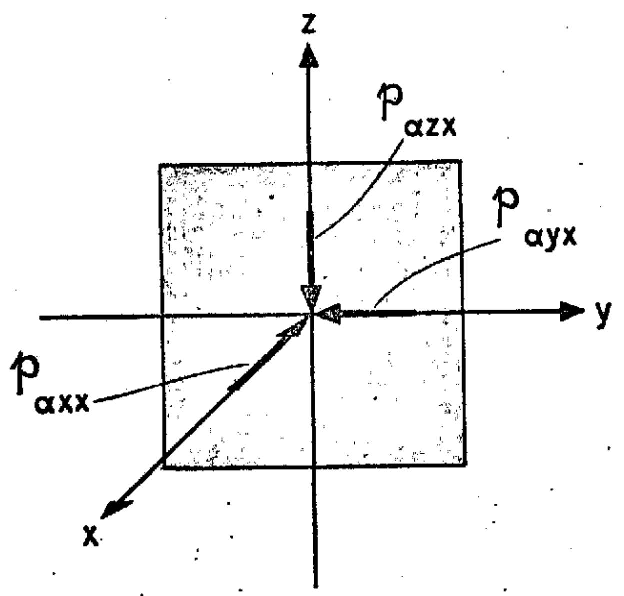

Fig. 5 - Components of the pressure dyad corresponding to the tangential shear stresses, $p_{\alpha yx}$ and $p_{\alpha zx}$, and to the normal stress, $p_{\alpha xx}$, acting on a surface element whose normal is oriented along $\hat{\underset{\sim}{x}}$.

6.3 - Force per unit volume

The force per unit volume inside the plasma, due to the random motion of the particles of type α, can be obtained by integrating Eq. (6.5) over the closed surface S bounding the volume element V, dividing the result by V, and then taking the limit as V tends to zero. This procedure is just the definition of the divergence,

$$-\nabla \cdot \mathbf{p}_\alpha = - \lim_{V \to 0} \left[\frac{1}{V} \oint \mathbf{p}_\alpha \cdot \hat{\mathbf{n}} \, dS \right]$$

and, from Gauss' divergence theorem,

$$- \oint \mathbf{p}_\alpha \cdot \hat{\mathbf{n}} \, dS = - \int \nabla \cdot \mathbf{p}_\alpha \, d^3r$$

We conclude, therefore, that the negative divergence of the kinetic pressure dyad $(-\nabla \cdot \mathbf{p}_\alpha)$ is the force exerted on a unit volume of the plasma due to the random motion of the particles of type α, and $- \mathbf{p}_\alpha \cdot \hat{\mathbf{n}}$ is the force acting on a unit area of a surface normal to the unit vector $\hat{\mathbf{n}}$.

6.4 - Scalar pressure and absolute temperature

An important macroscopic variable is the *scalar pressure*, or *mean hydrostatic pressure*, p_α. It is defined as one third the trace of the pressure tensor,

$$p_\alpha = \frac{1}{3} \sum_{i,j} p_{\alpha ij} \delta_{ij} = \frac{1}{3} \sum_i p_{\alpha ii}$$

$$= \frac{1}{3} (p_{\alpha xx} + p_{\alpha yy} + p_{\alpha zz}) \tag{6.9}$$

where δ_{ij} is the Kronecker delta, such that $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$. The pressure elements $p_{\alpha ii}$, with $i = x, y, z$, are just the hydrostatic pressures *normal* to the surfaces described by i = constant. Using Eq. (6.1),

$$p_\alpha = \frac{1}{3} \rho_\alpha < c_{\alpha x}^2 + c_{\alpha y}^2 + c_{\alpha z}^2 > \tag{6.10}$$

Since $c^2 = c_{\alpha x}^2 + c_{\alpha y}^2 + c_{\alpha z}^2$, we have

$$p_\alpha = \frac{1}{3} \rho_\alpha < c_\alpha^2 > \tag{6.11}$$

Another important parameter for a macroscopic description of a gas is its temperature. The *absolute temperature* T_α, for the type α particles, is a measure of the *mean kinetic energy* of the random motion of the particles of type α. According to the thermodynamic definition of absolute temperature, there is a thermal energy of $k\, T_\alpha / 2$ associated with each translational degree of freedom, so that

$$\frac{1}{2}\, k\, T_\alpha = \frac{1}{2}\, m_\alpha < c_{\alpha i}^2 > \qquad (6.12)$$

where k is Boltzmann's constant.

