1. Classification *INPE-COM.4/RPE* *C.D.U.: 533.9*	2. Period	4. Distribution Criterion internal [ ] external [X]
3. Key Words (selected by the author) *MOTION OF CHARGED PARTICLES* *CHARGED PARTICLE ORBIT THEORY*		
5. Report No. *INPE-1506-RPE/046*	6. Date *July, 1979*	7. Revised by *W. D. Gonzalez Alarcon*
8. Title and Sub-title *MOTION OF CHARGED PARTICLES IN CONSTANT AND UNIFORM ELECTROMAGNETIC FIELDS*		9. Authorized by *Nelson de Jesus Parada* *Director*
10. Sector *DCE/DGA/GIO* Code *30371*		11. No. of Copies *13*
12. Authorship *J. A. Bittencourt*		14. No. of Pages *52*
13. Signature of first author Bittencourt		15. Price

16. Summary/Notes

This is the second chapter, in a series of twenty two, written as an introduction to the fundamentals of plasma physics. This chapter analyses the motion of individual charged particles in the presence of electric and magnetic fields, which are constant in time and uniform in space. The fields are assumed to be prescribed and, therefore, are not affected by the presence of the charged particles. The problem is analyzed in steps, using the nonrelativistic equations of motion, incorporating separately in the theory the effects associated with only the electric field, or only the magnetic field, and the effects which are due to both electric and magnetic fields.

17. Remarks

INDEX

CHAPTER 2

MOTION OF CHARGED PARTICLES IN CONSTANT AND UNIFORM ELECTROMAGNETIC FIELDS

LIST OF FIGURES

CHAPTER 2

MOTION OF CHARGED PARTICLES IN CONSTANT AND UNIFORM ELECTROMAGNETIC FIELDS

1. INTRODUCTION

In this and in the following two chapters, we investigate the motion of charged particles in the presence of electric and magnetic fields known as functions of position and time. Thus, the electric and magnetic fields are assumed to be prescribed and not affected by the charged particles. This chapter, in particular, considers the fields to be constant in time and uniform in space. This subject is considered in some detail since many of the more complex situations, considered in chapters 3 and 4, can be treated as perturbations to this problem.

The study of the motion of charged particles in specified fields is important since it provides good physical insight for understanding some dynamical processes in plasmas. It also permits to obtain information on some macroscopic phenomena which are due to the collective behavior of a large number of particles. Not all of the components of the detailed microscopic motion of the particles contribute to macroscopic effects. It is possible, however, to isolate the components of the individual motion that contribute to the collective behavior of the plasma. Nevertheless, it should be mentioned that the macroscopic parameters can be obtained much more easily and conveniently from the macroscopic equations presented in chapters 8 and 9.

The equation of motion for a particle of charge q, under the action of the Lorentz force $\underset{\sim}{F}$ due to electric ($\underset{\sim}{E}$) and magnetic ($\underset{\sim}{B}$) fields, can be written as

$$\frac{d\underset{\sim}{p}}{dt} = \underset{\sim}{F} = q(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) \tag{1.1}$$

where $\underset{\sim}{p}$ represents the momentum of the particle and $\underset{\sim}{v}$ its velocity.

Eq.(1.1) is relativistically correct if we take

$$\underset{\sim}{p} = \gamma m \underset{\sim}{v} \tag{1.2}$$

where m is the rest mass of the particle and γ is the so-called *Lorentz factor* defined by

$$\gamma = (1 - v^2/c^2)^{-1/2} \tag{1.3}$$

where c is the speed of light in vacuum. In the relativistic case Eq. (1.1) can also be written in the form

$$\gamma m \frac{d\underset{\sim}{v}}{dt} + q \frac{\underset{\sim}{v}}{c^2} (\underset{\sim}{v} \cdot \underset{\sim}{E}) = q(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) \tag{1.4}$$

using the fact that the time rate of change of the total relativistic energy ($U = \gamma mc^2$) is given by $dU/dt = q(\mathbf{v} \cdot \mathbf{E})$ and that $d\mathbf{p}/dt = d(U\mathbf{v}/c^2)/dt$.

In many situations of practical interest, however, the term v^2/c^2 is negligible compared to unity. For $v^2/c^2 \ll 1$ we have $\gamma \simeq 1$ and m can be considered constant, independent of the particle's velocity, so that Eq. (1.4) reduces to the following non-relativistic expression

$$m \frac{d\mathbf{v}}{dt} = q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad (1.5)$$

If the velocity obtained using this equation does not satisfy the condition $v^2 \ll c^2$, then the corresponding result is not valid and the relativistic expression (1.4) must be used instead of (1.5). Therefore, (1.5) applies only to charged particles moving with velocities much smaller than the velocity of light. Relativistic effects become important only for highly energetic particles (a 1 MeV proton, for instance, has a velocity of 1.4×10^7 m/s, with $v^2/c^2 = 0.002$). For the situations to be considered here it is assumed that the restriction $v^2 \ll c^2$, implicit in Eq. (1.5), is not violated. Also, all radiation effects are neglected.

2. ENERGY CONSERVATION

Consider a particle of charge q and mass m moving in a *magnetostatic* field $\underset{\sim}{B}$ with velocity $\underset{\sim}{v}$. In the absence of an electric field ($\underset{\sim}{E} = 0$) the equation of motion (1.5) reduces to

$$m \frac{d\underset{\sim}{v}}{dt} = q(\underset{\sim}{v} \times \underset{\sim}{B}) \tag{2.1}$$

Since the magnetic force is perpendicular to the velocity $\underset{\sim}{v}$ it does no work on the particle. Taking the dot product of (2.1) with $\underset{\sim}{v}$ and noting that $(\underset{\sim}{v} \times \underset{\sim}{B}) \cdot \underset{\sim}{v} = 0$ for any vector $\underset{\sim}{v}$, we obtain

$$m \left(\frac{d\underset{\sim}{v}}{dt} \right) \cdot \underset{\sim}{v} = \frac{d}{dt} \left(\frac{1}{2} mv^2 \right) = 0 \tag{2.2}$$

which shows that the particle's kinetic energy ($mv^2/2$) and the magnitude of its velocity (speed v) are both constants. Therefore, a *static* magnetic field does not change the particle's kinetic energy. This result is valid whatever the spatial dependence of the magnetic flux density $\underset{\sim}{B}$. However, if $\underset{\sim}{B}$ varies with time then, according to Maxwell equations, an electric field such that $\underset{\sim}{\nabla} \times \underset{\sim}{E} = -\partial \underset{\sim}{B}/\partial t$ is also present which does work on the particle changing its kinetic energy.

When both *magnetostatic* and *electrostatic* fields are present we obtain from (1.5)

$$\frac{d}{dt}\left(\frac{1}{2}\,mv^2\right) = q\underset{\sim}{E}\cdot\underset{\sim}{v} \tag{2.3}$$

Since $\underset{\sim}{\nabla}\times\underset{\sim}{E} = 0$, we can express the electrostatic field in terms of the electrostatic potential ϕ,

$$\underset{\sim}{E} = -\,\underset{\sim}{\nabla}\phi \tag{2.4}$$

to obtain

$$\frac{d}{dt}\left(\frac{1}{2}\,mv^2\right) = -\,q(\underset{\sim}{\nabla}\phi)\cdot\underset{\sim}{v}$$

$$= -\,q(\underset{\sim}{\nabla}\phi)\cdot\frac{d\underset{\sim}{r}}{dt}$$

$$= -\,q\,\frac{d\phi}{dt} \tag{2.5}$$

This result can be written as

$$\frac{d}{dt}\left(\frac{1}{2}\,mv^2 + q\phi\right) = 0 \tag{2.6}$$

which shows that the sum of the kinetic energy and the electric potential energy of the particle remains constant in the presence of static electromagnetic fields. The electric potential ϕ can be considered as the potential energy per unit charge.

When the fields are time-dependent we have $\nabla \times E \neq 0$ and E is not the gradient of a scalar function. But since $\nabla \cdot B = 0$, we can define a magnetic vector potential A by $B = \nabla \times A$, and write Eq.(1.5.2) as

$$\nabla \times E + \frac{\partial B}{\partial t} = \nabla \times E + \frac{\partial}{\partial t}(\nabla \times A) = \nabla \times \left(E + \frac{\partial A}{\partial t} \right) = 0 \qquad (2.7)$$

Hence, we can express the electric field in the form

$$E = -\nabla\phi - \frac{\partial A}{\partial t} \qquad (2.8)$$

In this case the system is not conservative in the usual sense and there is no integral of the energy, but the analysis can be performed using a Lagrangian function L for a charged particle in electromagnetic fields, defined by

$$L = \frac{1}{2} m v^2 - U \qquad (2.9)$$

where U is a velocity-dependent potential energy given by

$$U = q(\phi - \underset{\sim}{v} \cdot \underset{\sim}{A}) \tag{2.10}$$

The energy considerations presented in this section assume that the energy of the particle changes only as a result of the work done by the fields. This assumption is not strictly correct since every charged particle when accelerated irradiates energy in the form of electromagnetic waves. For the situations to be considered here this effect is usually small and can be neglected.

