1. Classification *INPE-COM.4/RPE*
C.D.U.: 533-9

2. Period

4. Distribution Criterion

internal [ ]

external [X]

3. Key Words (selected by the author)

WAVE PROPAGATION
HOT PLASMAS

5. Report Nº

INPE-1575-RPE/072

6. Date

September, 1979

7. Revised by

René A. Medrano-B.

8. Title and Sub-title

WAVES IN HOT MAGNETIZED PLASMAS

9. Authorized by

Nelson de Jesus Parada
Director

10. Sector *DCE* Code

11. Nº of Copies *09*

12. Authorship *J. A. Bittencourt*

14. Nº of Pages *81*

15. Price

13. Signature of the responsible

16. Summary/Notes

This is the nineteenth chapter, in a series of twenty two, written on the fundamentals of plasma physics. It extends the theory of wave propagation in a hot plasma, presented in the previous chapter, to the case of a hot plasma immersed in an externally applied magnetic field. The characteristics of wave propagation in a plasma consisting of mobile electrons in a neutralizing background of stationary ions are analysed in some detail for directions either parallel or perpendicular to the magnetic field.

17. Remarks

INDEX

CHAPTER 19

WAVES IN HOT MAGNETIZED PLASMAS

List of Figures .... *v*

1. Introduction 1

2. Wave Propagation Along the Magnetostatic Field in a Hot Plasma 2

   2.1 - Linearized Vlasov equation 3

   2.2 - Solution of the linearized Vlasov equation 4

   2.3 - Perturbation current density 12

   2.4 - Separation into the various modes 16

   2.5 - Longitudinal plasma wave 18

   2.6 - Transverse electromagnetic waves 20

   2.7 - Temporal damping of the transverse electromagnetic waves 25

   2.8 - Cyclotron damping of the right circularly polarized transverse wave 28

   2.9 - Instabilities in the right circularly polarized transverse wave 31

3. Wave Propagation Across the Magnetostatic Field in a Hot Plasma 34

   3.1 - Solution of the linearized Vlasov equation 36

   3.2 - Current density and the conductivity tensor 40

   3.3 - Evaluation of the integrals 44

   3.4 - Separation into the various modes 51

3.5 - Dispersion relations 54

3.6 - The quasistatic mode 55

3.7 - The TEM mode 62

4. Summary 64

4.1 - Propagation along $\underset{\sim}{B}_o$ in hot magnetoplasmas 64

4.2 - Propagation across $\underset{\sim}{B}_o$ in hot magnetoplasmas 66

Problems 69

LIST OF FIGURES

Fig. 1 - Cylindrical coordinate system ($v_\perp$, ϕ, $v_\parallel$) in velocity space, with the $v_\parallel$ axis along the magnetostatic field $\underset{\sim}{B}_o$ and $v_\perp$ in the v_x - v_y plane normal to $\underset{\sim}{B}_o$ 5

Fig. 2 - Illustrating the resonance which occurs at $\omega = \omega_{ce}$ between the electrons and the electric field of the right circularly polarized wave propagating along $\underset{\sim}{B}_o$... 30

Fig. 3 - Decomposition of the wave electric field vector into components parallel and perpendicular to $\underset{\sim}{B}_o$, or in components longitudinal and transverse with respect to $\underset{\sim}{k}$ 35

Fig. 4 - Dependence of the function F (ω/ω_{ce}, $\tilde{\nu}$), given by (3.93), in terms of ω/ω_{ce} for a fixed value of $\tilde{\nu}$ (here $\tilde{\nu} = 0.1$), for the quasistatic waves 61

Fig. 5 - Curves of resonant frequencies for the quasistatic waves propagating across the magnetostatic field, as a function of $(\tilde{\nu})^{1/2}$ when $(\omega_{ce}/\omega_{pe})^2 = 0.2$. The resonant frequency, denoted by X, is the normalized upper hybrid frequency, $X = (\omega_{pe}^2 + \omega_{ce}^2)^{1/2}/\omega_{ce}$ 62

CHAPTER 19

WAVES IN HOT MAGNETIZED PLASMAS

1. INTRODUCTION

The analysis of small amplitude waves propagating in a plasma, presented in the previous chapter, is now extended to *anisotropic* plasmas immersed in an externally applied magnetic field. Emphasis is given to the study of the characteristics of plasma waves having their propagation vector, $\underset{\sim}{k}$, either parallel or perpendicular to the externally applied magnetostatic field.