When the distribution of random velocites is isotropic, as is the case of the Maxwell-Boltzmann distribution function (to be considered in the next chapter) which characterizes the state of *thermal equilibriam* of a gas, we have $c_{\alpha x}^2 = c_{\alpha y}^2 = c_{\alpha z}^2 = c_\alpha^2 / 3$, and therefore,

$$p_\alpha = p_{\alpha xx} = p_{\alpha yy} = p_{\alpha zz} = \rho_\alpha < c_{\alpha i}^2 > \qquad (6.13)$$

Combining (6.13) and (6.12), gives

$$p_\alpha = n_\alpha\, k\, T_\alpha \qquad (6.14)$$

which is the equation of state of an *ideal gas*. For the Maxwell--Boltzmann distribution function the non-diagonal elements of the kinetic pressure dyad $\underset{\sim}{p}_\alpha$ are all zero and the pressure dyad reduces to

$$\underset{\sim}{p}_\alpha = (\hat{\underset{\sim}{x}}\hat{\underset{\sim}{x}} + \hat{\underset{\sim}{y}}\hat{\underset{\sim}{y}} + \hat{\underset{\sim}{z}}\hat{\underset{\sim}{z}}) \, p_\alpha = \underset{\approx}{1} \, p_\alpha \tag{6.15}$$

where $\underset{\approx}{1}$ stands for the *unit dyad*, which in matrix form is

$$\underset{\approx}{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.16}$$

In this case the negative divergence of the pressure dyad becomes

$$\begin{aligned} - \underset{\sim}{\nabla} \cdot \underset{\sim}{p}_\alpha &= - \left(\hat{\underset{\sim}{x}} \frac{\partial}{\partial x} p_\alpha + \hat{\underset{\sim}{y}} \frac{\partial}{\partial y} p_\alpha + \hat{\underset{\sim}{z}} \frac{\partial}{\partial z} p_\alpha \right) \\ &= - \underset{\sim}{\nabla} p_\alpha \end{aligned} \tag{6.17}$$

Thus, for an isotropic velocity distribution, the force per unit volume, due to the random variations of the peculiar velocities, is given by the negative gradient of the scalar pressure.

In some problems, a simplification of practical interest for the general form of the kinetic pressure dyad consists in taking

$$\underset{\approx}{p}_\alpha = \underset{\sim\sim}{\hat{x}\hat{x}}\, p_{\alpha xx} + \underset{\sim\sim}{\hat{y}\hat{y}}\, p_{\alpha yy} + \underset{\sim\sim}{\hat{z}\hat{z}}\, p_{\alpha zz} \tag{6.18}$$

or, in matrix form,

$$\underset{\approx}{p}_\alpha = \begin{pmatrix} p_{\alpha xx} & 0 & 0 \\ 0 & p_{\alpha yy} & 0 \\ 0 & 0 & p_{\alpha zz} \end{pmatrix} \tag{6.19}$$

where the diagonal elements are different from one another but all non-diagonal elements vanish. This corresponds to an anisotropy of the peculiar velocities and the absence of shear forces and viscous drag. The effects of viscosity and shear stresses are incorporated in the non-diagonal elements of the pressure dyad. Usually, the effects of viscosity are relatively unimportant for most plasmas and the non-diagonal elements of $\underset{\approx}{p}_\alpha$ can, in many cases, be neglected. In this anisotropic case, a different absolute temperature can be defined for each direction in space, according to (6.12).