3. UNIFORM ELECTROSTATIC FIELD

According to (1.1) the motion of a charged particle in an electric field obeys the following differential equation

$$\frac{d\underset{\sim}{p}}{dt} = q\,\underset{\sim}{E} \tag{3.1}$$

For a constant $\underset{\sim}{E}$ field, Eq. (3.1) can be integrated directly giving

$$\underset{\sim}{p}(t) = q\,\underset{\sim}{E}\,t + \underset{\sim}{p}_o \tag{3.2}$$

where $\underset{\sim}{p}_o = \underset{\sim}{p}(t = 0)$ denotes the initial momentum of the particle. Using the nonrelativistic expression

$$\underset{\sim}{p} = m\underset{\sim}{v} = m \frac{d\underset{\sim}{r}}{dt} \tag{3.3}$$

and performing a second integration in (3.2) we obtain the following expression for the particle's position as a function of time

$$\underset{\sim}{r}(t) = \frac{q\underset{\sim}{E}}{2m} t^2 + \underset{\sim}{v}_o t + \underset{\sim}{r}_o \tag{3.4}$$

where $\underset{\sim}{r}_o$ denotes the initial position of the particle and $\underset{\sim}{v}_o$ its initial velocity.

Therefore, the particle moves with a constant acceleration, $q\underset{\sim}{E}/m$, in the direction of $\underset{\sim}{E}$ if $q > 0$, and in the opposite direction if $q < 0$. In a direction perpendicular to the electric field there is no acceleration and the state of motion of the particle remains unchanged.

4. UNIFORM MAGNETOSTATIC FIELD

4.1 - Formal solution of the equation of motion

For a particle of charge q and mass m, moving with velocity $\underset{\sim}{v}$ in a region of space where there is only a magnetic field $\underset{\sim}{B}$, the equation of motion is

$$m \frac{d\underset{\sim}{v}}{dt} = q\,(\underset{\sim}{v} \times \underset{\sim}{B}) \qquad (4.1)$$

It is convenient to separate the velocity $\underset{\sim}{v}$ in a component parallel to the magnetic field, $\underset{\sim}{v}_{\parallel}$, and a component perpendicular to the magnetic field, $v_{\perp}$,

$$\underset{\sim}{v} = \underset{\sim}{v}_{\parallel} + \underset{\sim}{v}_{\perp} \qquad (4.2)$$

as indicated in Fig 1. Substituting this expression into (4.1) and noting that $\underset{\sim}{v}_{\parallel} \times \underset{\sim}{B} = 0$ we obtain

$$\frac{d\underset{\sim}{v}_{\parallel}}{dt} + \frac{d\underset{\sim}{v}_{\perp}}{dt} = \frac{q}{m}\,(\underset{\sim}{v}_{\perp} \times \underset{\sim}{B}) \qquad (4.3)$$

Since the term $\underset{\sim}{v}_{\perp} \times \underset{\sim}{B}$ is perpendicular to $\underset{\sim}{B}$, the parallel component equation can be written as

$$\frac{d\underset{\sim}{v}_{\parallel}}{dt} = 0 \tag{4.4}$$

and the perpendicular component equation as

$$\frac{d\underset{\sim}{v}_{\perp}}{dt} = \frac{q}{m} (\underset{\sim}{v}_{\perp} \times \underset{\sim}{B}) \tag{4.5}$$

Eq. (4.4) shows that the particle's velocity along $\underset{\sim}{B}$ does not change and is equal to the particle's initial velocity. For the motion in the plane perpendicular to $\underset{\sim}{B}$, we can write (4.5) in the form

$$\frac{d\underset{\sim}{v}_{\perp}}{dt} = - \underset{\sim}{v} \times \underset{\sim}{\omega}_c = \underset{\sim}{\omega}_c \times \underset{\sim}{v}_{\perp} \tag{4.6}$$

where $\underset{\sim}{\omega}_c$ is a vector defined by

$$\underset{\sim}{\omega}_c = - \frac{q\underset{\sim}{B}}{m} = \frac{|q|B}{m} \hat{\underset{\sim}{\omega}}_c = \omega_c \hat{\underset{\sim}{\omega}}_c \tag{4.7}$$

Thus, $\underset{\sim}{\omega}_c$ points in the direction of $\underset{\sim}{B}$ for a negatively charged particle ($q < 0$), and in the opposite direction for a positively charged particle ($q > 0$). Its magnitude ω_c is always positive ($\omega_c = |q|\ B/m$). The unit vector $\hat{\underset{\sim}{\omega}}_c$ points along $\underset{\sim}{\omega}_c$.

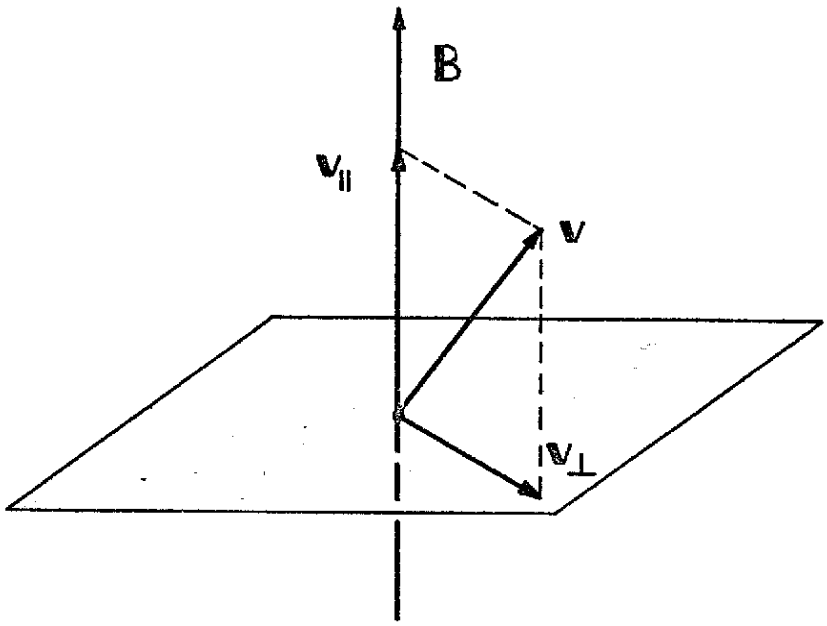

Fig. 1 - Decomposition of the velocity vector into components parallel ($v_{\parallel}$) and perpendicular ($v_{\perp}$) to the magnetic field.

Since ω_c is constant and (from conservation of kinetic energy) $v_{\perp} = |v_{\perp}|$ is also constant, Eq (4.6) shows that the acceleration of the particle is constant in magnitude and its direction is perpendicular to both $v_{\perp}$ and B. Thus, this acceleration corresponds to a rotation of the velocity vector $v_{\perp}$ in the plane perpendicular to B with the constant angular velocity ω_c. We can integrate (4.6) directly, noting that ω_c is constant and taking $v_{\perp} = dr_c/dt$, to obtain

$$\underset{\sim}{v}_\perp = \underset{\sim}{\omega}_c \times \underset{\sim}{r}_c \quad (4.8)$$

where the vector $\underset{\sim}{r}_c$ is interpreted as the position vector of the particle with respect to a point G (the center of gyration) in the plane perpendicular to $\underset{\sim}{B}$ which contains the particle. Since the particle speed $v_\perp$ is constant, the magnitude r_c of the position vector is also constant. Therefore, Eq. (4.8) shows that the velocity $\underset{\sim}{v}_\perp$ corresponds to a rotation of the position vector $\underset{\sim}{r}_c$ about the point G in the plane perpendicular to $\underset{\sim}{B}$ with constant angular velocity $\underset{\sim}{\omega}_c$. The component of the motion in the plane perpendicular to $\underset{\sim}{B}$ is therefore a circle of radius r_c. The instantaneous center of gyration of the particle, i. e., the point G at the distance r_c from the particle, is called the *guiding center*. This circular motion about the guiding center in the plane perpendicular to $\underset{\sim}{B}$ is shown in Fig. 2.