For propagation along the magnetostatic field the plasma waves separate again into three independent groups. The first group is the *longitudinal plasma wave,* and the second and third groups are the *left* and the *right circularly polarized transverse electromagnetic waves.* For propagation across the magnetostatic field the plasma waves separate into two groups, which are designated as the *TM (Transverse Magnetic)* and the *TEM (Transverse Electric-Magnetic)* modes. The longitudinal plasma wave does not exist independently for any orientation of the magnetostatic field other than parallel to $\underset{\sim}{k}$.

The mathematical analysis of the problem of wave propagation at an arbitrary direction relative to the magnetostatic field is more complicated insofar as the details are concerned, and will not be presented here.

2. WAVE PROPAGATION ALONG THE MAGNETOSTATIC FIELD IN A HOT PLASMA

In this section we study the problem of wave propagation in an unbounded plasma consisting of mobile electrons in a neutralizing background of stationary ions, immersed in a uniform magnetostatic field $\underset{\sim}{B}_0$. In the equilibrium state, the number density of the electrons (which is the same as that of the ions) is denoted by n_0.

In the absence of perturbations, the homogeneous equilibrium distribution function of the electrons has to satisfy the zero-order Vlasov equation

$$(\underset{\sim}{v} \times \underset{\sim}{B}_0) \cdot \underset{\sim}{\nabla}_v f_0 (\underset{\sim}{v}) = 0 \tag{2.1}$$

The presence of the magnetostatic field introduces an anisotropy in the distribution function, so that the equilibrium distribution function is denoted by $f_0 (v_{\parallel}, v_{\perp})$, where $v_{\parallel}$ and $v_{\perp}$ represent the velocity of the electrons in directions parallel and perpendicular to $\underset{\sim}{B}_0$, respectively.

2.1 - Linearized Vlasov equation

As before, the perturbed distribution function is assumed to consist of a small perturbation, $f_1(\underset{\sim}{r}, \underset{\sim}{v}, t)$, superimposed on the equilibrium distribution function, $f_o(v_\parallel, v_\perp)$, that is

$$f(\underset{\sim}{r}, \underset{\sim}{v}, t) = f_o(v_\parallel, v_\perp) + f_1(\underset{\sim}{r}, \underset{\sim}{v}, t) \tag{2.2}$$

where $|f_1| << f_o$. The electric field, $\underset{\sim}{E}(\underset{\sim}{r}, t)$, and the magnetic field, $\underset{\sim}{B}(\underset{\sim}{r}, t)$, related to the charge density and current density inside the plasma, and which are associated with the first order perturbation $f_1(\underset{\sim}{r}, \underset{\sim}{v}, t)$, are also first order quantities. Note, however, that $\underset{\sim}{E}(\underset{\sim}{r}, t)$ denotes the total electric field inside the plasma, whereas the total magnetic field $\underset{\sim}{B}_T(\underset{\sim}{r}, t)$ is given by

$$\underset{\sim}{B}_T(\underset{\sim}{r}, t) = \underset{\sim}{B}_o + \underset{\sim}{B}(\underset{\sim}{r}, t) \tag{2.3}$$

Substituting Eqs. (2.2) and (2.3) into the Vlasov equation (18.2.1), neglecting all second order terms, and noting that the equilibrium distribution function is homogeneous, results in the following *linearized* Vlasov equation

$$\frac{\partial}{\partial t} f_1(\underset{\sim}{r}, \underset{\sim}{v}, t) + \underset{\sim}{v} \cdot \underset{\sim}{\nabla} f_1(\underset{\sim}{r}, \underset{\sim}{v}, t) - \frac{e}{m_e} \left[\underset{\sim}{E}(\underset{\sim}{r}, t) + \underset{\sim}{v} \times \underset{\sim}{B}(\underset{\sim}{r}, t)\right] \cdot$$

$$\cdot \underset{\sim}{\nabla}_v f_o(v_\parallel, v_\perp) - \frac{e}{m_e} (\underset{\sim}{v} \times \underset{\sim}{B}_o) \cdot \underset{\sim}{\nabla}_v f_1(\underset{\sim}{r}, \underset{\sim}{v}, t) = 0 \tag{2.4}$$