7. HEAT FLOW VECTOR

The component of the heat flow vector, $q_{\alpha n}$, is defined as the flux of *random* or *thermal energy* across a surface whose normal points in the direction of the unit vector $\hat{\underset{\sim}{n}}$. Taking χ $(\underset{\sim}{r}, \underset{\sim}{v}, t)$, in Eq. (3.11), as the kinetic energy of random motion of the particles of type α, that is, $\chi = m_\alpha c_\alpha^2 / 2$, we obtain for the component of the heat flow vector along $\hat{\underset{\sim}{n}}$,

$$q_{\alpha n} = \underset{\sim}{q}_\alpha \cdot \hat{\underset{\sim}{n}} = \frac{1}{2} \rho_\alpha < c_\alpha^2 \underset{\sim}{c}_\alpha \cdot \hat{\underset{\sim}{n}} > \qquad (7.1)$$

The *heat flow vector* is, therefore, given by

$$\underset{\sim}{q}_\alpha = \frac{1}{2} \rho_\alpha < c_\alpha^2 \underset{\sim}{c}_\alpha > \qquad (7.2)$$

8. HEAT FLOW TRIAD

It is convenient, at this point, to introduce a *triad* of *thermal energy flux*, defined by

$$\underset{\approx}{Q}_\alpha = \rho_\alpha < \underset{\sim}{c}_\alpha \underset{\sim}{c}_\alpha \underset{\sim}{c}_\alpha > \qquad (8.1)$$

Its components are, explicitly,

$$Q_{\alpha ijk} = \rho_\alpha < c_{\alpha i} c_{\alpha j} c_{\alpha k} > \qquad (8.2)$$

Using Cartesian coordinates, the thermal energy flux triad can be written in the form

$$\underset{\approx}{Q}_{\alpha} = \underset{\approx}{Q}_{\alpha x}\, \hat{\underset{\sim}{x}} + \underset{\approx}{Q}_{\alpha y}\, \hat{\underset{\sim}{y}} + \underset{\approx}{Q}_{\alpha z}\, \hat{\underset{\sim}{z}} \tag{8.3}$$

where each of the *dyads* $\underset{\approx}{Q}_{\alpha n}$, with n = x, y, z, can be expressed in matrix form as

$$\underset{\approx}{Q}_{\alpha n} = \begin{pmatrix} Q_{\alpha xxn} & Q_{\alpha xyn} & Q_{\alpha xzn} \\ Q_{\alpha yxn} & Q_{\alpha yyn} & Q_{\alpha yzn} \\ Q_{\alpha zxn} & Q_{\alpha zyn} & Q_{\alpha zzn} \end{pmatrix} \tag{8.4}$$

To obtain a relation between $\underset{\sim}{q}_{\alpha}$ and $\underset{\approx}{Q}_{\alpha}$, note that Eq. (7.1) can be written as

$$q_{\alpha n} = \frac{1}{2} \left(\rho_{\alpha} < c_{\alpha x}^{2}\, c_{\alpha n} > + \rho_{\alpha} < c_{\alpha y}^{2}\, c_{\alpha n} > + \rho_{\alpha} < c_{\alpha z}^{2}\, c_{\alpha n} > \right) \tag{8.5}$$

and comparing this equation with (8.2), we see that $q_{\alpha n}$ becomes

$$q_{\alpha n} = \frac{1}{2} \left(Q_{\alpha xxn} + Q_{\alpha yyn} + Q_{\alpha zzn} \right) \tag{8.6}$$

9. TOTAL ENERGY FLUX TRIAD

In analogy with the definition of the heat flow triad $\underset{\approx}{Q}$, consider now the quantity

$$E_{\alpha ijk}(\underset{\sim}{r}, t) = \rho_\alpha < v_i\, v_j\, v_k >_\alpha \tag{9.1}$$

which may be called the *total energy flux triad.* This quantity can be considered as the sum of three parts. Substituting $v_i = u_{\alpha i} + c_{\alpha i}$ for each component in (9.1), and expanding,

$$\begin{aligned}\rho_\alpha < v_i\, v_j\, v_k >_\alpha = \rho_\alpha < & c_{\alpha i}\, c_{\alpha j}\, c_{\alpha k} + u_{\alpha i}\, c_{\alpha j}\, c_{\alpha k} + u_{\alpha j}\, c_{\alpha k}\, c_{\alpha i} + \\ & + u_{\alpha k}\, c_{\alpha i}\, c_{\alpha j} + u_{\alpha i}\, u_{\alpha j}\, c_{\alpha k} + u_{\alpha j}\, u_{\alpha k}\, c_{\alpha i} + \\ & + u_{\alpha k}\, u_{\alpha i}\, c_{\alpha j} + u_{\alpha i}\, u_{\alpha j}\, u_{\alpha k} >\end{aligned} \tag{9.2}$$