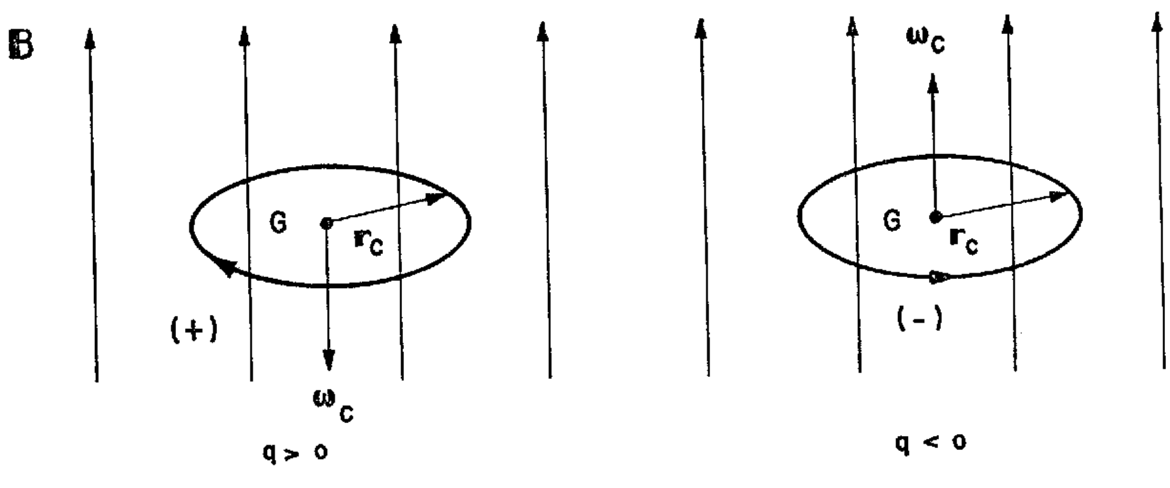

Fig. 2 - Circular motion of a charged particle about the guiding center G in a uniform magnetostatic field.

Note that, according to the definition of $\underset{\sim}{\omega}_c$ given in (4.7), $\underset{\sim}{\hat{\omega}}_c$ always points in the same direction as the particle's angular momentum vector ($\underset{\sim}{r}_c$ x $\underset{\sim}{p}$), irrespective of its charge.

The resulting trajectory of the particle consists in the superposition of a uniform motion along the magnetic field lines (with the constant velocity $\underset{\sim}{v}_{\parallel}$) and a circular motion in the plane normal to $\underset{\sim}{B}$ (with the constant speed $v_{\perp}$). Hence, the particle describes a helix (see Fig. 3). The angle between the direction of motion of the particle and the magnetic field is called the *pitch angle,* α, given by

$$\alpha = \sin^{-1} (v_{\perp}/v) = \tan^{-1} (v_{\perp}/v_{\parallel}) \qquad (4.9)$$

where v is the total speed of the particle, $v^2 = v_{\parallel}^2 + v_{\perp}^2$. When $v_{\parallel} = 0$ but $v_{\perp} \neq 0$, we have $\alpha = \pi/2$ and the particle's trajectory is a circle in the plane normal to $\underset{\sim}{B}$. On the other hand, when $v_{\perp} = 0$ but $v_{\parallel} \neq o$, we have $\alpha = 0$ and the particle moves along $\underset{\sim}{B}$ with the velocity $v_{\parallel}$.

The magnitude of the angular velocity,

$$\omega_c = |q| \, B/m \qquad (4.10)$$

is known as the *angular frequency of gyration,* or the *gyrofrequency,* or the *cyclotron frequency.* For an electron we have $|q| = 1.602 \times 10^{-19}$ Coulombs and $m = 9.109 \times 10^{-31}$ kg, which gives

$$\omega_c \text{ (electron)} = 1.76 \times 10^{11} B \qquad \text{(rad/sec)} \qquad (4.11)$$

the units of B being Tesla (or, equivalently, Weber/m2).
Similarly, for a proton $m = 1.673 \times 10^{-27}$ kg, which gives

$$\omega_c(\text{proton}) = 9.58 \times 10^7 \, B \text{ (rad/sec)} \tag{4.12}$$

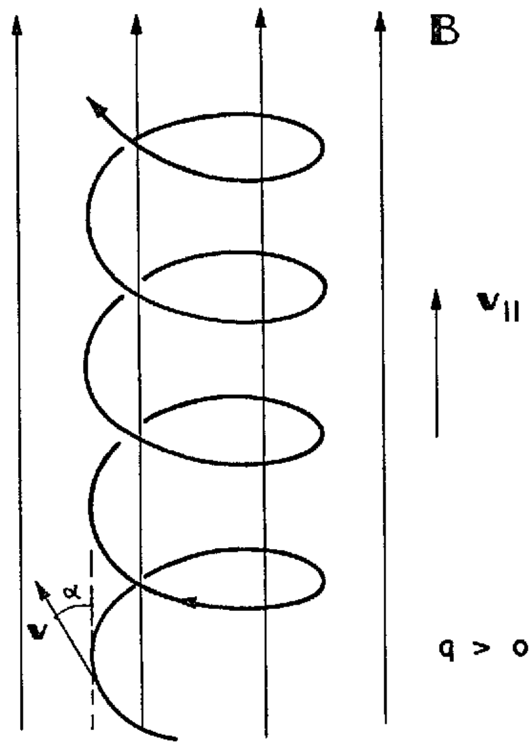

Fig. 3 - Helicoidal trajectory of a positively charged particle in a uniform magnetostatic field.

The radius of the circular orbit, given by

$$r_c = \frac{v_\perp}{\omega_c} = \frac{mv_\perp}{|q|\,B} \tag{4.13}$$

is called the *radius of gyration*, or *gyroradius*, or *cyclotron radius*, or *Larmor radius*.

It is important to note that ω_c is directly proportional to B whereas r_c is inversely proportional to B. Consequently, as the magnetic field increases, the gyrofrequency increases and the radius decreases. Also, the smaller the mass of the particle the larger will be its gyrofrequency and the smaller its gyroradius.

Multiplying (4.13) by B gives

$$Br_c = \frac{mv_\perp}{|q|} = \frac{p_\perp}{|q|} \tag{4.14}$$

which shows that the magnitude of $\underset{\sim}{B}$ times the gyroradius of the particle is equal to its momentum per unit charge. This quantity is often called the *magnetic rigidity*.

4.2 - Solution in Cartesian coordinates

The treatment presented so far in this section is not related to any particular frame of reference. Consider now a Cartesian coordinate system (x,y,z) such that the z-axis is parallel to the magnetic flux density, i. e., $\underset{\sim}{B} = B\hat{\underset{\sim}{z}}$. In this case, the cross product between $\underset{\sim}{v}$ and $\underset{\sim}{B}$ can be written as

$$\underset{\sim}{v} \times \underset{\sim}{B} = \begin{vmatrix} \hat{\underset{\sim}{x}} & \hat{\underset{\sim}{y}} & \hat{\underset{\sim}{z}} \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix} = B(v_y\,\hat{\underset{\sim}{x}} - v_x\,\hat{\underset{\sim}{y}}) \tag{4.15}$$

and the equation of motion (4.1) becomes

$$\frac{d\underset{\sim}{v}}{dt} = \frac{qB}{m}\,(v_y\,\hat{\underset{\sim}{x}} - v_x\,\hat{\underset{\sim}{y}})$$

$$= (\pm\,\omega_c)\,(v_y\,\hat{\underset{\sim}{x}} - v_x\,\hat{\underset{\sim}{y}}) \tag{4.16}$$

The (+) sign in front of ω_c applies to a positively charged particle ($q > 0$) and the (-) sign to a negatively charged particle ($q < 0$), since ω_c is always positive ($\omega_c = |q|\,B/m$).

In what follows we will consider a *positively charged particle*. The results for a negative charge can be obtained by changing the sign of ω_c in the results for the positive charge.

The Cartesian components of (4.16) are (for $q > 0$)

$$dv_x/dt = \omega_c v_y \tag{4.17}$$

$$dv_y/dt = -\omega_c v_x \tag{4.18}$$

$$dv_z/dt = 0 \tag{4.19}$$

The last of these equations gives $v_z(t) = v_z(0) = v_{\parallel}$, which is the initial value of the velocity component parallel to $\underset{\sim}{B}$. To obtain the solution of Eqs. (4.17) and (4.18) we take the derivative of (4.17) with respect to time and substitute the result into (4.18), getting

$$\frac{d^2 v_x}{dt^2} + \omega_c^2 v_x = 0 \tag{4.20}$$

This is the homogeneous differential equation for a harmonic oscillator of frequency ω_c, whose solution is

$$v_x(t) = v_\perp \sin(\omega_c t + \theta_o) \tag{4.21}$$

where $v_\perp$ is the constant speed of the particle in the (x,y) plane (normal to $\underset{\sim}{B}$) and θ_o is a constant of integration which depends on the relation between the initial velocities $v_x(0)$ and $v_y(0)$, according to

$$\tan(\theta_o) = v_x(0)/v_y(0) \tag{4.22}$$

To determine $v_y(t)$ we substitute (4.21) in the left hand side of (4.17), obtaining

$$v_y(t) = v_\perp \cos(\omega_c t + \theta_0) \tag{4.23}$$

Note that $v_x^2 + v_y^2 = v_\perp^2$. The equations for the components of $\underset{\sim}{v}$ can be further integrated with respect to time, yielding