2.2 - Solution of the linearized Vlasov equation

For the purpose of investigating the characteristics of plane waves propagating along the magnetostatic field, we shall assume that the space-time dependence of all physical quantities is a periodic harmonic dependence of the form exp $(i\,\underset{\sim}{k} \cdot \underset{\sim}{r} - i\,\omega\,t)$, that is,

$$\underset{\sim}{E}(\underset{\sim}{r}, t) = \underset{\sim}{E} \exp(i\,\underset{\sim}{k}\cdot\underset{\sim}{r} - i\,\omega\,t) \tag{2.5}$$

$$\underset{\sim}{B}(\underset{\sim}{r}, t) = \underset{\sim}{B} \exp(i\,\underset{\sim}{k}\cdot\underset{\sim}{r} - i\,\omega\,t) \tag{2.6}$$

$$f_1(\underset{\sim}{r}, \underset{\sim}{v}, t) = f_1(\underset{\sim}{v}) \exp(i\,\underset{\sim}{k}\cdot\underset{\sim}{r} - i\,\omega\,t) \tag{2.7}$$

where $\underset{\sim}{E}$, $\underset{\sim}{B}$ and $f_1(\underset{\sim}{v})$ are phasor amplitudes (which in general may be complex quantities), independent of space and time. With this space-time dependence, the differential operators $\partial/\partial t$ and $\underset{\sim}{\nabla}$ in Eq. (2.4) are replaced by $-i\,\omega$ and $i\,\underset{\sim}{k}$, respectively, so that the linearized Vlasov equation (2.4) reduces to

$$-i(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v})\, f_1(\underset{\sim}{v}) - \frac{e}{m_e}(\underset{\sim}{v} \times \underset{\sim}{B}_o)\cdot\underset{\sim}{\nabla}_v f_1(\underset{\sim}{v}) =$$

$$= \frac{e}{m_e}(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B})\cdot\underset{\sim}{\nabla}_v f_o(v_{\parallel}, v_{\perp}) \tag{2.8}$$

To solve this differential equation for $f_1(\underset{\sim}{v})$ in velocity space, we introduce cylindrical coordinates $(v_{\perp}, \phi, v_{\parallel})$

with the vector component $\underset{\sim}{v}_{\parallel}$ along the magnetostatic field, as shown in Fig. 1. Therefore, $\underset{\sim}{B}_o = B_o \hat{\underset{\sim}{z}}$ and

$$v_x = v_\perp \cos\phi; \qquad v_y = v_\perp \sin\phi; \qquad v_z = v_\parallel \tag{2.9}$$

Also, using these relations, we have

$$\frac{d\, f_1(\underset{\sim}{v})}{d\phi} = \left(\frac{d\, v_x}{d\phi}\frac{\partial}{\partial v_x} + \frac{dv_y}{d\phi}\frac{\partial}{\partial v_y} + \frac{dv_z}{d\phi}\frac{\partial}{\partial v_z}\right) f_1(\underset{\sim}{v})$$

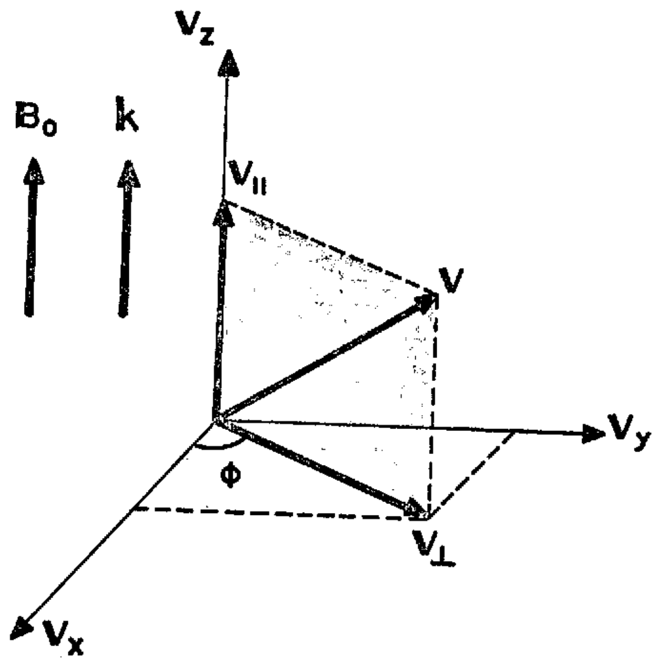

Fig. 1 - Cylindrical coordinate system ($v_\perp$, ϕ, $v_\parallel$) in velocity space, with the $v_\parallel$ axis along the magnetostatic field $\underset{\sim}{B}_o$ and $v_\perp$ in the v_x - v_y plane normal to $\underset{\sim}{B}_o$.