Noting that $< u_{\alpha i} > = u_{\alpha i}$ and $< c_{\alpha i} > = 0$, and using Eqs. (8.2) and (6.1), we obtain

$$\rho_\alpha < v_i\, v_j\, v_k >_\alpha = \rho_\alpha\, u_{\alpha i}\, u_{\alpha j}\, u_{\alpha k} + (\underset{\sim}{u}\, ,\, \underset{\approx}{p}_\alpha)_{ijk} + Q_{\alpha ijk} \tag{9.3}$$

where the following notation was employed

$$(\underset{\sim}{u}_\alpha, \underset{\approx}{p}_\alpha)_{ijk} = u_{\alpha i}\, p_{\alpha jk} + u_{\alpha j}\, p_{\alpha ki} + u_{\alpha k}\, p_{\alpha ij} \tag{9.4}$$

Therefore, we can write Eq. (9.3) in triadic form as

$$\rho_\alpha < \underset{\sim}{v}\,\underset{\sim}{v}\,\underset{\sim}{v} >_\alpha \; = \; \rho_\alpha\, \underset{\sim}{u}_\alpha\, \underset{\sim}{u}_\alpha\, \underset{\sim}{u}_\alpha \; + \; (\underset{\sim}{u}_\alpha,\, \underset{\approx}{p}_\alpha) \; + \; \underset{\approx}{Q}_\alpha \tag{9.5}$$

The total energy flux triad $\rho_\alpha < \underset{\sim}{v}\,\underset{\sim}{v}\,\underset{\sim}{v} >_\alpha$ can, therefore, be considered as the sum of the energy flux density transported by the *convective* particle motions, represented by the first two terms in the right-hand side of (9.5), and the *thermal* energy flux $\underset{\approx}{Q}_\alpha$ due to the random thermal motions of the particles of type α.

The physical interpretation of the heat flow triad $\underset{\approx}{Q}_\alpha$ is, in some sense, analogous to the physical interpretation of the heat flow vector $\underset{\sim}{q}_\alpha$. For this purpose, consider the quantity

$$\frac{1}{2}\, \rho_\alpha < v^2\, \underset{\sim}{v} >_\alpha \tag{9.6}$$

which represents the *average energy flux* transported by the particles of type α. This quantity can be written as the sum of three terms. Substituting $\underset{\sim}{v} = \underset{\sim}{c}_\alpha + \underset{\sim}{u}_\alpha$ in expression (9.6) and expanding,

$$\frac{1}{2}\rho_\alpha < v^2\, \underset{\sim}{v} >_\alpha \; = \; \frac{1}{2}\rho_\alpha < u_\alpha^2\, \underset{\sim}{u}_\alpha + 2\,(\underset{\sim}{u}_\alpha \cdot \underset{\sim}{c}_\alpha)\, \underset{\sim}{u}_\alpha + c_\alpha^2\, \underset{\sim}{u}_\alpha + u_\alpha^2\, \underset{\sim}{c}_\alpha + $$

$$+ 2\,(\underset{\sim}{u}_\alpha \cdot \underset{\sim}{c}_\alpha)\, \underset{\sim}{c}_\alpha + c_\alpha^2\, \underset{\sim}{c}_\alpha > \tag{9.7}$$

and since $< \underset{\sim}{c}_\alpha > = 0$ and $< \underset{\sim}{u}_\alpha > = \underset{\sim}{u}_\alpha$, we obtain

$$\frac{1}{2}\rho_\alpha < v^2 \underset{\sim}{v} >_\alpha = \frac{1}{2}\rho_\alpha (u_\alpha^2 + < c_\alpha^2 >) \underset{\sim}{u}_\alpha + \rho_\alpha \underset{\sim}{u}_\alpha \cdot < \underset{\sim}{c}_\alpha \underset{\sim}{c}_\alpha > + $$

$$+ \frac{1}{2}\rho_\alpha < c_\alpha^2 \underset{\sim}{c}_\alpha > \tag{9.8}$$

If we now use Eqs. (6.2) and (7.2), which define $\underset{\approx}{p}_\alpha$ and $\underset{\sim}{q}_\alpha$, respectively, we obtain the identity

$$\frac{1}{2}\rho_\alpha < v^2 \underset{\sim}{v} >_\alpha = W_\alpha \underset{\sim}{u}_\alpha + \underset{\sim}{u}_\alpha \cdot \underset{\approx}{p}_\alpha + \underset{\sim}{q}_\alpha \tag{9.9}$$

where W_α is the *mean kinetic energy density* of the type α particles, defined by

$$W_\alpha = \frac{1}{2}\rho_\alpha u_\alpha^2 + \frac{1}{2}\rho_\alpha < c_\alpha^2 > \tag{9.10}$$