$$x(t) = -(v_\perp/\omega_c) \quad \cos(\omega_c t + \theta_0) + X_0 \tag{4.24}$$

$$y(t) = (v_\perp/\omega_c) \quad \sin(\omega_c t + \theta_0) + Y_0 \tag{4.25}$$

$$z(t) = v_\parallel t + z_0 \tag{4.26}$$

where we have defined

$$X_0 = x_0 + (v_\perp/\omega_c) \cos\theta_0 \tag{4.27}$$

$$Y_0 = y_0 - (v_\perp/\omega_c) \sin\theta_0 \tag{4.28}$$

The vector $\underset{\sim}{r}_0 = x_0\hat{\underset{\sim}{x}} + y_0\hat{\underset{\sim}{y}} + z_0\hat{\underset{\sim}{z}}$ gives the initial position of the particle. From (4.24) and (4.25) we see that

$$(x - X_0)^2 + (y - Y_0)^2 = (v_\perp/\omega_c)^2 = r_c^2 \tag{4.29}$$

The trajectory of the particle in the plane normal to $\underset{\sim}{B}$ is, therefore, a circle with center at (X_o, Y_o) and radius equal to $(v_\perp/\omega_c)$. The motion of the point $[X_o, Y_o, z(t)]$, at the instantaneous center of gyration, corresponds to the *trajectory of the guiding center*. Thus, the guiding center moves with constant velocity $\underset{\sim}{v}_\parallel$ along $\underset{\sim}{B}$.

In the (x, y) plane, the argument $\phi(t)$, defined by

$$\phi(t) = \tan^{-1}\left(\frac{y - Y_o}{x - X_o}\right) = -(\omega_c t + \theta_o); \quad \phi_o = -\theta_o \tag{4.30}$$

decreases with time for a positively charged particle. For a magnetic field pointing toward the observer, a positive charge describes a circle in the clockwise direction. For a negatively charged particle we must replace ω_c by $-\omega_c$ in the results of this sub-section. Hence, (4.30) shows that for a negative charge $\phi(t)$ increases with time and the particle moves in a circle in the counterclockwise direction, as shown in Fig. 4. The resulting motion of the particle is a cylindrical helix of constant pitch angle. Fig. 5 shows the parameters of the helix with reference to a Cartesian coordinate system.

4.3 - Magnetic moment

To the circular motion of a charged particle in a magnetic field, there is associated a circulating electric current I. This current flows in the clockwise direction for a $\underset{\sim}{B}$ field pointing toward the observer (Fig. 4).

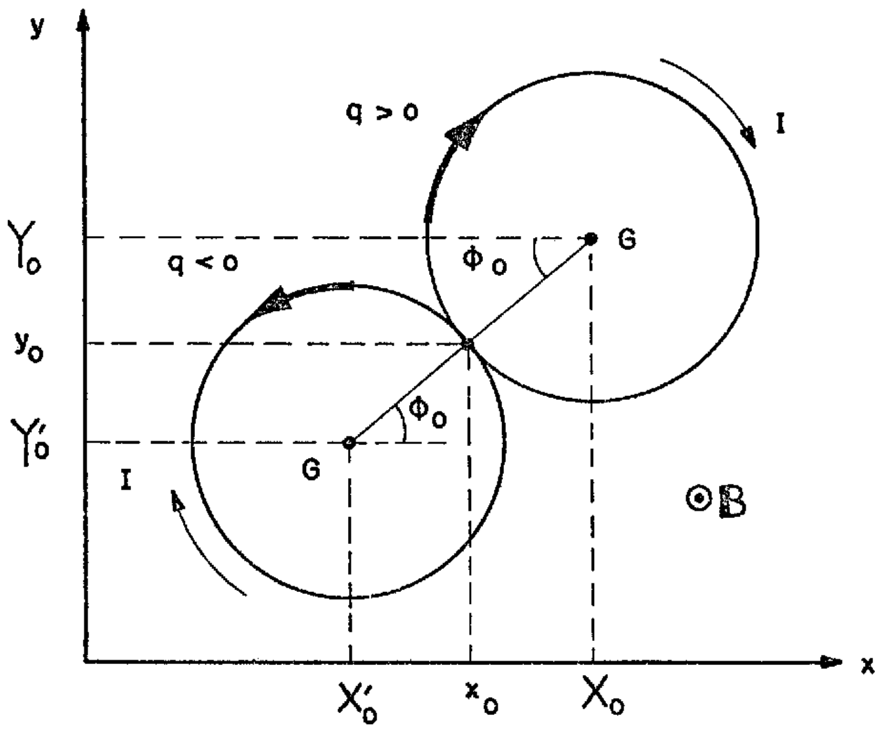

Fig. 4 - Circular trajectory of a charged particle in a uniform and constant $\underset{\sim}{B}$ field (directed out of the paper), and the direction of the associated electric current.

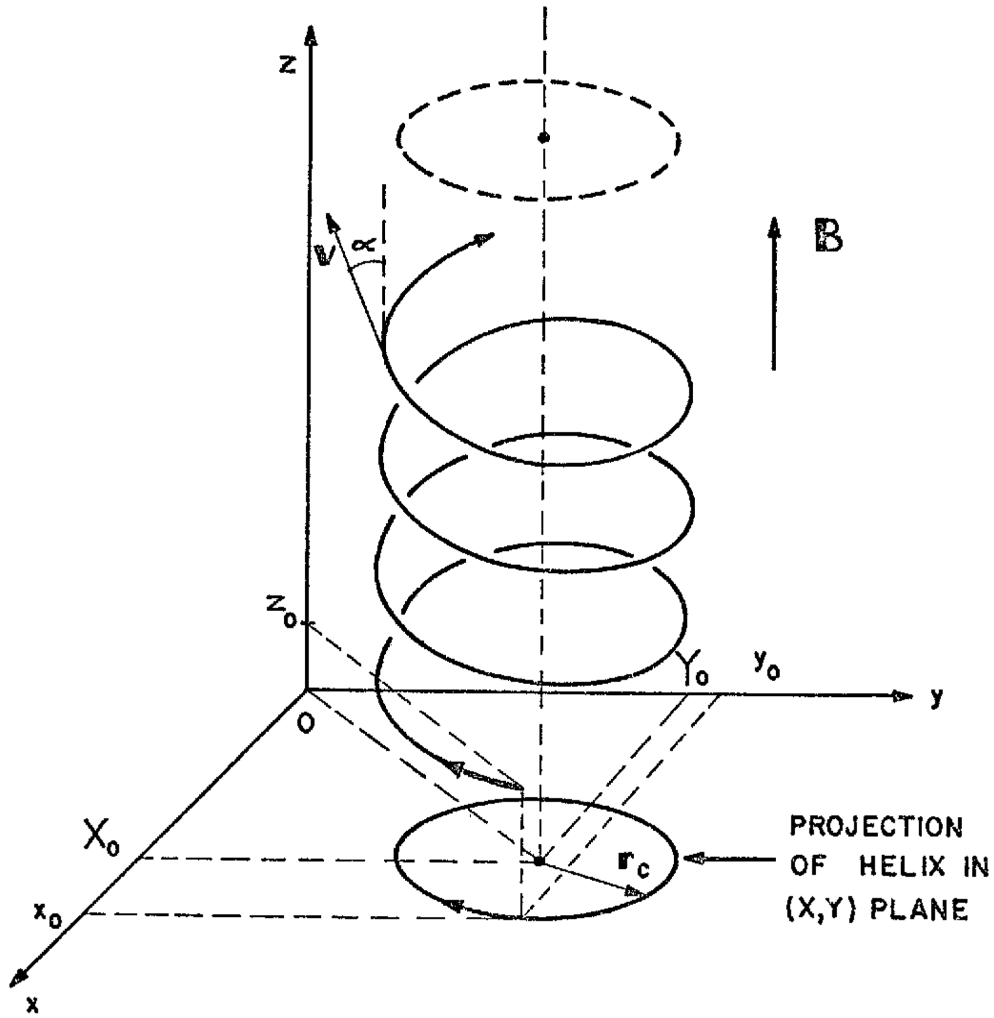

Fig. 5 - Parameters of the helicoidal trajectory of a positively charged particle with reference to a Cartesian coordinate system.