$$= \left(- v_y \frac{\partial}{\partial v_x} + v_x \frac{\partial}{\partial v_y}\right) f_1 (\underset{\sim}{v})$$

$$= - (\underset{\sim}{v} \times \hat{\underset{\sim}{z}}) \cdot \underset{\sim}{\nabla}_v f_1 (\underset{\sim}{v}) \tag{2.10}$$

Substituting this result into Eq. (2.8), we obtain

$$- i (\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}) f_1 (\underset{\sim}{v}) + \frac{e B_o}{m_e} \frac{d f_1 (\underset{\sim}{v})}{d \phi} = \frac{e}{m_e} (\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) \cdot \underset{\sim}{\nabla}_v f_o (v_{\parallel}, v_{\perp}) \tag{2.11}$$

Using the electron cyclotron frequency $\omega_{ce} = e B_o/m_e$, (2.11) can be rewritten as

$$\frac{d f_1(\underset{\sim}{v})}{d \phi} - \frac{i (\omega - \underset{\sim}{k} \cdot \underset{\sim}{v})}{\omega_{ce}} f_1 (\underset{\sim}{v}) = \frac{e}{m_e \omega_{ce}} (\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) \cdot \underset{\sim}{\nabla}_v f_o (v_{\parallel}, v_{\perp}) \tag{2.12}$$

From Maxwell $\underset{\sim}{\nabla} \times \underset{\sim}{E}$ equation we can express the magnetic field as

$$\underset{\sim}{B} = (\underset{\sim}{k} \times \underset{\sim}{E})/\omega \tag{2.13}$$

Substituting (2.13) into (2.12), and making use of the vector identity $\underset{\sim}{v} \times (\underset{\sim}{k} \times \underset{\sim}{E}) = (\underset{\sim}{v} \cdot \underset{\sim}{E}) \underset{\sim}{k} - (\underset{\sim}{k} \cdot \underset{\sim}{v}) \underset{\sim}{E}$, we obtain for the right-hand side of (2.12),

$$\frac{e}{m_e \omega_{ce}} (\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) \cdot \underset{\sim}{\nabla}_v f_o = \frac{e}{m_e \omega_{ce}} \left[\left(1 - \frac{k v_{\parallel}}{\omega}\right) \underset{\sim}{E} \cdot \underset{\sim}{\nabla}_v f_o + \right.$$

$$+ \frac{k\,(\underset{\sim}{v}\cdot\underset{\sim}{E})}{\omega}\;\frac{\partial f_o}{\partial v_\parallel}\Bigg]$$

$$= \frac{e}{m_e\,\omega_{ce}}\left\{\left(1-\frac{kv_\parallel}{\omega}\right)\left[(E_x\cos\phi + E_y\sin\phi)\,\frac{\partial f_o}{\partial v_\perp} + E_\parallel\,\frac{\partial f_o}{\partial v_\parallel}\right] + \right.$$

$$\left. + \left(\frac{k}{\omega}\right)\left[(E_x\cos\phi + E_y\sin\phi)\,v_\perp + E_\parallel\,v_\parallel\right]\frac{\partial f_o}{\partial v_\parallel}\right\}$$

$$= \frac{e}{m_e\,\omega_{ce}}\left\{\left[\left(1-\frac{kv_\parallel}{\omega}\right)\frac{\partial f_o}{\partial v_\perp} + \left(\frac{kv_\perp}{\omega}\right)\frac{\partial f_o}{\partial v_\parallel}\right](E_x\cos\phi + E_y\sin\phi) + \right.$$

$$\left. + E_\parallel\,\frac{\partial f_o}{\partial v_\parallel}\right\} \qquad (2.14)$$

where we have taken $\underset{\sim}{k}\cdot\underset{\sim}{v} = k\,v_\parallel$ and $\underset{\sim}{k}\cdot\underset{\sim}{\nabla}_v = k\,\partial/\partial v_\parallel$, since we are considering wave propagation parallel to the magnetic field ($\underset{\sim}{k} \parallel \underset{\sim}{B}_o$).