Eq. (9.9) is written in a form analogous to Eq. (9.5). It shows that the rate of transport per unit area (flux) of the average energy of the type α particles, $\rho_\alpha < v^2 \underset{\sim}{v} >_\alpha / 2$, can be separated into three parts: the first term in the right-hand side of (9.9) represents the flux of the mean kinetic energy transported *convectively*, the second term is the rate of work per unit area done by the kinetic pressure dyad, and the third one is the random thermal energy flux transported by the particles of type α due to their random *thermal* motions. It is instructive to note that in a frame of reference moving with the average velocity $\underset{\sim}{u}_\alpha$, the particle velocities become identical to their random velocities, that is $\underset{\sim}{v} = \underset{\sim}{c}_\alpha$, so that Eq. (9.9) reduces to Eq. (7.2) which defines the thermal energy flux vector $\underset{\sim}{q}_\alpha$. When the thermal velocities $\underset{\sim}{c}_\alpha$ are distributed uniformly in all directions, that is isotropically, it turns out that $\underset{\sim}{q}_\alpha = 0$ (since the integrand in $< c_\alpha^2 \underset{\sim}{c}_\alpha >$ is an odd function of $\underset{\sim}{c}_\alpha$). Consequently, $\underset{\sim}{q}_\alpha$ can be considered as a partial measure of the anisotropies in the distribution of the thermal velocities. The thermal energy flux triad $\underset{\approx}{Q}_\alpha$ considerably extends the concept of the heat flux vector and in this sense can be considered as a complete measure of the anisotropies in the distribution of the thermal velocities of the particles.

10. HIGHER MOMENTS OF THE DISTRIBUTION FUNCTION

The first four moments of the distribution function $f_\alpha(\underset{\sim}{r}, \underset{\sim}{v}, t)$ are related to the number density n_α, the average velocity $\underset{\sim}{u}_\alpha$, the momentum flow dyad $\underset{\approx}{P}_\alpha$ and the energy flow triad $\underset{\approx}{E}_\alpha$. For reference, it is convenient to collect them here,

$$n_\alpha(\underset{\sim}{r}\ t) = \int_V f_\alpha(\underset{\sim}{r}, \underset{\sim}{v}, t)\, d^3v \tag{10.1}$$

$$u_{\alpha i} (\underset{\sim}{r}, t) = < v_i >_\alpha = \frac{1}{n_\alpha (\underset{\sim}{r}, t)} \int_V v_i f_\alpha (\underset{\sim}{r}, \underset{\sim}{v}, t) d^3v \qquad (10.2)$$

$$P_{\alpha ij} (\underset{\sim}{r}, t) = \rho_\alpha < v_i v_j >_\alpha = m_\alpha \int_V v_i v_j f_\alpha (\underset{\sim}{r}, \underset{\sim}{v}, t) d^3v \qquad (10.3)$$

$$E_{\alpha ijk} (\underset{\sim}{r}, t) = \rho_\alpha < v_i v_j v_k >_\alpha = m_\alpha \int_V v_i v_j v_k f_\alpha (\underset{\sim}{r}, \underset{\sim}{v}, t) d^3v \qquad (10.4)$$

When the average velocity $\underset{\sim}{u}_\alpha$ vanishes, we have $\underset{\sim}{v} = \underset{\sim}{c}_\alpha$, $\underset{\approx}{P}_\alpha$ becomes the same as the pressure dyad $\underset{\approx}{p}_\alpha$, and $\underset{\approx}{E}_\alpha$ becomes the same as the thermal energy flux triad $\underset{\approx}{Q}_\alpha$.