From Ampère's law, the direction of the magnetic field associated with this circulating current is given by the right-hand rule, that is, with the right thumb pointing in the direction of the current I, the right fingers curl in the direction of the associated magnetic field. Therefore, the $\underset{\sim}{B}$ field produced by the circular motion of a charged particle is *opposite* to the externally applied $\underset{\sim}{B}$ field *inside the particle's orbit*, but in the same direction outside the orbit. The magnetic field generated by the ring current I, at distances much larger than r_c, is similar to that of a dipole (Fig. 6). Since a plasma is a collection of charged particles it possesses therefore *diamagnetic properties.*

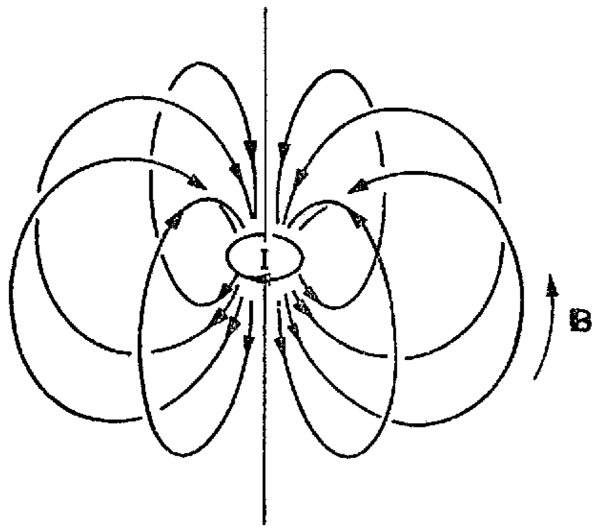

Fig. 6 - The magnetic field generated by a small ring current is that of a magnetic dipole.

The *magnetic moment* $\underset{\sim}{m}$ associated with the circulating current is normal to the area A bounded by the orbit of the particle and points in the direction opposite to the externally applied $\underset{\sim}{B}$ field, as shown in Fig. 7. Its magnitude is given by

$$|\underset{\sim}{m}| = (\text{Current}) \,.\, (\text{Orbital Area}) = IA \tag{4.31}$$

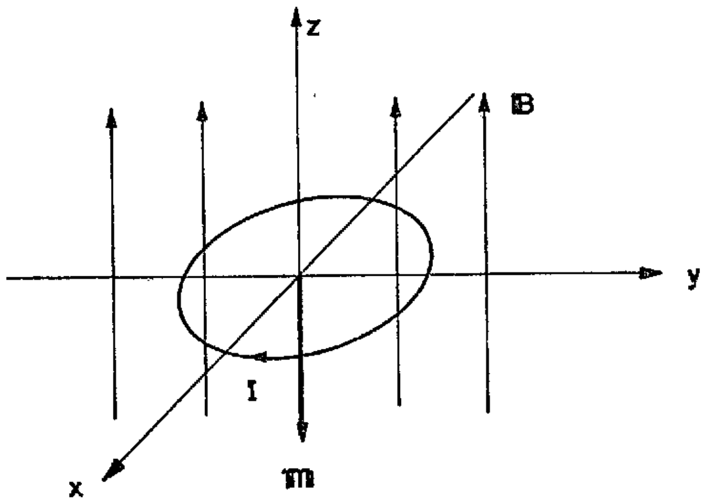

Fig. 7 - Magnetic moment $\underset{\sim}{m}$ associated with a circulating current due to the circular motion of a charged particle in an external $\underset{\sim}{B}$ field.

The circulating current corresponds to a flow of charge and is given by

$$I = \frac{|q|}{T_c} = \frac{|q|\ \omega_c}{2\pi} \tag{4.32}$$

where $T_c = 2\pi/\omega_c$ is the period of the particle's orbit, known as the *cyclotron period* or *Larmor period*. The magnitude of $\underset{\sim}{m}$ is, therefore,

$$|\underset{\sim}{m}| = \frac{|q|\ \omega_c}{2\pi}(\pi r_c^2) = \frac{1}{2}|q|\ \omega_c\ r_c^2 \tag{4.33}$$

Using the relations $\omega_c = |q|\ B/m$ and $r_c = v_\perp/\omega_c$, (4.33) becomes

$$|\underset{\sim}{m}| = \frac{1}{2}\ m\ v_\perp^2/B = W_\perp/B \tag{4.34}$$

where $W_\perp$ denotes the part of the kinetic energy of the particle which is associated with its transversal velocity $v_\perp$. Thus, in vector form we have

$$\underset{\sim}{m} = -(W_\perp/B^2)\ \underset{\sim}{B} \tag{4.35}$$

4.4 - Magnetization current

Consider now a collection of charged particles, positive and negative in equal numbers (in order to have no internal macroscopic electrostatic fields), instead of just one single particle. For instance, consider the case of a low-density plasma, for which we can neglect the collisions between the particles (collisionless plasma). The condition for this is that the average time between collisions be much greater than the cyclotron period. This condition is fulfilled for many space plasmas, for example.

For a collisionless plasma in an external magnetic field, the magnetic moments due to the orbital motion of the charged particles act together, giving rise to a resultant magnetic field wich may be strong enough to appreciably change the externally applied $\underset{\sim}{B}$ field. The mean magnetic field produced by the orbital motion of the charged particles can be determined from the net electric current density associated with their motion.

To calculate the resultant electric current density, let us consider a macroscopic volume containing a large number of particles. Let S be an element of area in this volume, bounded by the curve C (Fig. 8-a). Orbits such as (1), which encircle the bounded surface only once, contribute to the resultant current whereas orbits such as (2), which cross the surface twice, do not contribute to the net current. If $\underset{\sim}{dl}$ is an element of arc along the curve C, the number of orbits encircling $\underset{\sim}{dl}$ is given by $n\underset{\sim}{A}.\underset{\sim}{dl}$, where n is the number of orbits of current I per unit volume, and $\underset{\sim}{A}$ is the vector area bounded by *each* orbit. The direction of $\underset{\sim}{A}$ is that of the normal to the orbital area A, the positive sense being related to the sense of circulation in the way the linear motion of a right-hand screw is related to its rotary motion. Thus, $\underset{\sim}{A}$ points in the direction of the observer when I flows counterclockwise (Fig. 8-b). The net resultant current crossing S is therefore given by the current encircling $d\underset{\sim}{l}$ integrated along the curve C, i. e.,

$$I_n = \oint_C I\, n\, \underset{\sim}{A} \cdot d\underset{\sim}{l} \qquad (4.36)$$

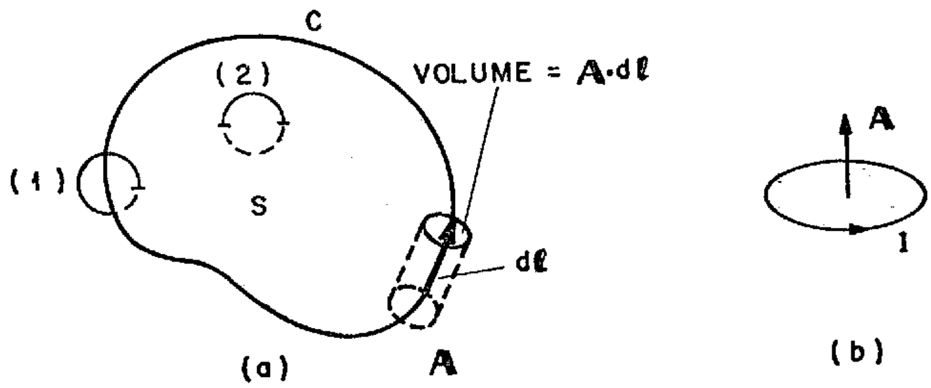

Fig. 8 - (a) Electric current orbits crossing the surface element S bounded by the curve C, in a macroscopic volume containing a large number of particles. (b) Positive direction of the area vector $\underset{\sim}{A}$.

Since $\underset{\sim}{m} = I\,\underset{\sim}{A}$, the magnetic moment per unit volume, $\underset{\sim}{M}$, (also called the *magnetization* vector) is given by

$$\underset{\sim}{M} = n\,\underset{\sim}{m} = n\, I\, \underset{\sim}{A} \qquad (4.37)$$

Hence, (4.36) can be written as

$$I_n = \oint_C \underset{\sim}{M} \cdot d\underset{\sim}{l} = \int (\underset{\sim}{\nabla} \times \underset{\sim}{M}) \cdot d\underset{\sim}{S} \qquad (4.38)$$

where we have applied Stoke's theorem.

We may define an average *magnetization current density*, $\underset{\sim}{J}_M$, crossing the surface S, by

$$I_n = \int_S \underset{\sim}{J}_M \cdot d\,\underset{\sim}{S} \qquad (4.39)$$

Consequently, from (4.38) and (4.39) we obtain the magnetization current density as

$$\underset{\sim}{J}_M = \underset{\sim}{\nabla} \times \underset{\sim}{M} \qquad (4.40)$$

where, from (4.37) and (4.35),

$$\underset{\sim}{M} = n\,\underset{\sim}{m} = -\,(nW_{\perp}/B^2)\,\underset{\sim}{B} \qquad (4.41)$$

and $nW_{\perp}$ denotes the kinetic energy per unit volume, associated with the transverse velocity of the particle.