At this point it is convenient to express the component of the electric field vector in the plane perpendicular to $\underset{\sim}{B}_o$ as a linear superposition of two, oppositely directed, circularly polarized components. Noting that $(\hat{\underset{\sim}{x}} + i\,\hat{\underset{\sim}{y}})/\sqrt{2}$ and $(\hat{\underset{\sim}{x}} - i\,\hat{\underset{\sim}{y}})/\sqrt{2}$ are unit complex vectors, the Cartesian components of the electric field vector

$$\underset{\sim}{E} = \hat{\underset{\sim}{x}}\,E_x + \hat{\underset{\sim}{y}}\,E_y + \hat{\underset{\sim}{z}}\,E_\parallel \qquad (2.15)$$

can be appropriately rewritten as

$$\underset{\sim}{E} = E_+ \frac{(\hat{\underset{\sim}{x}} + i\hat{\underset{\sim}{y}})}{\sqrt{2}} + E_- \frac{(\hat{\underset{\sim}{x}} - i\hat{\underset{\sim}{y}})}{\sqrt{2}} + \hat{\underset{\sim}{z}}\, E_\parallel \tag{2.16}$$

where the following notation is used

$$E_\pm = \frac{1}{\sqrt{2}} \; (E_x \mp iE_y) \tag{2.17}$$

The first term in the right-hand side of (2.16) represents a circularly polarized field with the electric field vector rotating in the clockwise direction, whereas the second term represents a circularly polarized field with the electric field vector rotating in the counterclockwise direction. For the right (left) circularly polarized field, with the thumb of the right (left) hand pointing in the direction of propagation ($\hat{\underset{\sim}{z}}$), the fingers curl in the direction of rotation of the electric field vector. Thus, the two linearly polarized perpendicular components of the electric field in the plane (x, y), normal to $\underset{\sim}{B}_0$, can be recast as a superposition of two circularly polarized components with opposite directions of rotation. The advantage of using the two circularly polarized components is that it permits the final equations, involving the transverse modes of propagation, to be separated into two independent sets of transverse fields.

It is a trivial matter to verify the relation

$$E_x \cos\phi + E_y \sin\phi = \frac{1}{\sqrt{2}} (E_+ e^{i\phi} + E_- e^{-i\phi}) \tag{2.18}$$

so that Eq. (2.12) can be rewritten as

$$\frac{d f_1(\underset{\sim}{v})}{d\phi} - \frac{i(\omega - kv_\parallel)}{\omega_{ce}} f_1(\underset{\sim}{v}) = \frac{e}{m_e \omega_{ce}} \left\{ \left[(1 - \frac{kv_\parallel}{\omega}) \frac{\partial f_o}{\partial v_\perp} + (\frac{k v_\perp}{\omega}) \frac{\partial f_o}{\partial v_\parallel} \right] \frac{1}{\sqrt{2}} (E_+ e^{i\phi} + E_- e^{-i\phi}) + E_\parallel \frac{\partial f_o}{\partial v_\parallel} \right\} \tag{2.19}$$

Introducing the notation

$$F_+(\underset{\sim}{v}) = F_+(v_\parallel, v_\perp)\, e^{i\phi} \tag{2.20}$$

$$F_-(\underset{\sim}{v}) = F_-(v_\parallel, v_\perp)\, e^{-i\phi} \tag{2.21}$$

$$F_\parallel(\underset{\sim}{v}) = F_\parallel(v_\parallel, v_\perp) \tag{2.22}$$

where

$$F_+(v_\parallel, v_\perp) = \frac{e}{m_e \omega_{ce}} \left[(1 - \frac{kv_\parallel}{\omega}) \frac{\partial f_o}{\partial v_\perp} + (\frac{kv_\perp}{\omega}) \frac{\partial f_o}{\partial v_\parallel} \right] \frac{E_+}{\sqrt{2}} \tag{2.23}$$

$$F_-(v_\parallel, v_\perp) = \frac{e}{m_e\,\omega_{ce}} \left[\left(1 - \frac{kv_\parallel}{\omega}\right) \frac{\partial f_o}{\partial v_\perp} + \left(\frac{kv_\perp}{\omega}\right) \frac{\partial f_o}{\partial v_\parallel} \right] \frac{E_-}{\sqrt{2}} \tag{2.24}$$

$$F_\parallel(v_\parallel, v_\perp) = \frac{e}{m_e\,\omega_{ce}} \left(\frac{\partial f_o}{\partial v_\parallel}\right) E_\parallel \tag{2.25}$$