As a formal extension of these definitions we can, whenever necessary, consider higher moments of the distribution function. The moment of order N can be defined by the expression

$$M^{(N)}_{ij\ldots k} (\underset{\sim}{r}, t) = \int_V \underbrace{v_i v_j \ldots v_k}_{N \text{ times}} f_\alpha (\underset{\sim}{r}, \underset{\sim}{v}, t) d^3v \qquad (10.5)$$

PROBLEMS

6.1 - Consider a system of particles characterized by the distribution function of Problem 5.1.

(a) Show that the absolute temperature of the system is given by $T = m\, v_0^2/3k$, where m is the mass of each particle and k is Boltzmann's constant.

(b) Obtain the following expression for the pressure dyad

$$\underset{\approx}{p} = \frac{1}{3} \rho\, v_0^2\, \underset{\approx}{1}$$

where $\rho = n\, m$ and $\underset{\approx}{1}$ is the unit dyad.

(c) Verify that the heat flow vector $\underset{\sim}{q} = 0$.

6.2 - Suppose that the peculiar (random) velocities of the electrons in a given plasma, satisfy the following (modified Maxwell-Boltzmann) distribution function

$$f(\underset{\sim}{c}) = n_0 \left(\frac{m}{2\pi k T_\perp}\right) \left(\frac{m}{2\pi k T_\parallel}\right)^{1/2} \exp\left[- \frac{m}{2k} \left(\frac{c_x^2 + c_y^2}{T_\perp} + \frac{c_z^2}{T_\parallel}\right)\right]$$

(a) Verify that the electron number density is given by n_0.

(b) Show that the kinetic pressure dyad is given by

$$\underset{\approx}{p} = n_0 \, k \, [T_\perp (\hat{\underset{\sim}{x}} \hat{\underset{\sim}{x}} + \hat{\underset{\sim}{y}} \hat{\underset{\sim}{y}}) + T_\parallel \, \hat{\underset{\sim}{z}} \hat{\underset{\sim}{z}}]$$

which indicates the presence of an anisotropy in the z-direction.

(c) Calculate the heat flow vector $\underset{\sim}{q}$.

(d) Show that $m<v_\parallel^2>/2 = kT_\parallel/2$ and $m<v_\perp^2>/2 = kT_\perp$

6.3 - For the loss-cone distribution function of Problem 5.3, show that $m<v_\parallel^2>/2 = m\alpha_\parallel^2/4$ and $m<v_\perp^2>/2 = m\alpha_\perp^2$.
Compare these results with those of Problem 6.2(d), and provide physical arguments to justify the difference in the perpendicular part of the thermal energy.

6.4 - Convince yourself that there are only *ten* independent elements in the thermal energy flux triad $\underset{\approx}{Q}$. Note that $Q_{ijk} =$
$= nm <c_i \, c_j \, c_k>$ is symmetric under the the interchange of any two of its three indices.

6.5 - A plasma is made up of a mixture of various particle species, the type α species having mass m_α, number density n_α, average velocity $\underset{\sim}{u}_\alpha = \langle \underset{\sim}{v} \rangle_\alpha$, random velocity $\underset{\sim}{c}_\alpha = \underset{\sim}{v} - \underset{\sim}{u}_\alpha$, temperature $T_\alpha = (m_\alpha/3k) \langle c_\alpha^2 \rangle$, pressure dyad $\underset{\approx}{p}_\alpha = n_\alpha m_\alpha \langle \underset{\sim}{c}_\alpha \underset{\sim}{c}_\alpha \rangle$, and heat flow vector $\underset{\sim}{q}_\alpha = (n_\alpha m_\alpha/2) \langle c_\alpha^2 \underset{\sim}{c}_\alpha \rangle$. Similar quantities can be defined for the plasma *as a whole*, for example:

total number density $n_o = \sum_\alpha n_\alpha$

average mass $m_o = \dfrac{1}{n_o} \sum_\alpha n_\alpha m_\alpha$

average velocity $\underset{\sim}{u}_o = \dfrac{1}{n_o m_o} \sum_\alpha n_\alpha m_\alpha \underset{\sim}{u}_\alpha$

We can also define an *alternative* random velocity for the type α species as $\underset{\sim}{c}_{\alpha o} = \underset{\sim}{v} - \underset{\sim}{u}_o$, as well as an alternative temperature $T_{\alpha o} = (m_\alpha/3k) \langle c_{\alpha o}^2 \rangle$, pressure dyad $\underset{\approx}{p}_{\alpha o} = n_\alpha m_\alpha \langle \underset{\sim}{c}_{\alpha o} \underset{\sim}{c}_{\alpha o} \rangle$, and heat flow vector $\underset{\sim}{q}_{\alpha o} = (n_\alpha m_\alpha/2) \langle c_{\alpha o}^2 \underset{\sim}{c}_{\alpha o} \rangle$.