The charge density, ρ_M, associated with the magnetization current density, $\underset{\sim}{J}_M$, can be deduced from the equation of continuity,

$$\frac{\partial \rho_M}{\partial t} + \underset{\sim}{\nabla} \cdot \underset{\sim}{J}_M = 0 \qquad (4.42)$$

Since $\underset{\sim}{J}_M = \underset{\sim}{\nabla} \times \underset{\sim}{M}$ and since for any vector $\underset{\sim}{a}$, $\underset{\sim}{\nabla} \cdot (\underset{\sim}{\nabla} \times \underset{\sim}{a}) = 0$, it follows that the charge density ρ_M is a constant.

In the following Maxwell equation

$$\underset{\sim}{\nabla} \times \underset{\sim}{B} = \mu_o (\underset{\sim}{J} + \varepsilon_o \, \partial \underset{\sim}{E}/\partial t) \qquad (4.43)$$

we can separate the total current density, $\underset{\sim}{J}$, in two parts: a magnetization current density, $\underset{\sim}{J}_M$, and a current density, $\underset{\sim}{J}'$ due to other sources,

$$\underset{\sim}{J} = \underset{\sim}{J}_M + \underset{\sim}{J}' \qquad (4.44)$$

Expressing $\underset{\sim}{J}_M$ in terms of $\underset{\sim}{M}$, through (4.40), and substituting in (4.43), we obtain

$$\underset{\sim}{\nabla} \times \underset{\sim}{B} = \mu_o (\underset{\sim}{\nabla} \times \underset{\sim}{M} + \underset{\sim}{J}' + \varepsilon_o \, \partial \underset{\sim}{E}/\partial t) \qquad (4.45)$$

which can be rearranged as

$$\nabla \times \left(\frac{B}{\mu_0} - M \right) = J' + \varepsilon_0 \frac{\partial E}{\partial t} \tag{4.46}$$

Defining an effective magnetic field, H, by the relation

$$B = \mu_0 (H + M) \tag{4.47}$$

we can write (4.46) as

$$\nabla \times H = J' + \varepsilon_0 \, \partial E/\partial t \tag{4.48}$$

Thus, the effective magnetic field H is related to the current due to other sources J', in the way B is related to the total current J. Eqs. (4.40) and (4.47) constitute the basic relations for the classical treatment of magnetic materials.

A simple linear relation between B and H exists when M is proportional to B or H, i.e.,

$$M = \chi_m H \tag{4.49}$$

where the constant χ_m is called the *magnetic susceptibility* of the medium. However, for a plasma we have seen that $M \propto 1/B$ [see Eq. 4.41)], so that the relation between H and B (or M) is *not* linear. For this reason it is generally not convenient to consider a plasma as a magnetic medium.

5. UNIFORM ELECTROSTATIC AND MAGNETOSTATIC FIELDS

5.1 - Formal solution of the equation of motion

We consider now the motion of a charged particle in the presence of both electric and magnetic fields which are constant in time and uniform in space. The nonrelativistic equation of motion is

$$m \frac{d\underset{\sim}{v}}{dt} = q\,(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) \tag{5.1}$$

Taking components parallel and perpendicular to the magnetic flux density $\underset{\sim}{B}$, i. e.,

$$\underset{\sim}{v} = \underset{\sim}{v}_{\perp} + \underset{\sim}{v}_{\parallel} \tag{5.2}$$

$$\underset{\sim}{E} = \underset{\sim}{E}_{\perp} + \underset{\sim}{E}_{\parallel} \tag{5.3}$$

we can resolve (5.1) into two component equations

$$m \frac{d\underset{\sim}{v}_{\parallel}}{dt} = q\,\underset{\sim}{E}_{\parallel} \tag{5.4}$$

$$m \frac{d\underset{\sim}{v}_{\perp}}{dt} = q\,(\underset{\sim}{E}_{\perp} + \underset{\sim}{v}_{\perp} \times \underset{\sim}{B}) \tag{5.5}$$

Eq. (5.4) is similar to (3.1) and represents a motion with constant acceleration $q\,\underset{\sim}{E}_{\parallel}/m$ along the $\underset{\sim}{B}$ field. Hence, according to (3.2) and (3.4) we have

$$\underset{\sim}{v}_{\parallel}(t) = \frac{q}{m}\,\underset{\sim}{E}_{\parallel} t + \underset{\sim}{v}_{\parallel}(0) \tag{5.6}$$

$$\underset{\sim}{r}_{\parallel}(t) = \frac{q\,\underset{\sim}{E}_{\parallel}}{2m}\, t^2 + \underset{\sim}{v}_{\parallel}(0)t + \underset{\sim}{r}_{\parallel}(0) \tag{5.7}$$

To solve (5.5) it is convenient to separate $\underset{\sim}{v}_{\perp}$ in two components

$$\underset{\sim}{v}_{\perp}(t) = \underset{\sim}{v}_{\perp}'(t) + \underset{\sim}{v}_{E} \tag{5.8}$$

where $\underset{\sim}{v}_{E}$ is a constant velocity in the plane normal to $\underset{\sim}{B}$. Hence, $\underset{\sim}{v}_{\perp}'$ represents the velocity of the particle as seen by an observer in a frame of reference moving with the constant velocity $\underset{\sim}{v}_{E}$. Substituting (5.8) into (5.5), and writing the component of the electric field perpendicular to $\underset{\sim}{B}$ in the form (see Fig. 9)

$$\underset{\sim}{E}_{\perp} = - \frac{(\underset{\sim}{E}_{\perp} \times \underset{\sim}{B})}{B^2} \times \underset{\sim}{B} \tag{5.9}$$

we obtain

$$m \frac{d\underset{\sim}{v}'_\perp}{dt} = q \left[\underset{\sim}{v}'_\perp + \underset{\sim}{v}_E - \frac{(\underset{\sim}{E}_\perp \times \underset{\sim}{B})}{B^2} \right] \times \underset{\sim}{B} \qquad (5.10)$$

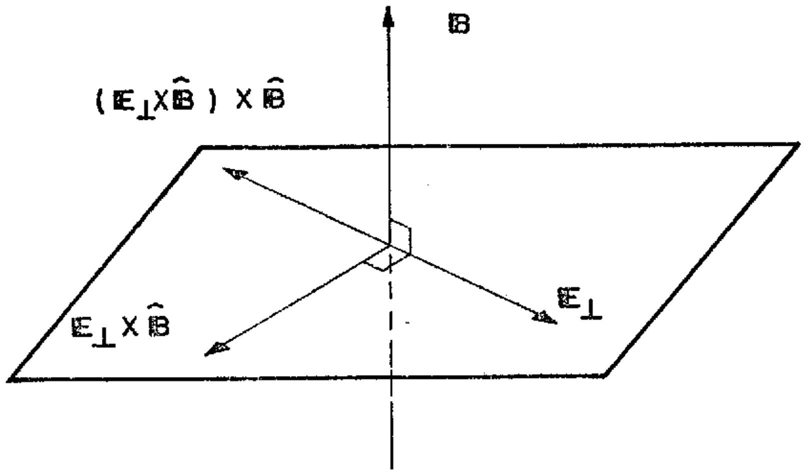

Fig. 9 - Vector products appearing in Eq. (5.9) ($\hat{\underset{\sim}{B}} = \underset{\sim}{B}/B$).

This equation shows that in a coordinate system moving with the constant velocity

$$\underset{\sim}{v}_E = \frac{(\underset{\sim}{E}_\perp \times \underset{\sim}{B})}{B^2} \qquad (5.11)$$

the motion of the particle in the plane normal to $\underset{\sim}{B}$ is governed entirely by the magnetic field, according to

$$m \frac{d\underset{\sim}{v}'_\perp}{dt} = q\ (\underset{\sim}{v}'_\perp \times \underset{\sim}{B}) \qquad (5.12)$$

Thus, in this frame of reference the electric field component $\underset{\sim}{E}_\perp$ is transformed away, whereas the magnetic field is left unchanged.

Eq. (5.12) is identical to (4.5) and implies that in the reference system moving with the constant velocity $\underset{\sim}{v}_E$, given by (5.11), the particle describes a circular motion at the cyclotron frequency ω_c and radius r_c i.e.,

$$\underset{\sim}{v}_\perp = \underset{\sim}{\omega}_c \times \underset{\sim}{r}_c \tag{5.13}$$

The results obtained so far indicate that the resulting motion of the particle is described by a superposition of a circular motion in the plane normal to $\underset{\sim}{B}$, with a uniform motion with the constant velocity $\underset{\sim}{v}_E$ perpendicular to both $\underset{\sim}{B}$ and $\underset{\sim}{E}_\perp$, plus a uniform acceleration $q\underset{\sim}{E}_\parallel/m$ along $\underset{\sim}{B}$. The velocity of the particle can be expressed in vector form, independently of a coordinate system, as

$$\underset{\sim}{v}(t) = \underset{\sim}{\omega}_c \times \underset{\sim}{r}_c + \frac{\underset{\sim}{E}_\perp \times \underset{\sim}{B}}{B^2} + \frac{q}{m} \underset{\sim}{E}_\parallel t + \underset{\sim}{v}_\parallel \tag{5.14}$$

The first term in the right hand side of (5.14) represents the cyclotron circular motion, the second term represents the constant drift velocity of the guiding center in the direction perpendicular to both $\underset{\sim}{E}_\perp$ and $\underset{\sim}{B}$, the third term represents the constant acceleration of the guiding center along $\underset{\sim}{B}$, and the last term is the initial velocity parallel to $\underset{\sim}{B}$.