(a) Show that, for the plasma as a whole, the *total* pressure dyad is given by

$$\underset{\approx}{p}_o = \sum_\alpha (\underset{\approx}{p}_\alpha + n_\alpha m_\alpha \underset{\sim}{w}_\alpha \underset{\sim}{w}_\alpha)$$

and the *total* scalar pressure by

$$p_o = \sum_\alpha (p_\alpha + \frac{1}{3} n_\alpha m_\alpha w_\alpha^2)$$

where $\underset{\sim}{w}_\alpha = \underset{\sim}{u}_\alpha - \underset{\sim}{u}_o$ is the diffusion velocity.

(b) Assuming that $\underset{\sim}{c}_\alpha$ is isotropic, that is, $<c_{\alpha i}^2> = <c_\alpha^2>/3$, for $i = x, y, z$, show that the *total* heat flow vector is given by

$$\underset{\sim}{q}_0 = \sum_\alpha \left(\underset{\sim}{q}_\alpha + \frac{5}{2} p_\alpha \underset{\sim}{w}_\alpha + \frac{1}{2} n_\alpha m_\alpha w_\alpha^2 \underset{\sim}{w}_\alpha\right)$$

(c) If an *average* temperature T_0, for the plasma as a whole, is defined by requiring that $p_0 = n_0 k T_0$, show that

$$T_0 = \frac{1}{n_0} \sum_\alpha n_\alpha \left(T_\alpha + \frac{m_\alpha w_\alpha^2}{3k}\right)$$

(d) Verify that

$$\frac{3}{2} n_0 k T_0 = \sum_\alpha \frac{1}{2} n_\alpha m_\alpha <c_{\alpha 0}^2>$$

6.6 - Consider an infinitesimal element of volume $d^3r = dx\, dy\, dz$ in a gas of number density n.

(a) Show that the time rate of increase of momentum in d^3r, as a result of particles of mass m entering d^3r with average velocity $\underset{\sim}{u}$, is given by $-\underset{\sim}{\nabla}.(n\, m\, \underset{\sim}{u}\, \underset{\sim}{u})\, d^3r$.

(b) If the infinitesimal volume element d^3r moves with the average particle velocity $\underset{\sim}{u}$, show that, because of the work done by the kinetic pressure dyad $\underset{\approx}{p}$, the energy of the particles inside d^3r increases at a time rate given by $- \underset{\sim}{\nabla} \cdot (\underset{\sim}{u} \cdot \underset{\approx}{p})\, d^3r$.

(c) Verify, by expansion, that

$$(\underset{\approx}{p} \cdot \hat{n}) \cdot \underset{\sim}{u} = (\underset{\sim}{u} \cdot \underset{\approx}{p}) \cdot \hat{n}$$

where $\hat{n}$ denotes an outward unit vector, normal to the surface bounding the volume element.

6.7 - Consider Eq. (5.6.4), which is the solution of the Boltzmann equation with the relaxation model for the collision term, in the absence of external forces and spatial gradients, and when $f_{0\alpha}$ and τ are time-independent. Show that, according to this result, we have

$$G_\alpha(t) = G_{0\alpha} + [\, G_\alpha(0) - G_{0\alpha}\,] \quad \exp(-t/\tau)$$

where

$$G_\alpha(t) = \int_v f_\alpha\, \chi\, d^3v = n_\alpha < \chi >_\alpha$$

$$G_{0\alpha} = \int_v f_{0\alpha}\, \chi\, d^3v = n_\alpha < \chi >_{0\alpha}$$

Thus, according to the relaxation model for the collision term, every average value $< \chi >_\alpha$ approaches equilibrium with the same relaxation time τ.