Note that the velocity $\underset{\sim}{v}_E$ is independent of the mass and the sign of the charge and therefore is the same for both positive and negative particles. It is usually called the *plasma drift velocity*. Since $\underset{\sim}{E}_{\parallel} \times \underset{\sim}{B} = 0$, (5.11) can also be written as

$$\underset{\sim}{v}_E = \frac{\underset{\sim}{E} \times \underset{\sim}{B}}{B^2} \tag{5.15}$$

The resulting motion of the particle in the plane normal to $\underset{\sim}{B}$ is a cycloid, as shown in Fig. 10. The physical explanation for this cycloidal motion is as follows. The electric force $q\underset{\sim}{E}_{\perp}$, acting simultaneously with the magnetic force, accelarates the particle so as to increase or decrease its velocity, depending on the relative direction of motion of the particle with respect to the direction of $\underset{\sim}{E}_{\perp}$ and on the sign of the charge. According to Eq. (4.13) the radius of gyration increases with velocity and, hence, the radius of curvature of the particle's path will vary under the action of $\underset{\sim}{E}_{\perp}$. This results in a cycloidal trajectory with a net drift in the direction perpendicular to $\underset{\sim}{E}_{\perp}$ and $\underset{\sim}{B}$. Different cycloidal trajectories are obtained, depending on the initial conditions and on the magnitude of the applied electric and magnetic fields.

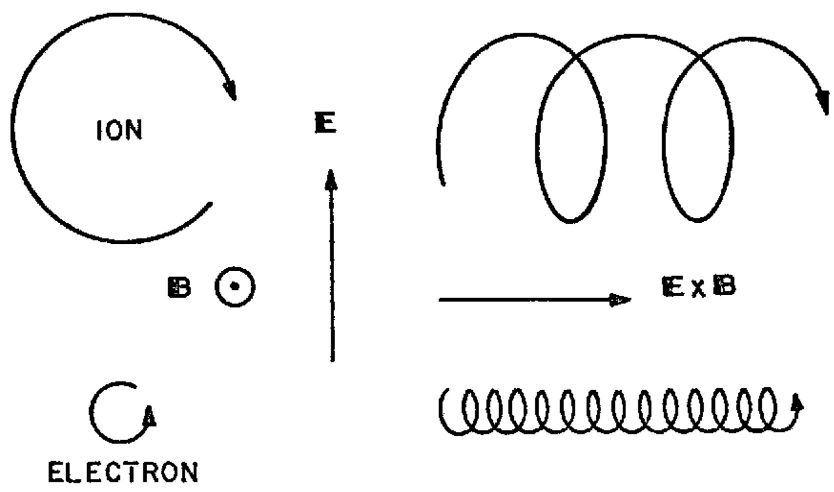

Fig. 10 - Cycloidal trajectories described by ions and electrons in crossed electric and magnetic fields. The electric field $\underset{\sim}{E}$ acting together with the magnetic flux density $\underset{\sim}{B}$ gives rise to a drift velocity in the direction $\underset{\sim}{E} \times \underset{\sim}{B}$.

The ions are much more massive than the electrons and, therefore, the Larmor radius for ions is correspondingly greater and the Larmor frequency correspondingly smaller than for electrons. Consequently, the arcs of cycloid for ions are greater than for electrons, but there is a larger number of arcs of cycloid per second for electrons, such that the drift velocity is the same for both.

In a collisionless plasma the drift velocity does not imply in an electric current, since both positive and negative particles move together. When collisions between charged and neutral particles are important, this drift gives rise to an electric current, since the ion-neutral collision frequency is greater than the electron-neutral collision frequency, causing the ions to move slower than the electrons. This current is normal to both $\underset{\sim}{E}$ and $\underset{\sim}{B}$, and is in the direction opposite to $\underset{\sim}{v}_E$. It is known as the *Hall current.*

5.2 - Solution in Cartesian coordinates

Let us choose a Cartesian coordinate system with the z-axis pointing in the direction of $\underset{\sim}{B}$, so that

$$\underset{\sim}{B} = B\,\hat{\underset{\sim}{z}} \tag{5.16}$$

$$\underset{\sim}{E} = E_x\,\hat{\underset{\sim}{x}} + E_y\,\hat{\underset{\sim}{y}} + E_z\,\hat{\underset{\sim}{z}} \tag{5.17}$$

Using (4.15), the equation of motion (5.1) can be written as

$$\frac{d\underset{\sim}{v}}{dt} = \frac{q}{m} [(E_x + v_y B) \hat{\underset{\sim}{x}} + (E_y - v_x B) \hat{\underset{\sim}{y}} + E_z \hat{\underset{\sim}{z}}] \quad (5.18)$$

As before, we consider, in what follows, a *positively charged particle.* The results for a negatively charged particle can be obtained by changing the sign of ω_c in the results for a positively charged particle. The z-component of (5.18) can be integrated directly and gives the same results expressed in Eqs. (5.6) and (5.7).

For the x-and y-components, we first take the derivative of dv_x/dt with respect to time and substitute the expression for dv_y/dt, which gives

$$\frac{d^2 v_x}{dt^2} + \omega_c^2 v_x = \omega_c^2 E_y/B \quad (5.19)$$

This is the inhomogeneous differential equation for a harmonic oscillator of frequency ω_c. Its solution is given by the sum of the solution of the homogeneous equation [given in (4.21)] with a particular solution (which is clearly E_y/B). Thus,

$$v_x(t) = v_\perp' \sin (\omega_c t + \theta_0) + E_y/B \quad (5.20)$$

where $v_\perp'$ and θ_0 are integration constants. The solution for $v_y(t)$ can be obtained by substituting (5.20) directly into (5.18). Hence, we obtain

$$v_y(t) = \frac{1}{\omega_c} \frac{dv_x}{dt} - \frac{E_x}{B} = v_\perp' \cos (\omega_c t + \theta_o) - \frac{E_x}{B} \qquad (5.21)$$

Therefore, the velocity components $v_x(t)$ and $v_y(t)$, in the plane perpendicular to $\underset{\sim}{B}$, oscillate at the cyclotron frequency ω_c with amplitude $v_\perp'$. This motion is superposed to a constant drift velocity $\underset{\sim}{v}_E$ given by

$$\underset{\sim}{v}_E = \frac{E_y}{B} \hat{\underset{\sim}{x}} - \frac{E_x}{B} \hat{\underset{\sim}{y}} \qquad (5.22)$$

This expression corresponds to (5.11) for the case when $\underset{\sim}{B} = B\hat{\underset{\sim}{z}}$.

One more integration of (5.20) and (5.21) gives the particle trajectory in the (x, y) plane

$$x(t) = - \left(\frac{v_\perp'}{\omega_c} \right) \cos (\omega_c t + \theta_o) + \left(\frac{E_y}{B} \right) t + X_o \qquad (5.23)$$

$$y(t) = \left(\frac{v_\perp'}{\omega_c} \right) \sin (\omega_c t + \theta_o) - \left(\frac{E_x}{B} \right) t + Y_o \qquad (5.24)$$

where X_o and Y_o are defined according to (4.27) and (4.28), but with $v_\perp$ replaced by $v_\perp'$.

In summary, the motion of a charged particle in uniform electrostatic and magnetostatic fields consists of three components:

(a) A constant acceleration $q\,\underset{\sim}{E}_{\parallel}/m$ along the $\underset{\sim}{B}$ field. If $\underset{\sim}{E}_{\parallel} = 0$, the particle moves along B with its initial velocity.

(b) A rotation about the direction of $\underset{\sim}{B}$ at the cyclotron frequency $\omega_c = |q|\,B/m$ and radius $r_c = v_{\perp}'/\omega c$.

(c) An electromagnetic drift velocity $\underset{\sim}{v}_E = (\underset{\sim}{E} \times \underset{\sim}{B})/B^2$, perpendicular to both $\underset{\sim}{B}$ and $\underset{\sim}{E}$.

6. DRIFT DUE TO AN EXTERNAL FORCE

If some additional force $\underset{\sim}{F}$ (gravitacional force, or inertial force if the motion is considered in a noninertial system, for example) is present, the equation of motion (1.5) must be modified to include this force,

$$m \frac{d\underset{\sim}{v}}{dt} = q\,(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) + \underset{\sim}{F} \tag{6.1}$$

The effect of this force is, in a formal sense, analogous to the effect of the electric field. We assume here that $\underset{\sim}{F}$ is uniform and constant. In analogy with the drift velocity $(\underset{\sim}{E} \times \underset{\sim}{B})/B^2$, the drift produced by the force $\underset{\sim}{F}$ having a component normal to the magnetic flux density $\underset{\sim}{B}$ is given by

$$\underset{\sim}{v}_F = \frac{\underset{\sim}{F} \times \underset{\sim}{B}}{qB^2} \tag{6.2}$$

In the case of a uniform gravitacional field, for example, we have $\underset{\sim}{F} = m\underset{\sim}{g}$, where $\underset{\sim}{g}$ is the acceleration due to gravity, and the drift velocity is given by

$$\underset{\sim}{v}_g = \frac{m}{q} \frac{\underset{\sim}{g} \times \underset{\sim}{B}}{B^2} \tag{6.3}$$

This drift velocity depends on the ratio m/q and therefore is in opposite directions for particles of opposite charge (Fig. 11). We have seen that in a coordinate system moving with the velocity $\underset{\sim}{v}_E = (\underset{\sim}{E} \times \underset{\sim}{B})/B^2$, the electric field component $\underset{\sim}{E}_\perp$ is transformed away leaving the magnetic field unchanged. The gravitational field however cannot, in this context, be transformed away.

In a collisionless plasma, associated with the gravitational drift velocity there is an electric current density, $\underset{\sim}{J}_g$, in the direction of $\underset{\sim}{g} \times \underset{\sim}{B}$, which can be expressed as

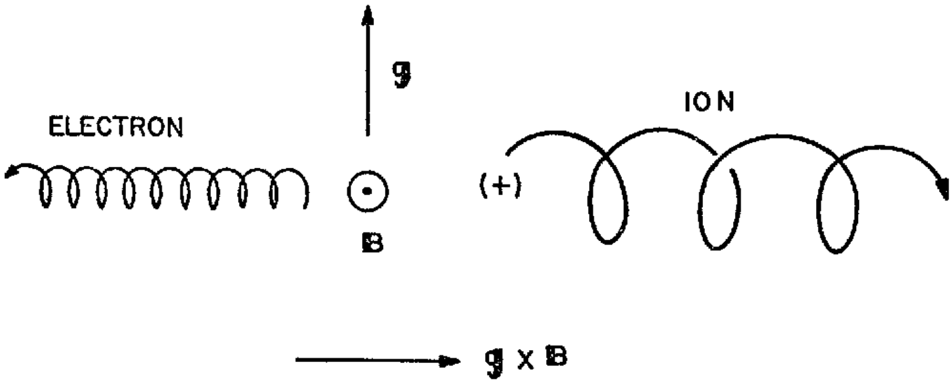

Fig. 11 - The drift of a gyrating particle in crossed gravitational and magnetic fields.

$$\underset{\sim}{J}_g = \frac{1}{\Delta V} \sum_i q_i \underset{\sim}{v}_{gi} \tag{6.4}$$

where the summation is over all charged particles contained in a suitably chosen small volume element ΔV. Using (6.3) we obtain

$$\underset{\sim}{J}_g = \left(\frac{1}{\Delta V} \sum_i m_i \right) \frac{(\underset{\sim}{g} \times \underset{\sim}{B})}{B^2} = \rho \, \frac{(\underset{\sim}{g} \times \underset{\sim}{B})}{B^2} \tag{6.5}$$

where ρ denotes the total mass density of the charged particles.

A comment on the validity of Eq. (6.2) is appropriate here. Since we have used the nonrelativistic equation of motion, there is a limitation on the magnitude of the force $\underset{\sim}{F}$ in order that (6.2) be applicable. The magnitude of the transverse drift velocity is given by

$$v_D = \frac{F_\perp}{qB} \tag{6.6}$$

Hence, for the nonrelativistic equation of motion to be applicable we must have

$$\frac{F_\perp}{qB} \ll c \tag{6.7}$$

or, if $\underset{\sim}{F}$ is due to an electrostatic field $\underset{\sim}{E}$,

$$\frac{E_\perp}{B} \ll c \tag{6.8}$$

For a magnetic field of 1 Tesla (10^4 Gauss) for example, Eq. (6.2) may be used as long as $E_\perp$ is much less than 10^8 Volts/m. If these conditions are not satisfied, the problem becomes a relativistic one. Although the relativistic equations of motion can be integrated exactly for contant $\underset{\sim}{B}$, $\underset{\sim}{E}$ and $\underset{\sim}{F}$, we shall not analyze this problem here. It is left as an exercise for the reader.

PROBLEMS

2.1 - Calculate the cyclotron frequency, ω_c, and the cyclotron radius, r_c, for:

(a) An electron in the Earth's ionosphere at 300 km altitude, where the magnetic flux density $B \simeq 0.5$ Gauss, considering that the electron moves at the thermal velocity $(kT/m)^{1/2}$ with $T = 1000$ K, where k is Boltzmann's constant.

(b) A 50 MeV proton in the Earth's inner Van Allen radiation belt at about 1.5 R_E (where $R_E = 6370$ km is the Earth's radius) from the center of the Earth in the equatorial plane, where $B \simeq 0.1$ Gauss.

(c) A 1 MeV electron in the Earth's outer Van Allen radiation belt at about 4 R_E from the center of the Earth in the equatorial plane, where $B \simeq 10^{-5}$ Gauss.

(d) A proton in the solar wind with a streaming velocity of 100 km/sec, in a magnetic flux density $B \simeq 10^{-5}$ Gauss.

(e) A 1 MeV proton in the solar atmosphere, in the region of a sunspot, in which $B = 1000$ Gauss.

2.2 - For an electron and an oxygen ion O^+ in the Earth's ionosphere, at 300 km altitude in the equatorial plane, where $B \simeq 0.5$ Gauss, calculate:

(a) The gravitational drift velocity $\underset{\sim}{V}_g$.

(b) The gravitational current density $\underset{\sim}{J}_g$, considering $n_e = n_i = 10^6\ cm^{-3}$.

Assume that $\underset{\sim}{g}$ is perpendicular to $\underset{\sim}{B}$.

2.3 - Consider a particle of mass m and charge q moving in the presence of constant and uniform electromagnetic fields given by $\underset{\sim}{E} = \hat{y}\, E_0$ and $\underset{\sim}{B} = \hat{z}\, B_0$. Assuming that initially, at t = 0, the particle is at rest at the origin of a Cartesian coordinate system, show that it moves on the cycloid

$$x(t) = \frac{E_0}{B_0} \left[t - \frac{\sin(\omega_c t)}{\omega_c} \right]$$

$$y(t) = \frac{E_0}{B_0\, \omega_c} \left[1 - \cos(\omega_c t) \right]$$

Plot the trajectory of the particle in the z = 0 plane, for $q > 0$ and $q < 0$, and consider the cases when $v_c > v_E$,

$v_c = v_E$ and $v_c < v_E$, where v_c denotes the particle velocity associated with only its cyclotron motion and v_E is the electromagnetic drift velocity.

2.4 - In general, the trajectory of a charged particle in crossed electric and magnetic fields is a cycloid. Show that, if $\underset{\sim}{v} = \hat{x}\, v_0$, $\underset{\sim}{B} = \hat{z}\, B$ and $\underset{\sim}{E} = \hat{y}\, E$, then for $v_0 = E/B$ the path is a straight line. Explain how this situation can be exploited to design a mass spectrometer.

2.5 - Derive the relativistic equation of motion in the form (1.4), starting from (1.1) and the relation (1.2).

2.6 - Write down, in vector form, the relativistic equation of motion for a charged particle in the presence of a uniform magnetostatic field $\underset{\sim}{B} = \hat{z}\, B_0$, and show that its Cartesian components are given by

$$\frac{d}{dt}\left[\frac{v_x}{(1-\beta^2)^{1/2}}\right] = \frac{q\ B_0}{m}\ v_y$$

$$\frac{d}{dt}\left[\frac{v_y}{(1-\beta^2)^{1/2}}\right] = -\frac{q\,B_o}{m}\,v_x$$

$$\frac{d}{dt}\left[\frac{v_z}{(1-\beta^2)^{1/2}}\right] = 0$$

were $\beta = v/c$. Show that the velocity and trajectory of the charged particle are given by the same formulas as in the nonrelativistic case, but with ω_c replaced by $(|q|\,B_o/m)\,(1-\beta^2)^{1/2}$.

2.7 - Study the motion of a relativistic charged particle in the presence of crossed electric ($\underset{\sim}{E}$) and magnetic ($\underset{\sim}{B}$) fields which are constant in time and uniform in space. What coordinate transformation must be made in order to transform away the transversal electric field? Derive equations for the velocity and trajectory of the charged particle.