Eq. (2.19) becomes

$$\frac{d\, f_1(\underset{\sim}{v})}{d\,\phi} - \frac{i\,(\omega - kv_\parallel)}{\omega_{ce}}\, f_1(\underset{\sim}{v}) = F_+(\underset{\sim}{v}) + F_-(\underset{\sim}{v}) + F_\parallel(\underset{\sim}{v}) \tag{2.26}$$

In order to solve this differential equation, let

$$f_1(\underset{\sim}{v}) = f_{1+}(\underset{\sim}{v}) + f_{1-}(\underset{\sim}{v}) + f_{1\parallel}(\underset{\sim}{v}) \tag{2.27}$$

where $f_{1+}(\underset{\sim}{v})$, $f_{1-}(\underset{\sim}{v})$ and $f_{1\parallel}(\underset{\sim}{v})$ are the solutions of (2.26) corresponding, respectively, to $F_+(\underset{\sim}{v})$, $F_-(\underset{\sim}{v})$ and $F_\parallel(\underset{\sim}{v})$, individually, in the right hand side of (2.26). Thus, the differential equation for $f_{1+}(\underset{\sim}{v})$, for example, can be written as

$$\frac{d}{d\phi} \left\{ f_{1+}(\underset{\sim}{v}) \exp\left[\frac{-i\,(\omega - kv_\parallel)}{\omega_{ce}}\, \phi \right] \right\} =$$

$$= F_+(v_\parallel, v_\perp) \exp\left[\frac{-i\,(\omega - kv_\parallel)}{\omega_{ce}}\, \phi + i\,\phi \right] \tag{2.28}$$

Integrating both sides of this equation with respect to ϕ, from $\phi = -\infty$ to an arbitrary value of ϕ, and noting that the exponential term vanishes at $\phi = -\infty$, since ω has a vanishingly small positive imaginary part, yields

$$f_{1+}(\underset{\sim}{v}) = \frac{i\,\omega_{ce}}{(\omega - kv_{\parallel} - \omega_{ce})} F_{+}(v_{\parallel}, v_{\perp})\, e^{i\phi} + C_{+} \exp\left[\frac{i\,(\omega - kv_{\parallel})}{\omega_{ce}}\,\phi\right] \quad (2.29)$$

The value of $f_{1+}(\underset{\sim}{v})$ must not change if ϕ is increased or decreased by integral multiples of 2π, since by physical arguments $f_{1+}(\underset{\sim}{v})$ must be a unique function of $\bar{v}$. This requirement can be satisfied only if $C_{+} = 0$. Therefore, we obtain

$$f_{1+}(\underset{\sim}{v}) = f_{1+}(v_{\parallel}, v_{\perp})\, e^{i\phi} \quad (2.30)$$

where

$$f_{1+}(v_{\parallel}, v_{\perp}) = \frac{i\,\omega_{ce}}{(\omega - kv_{\parallel} - \omega_{ce})} F_{+}(v_{\parallel}, v_{\perp}) \quad (2.31)$$

In a similar way, we find

$$f_{1-}(\underset{\sim}{v}) = f_{1-}(v_{\parallel}, v_{\perp})\, e^{-i\phi} \quad (2.32)$$

where

$$f_{1-}(v_{\parallel}, v_{\perp}) = \frac{i\,\omega_{ce}}{(\omega - kv_{\parallel} + \omega_{ce})} F_{-}(v_{\parallel}, v_{\perp}) \quad (2.33)$$

and

$$f_{1\parallel}(\underset{\sim}{v}) = f_{1\parallel}(v_\parallel, v_\perp) = \frac{i\,\omega_{ce}}{(\omega - kv_\parallel)}\, F_\parallel(v_\parallel, v_\perp) \tag{2.34}$$

Substituting expressions (2.23), (2.24) and (2.25) for $F_+(v_\parallel, v_\perp)$, $F_-(v_\parallel, v_\perp)$ and $F_\parallel(v_\parallel, v_\perp)$, respectively, into Eqs. (2.31), (2.33) and (2.34), yields the following explicit expression for the phasor amplitude $f_1(\underset{\sim}{v})$ of the perturbation of the velocity distribution function, in terms of the equilibrium distribution function of the electrons,

$$f_1(\underset{\sim}{v}) = \frac{i\,e}{m_e(\omega - kv_\parallel - \omega_{ce})}\left[(1 - \frac{kv_\parallel}{\omega})\frac{\partial f_o}{\partial v_\perp} + (\frac{kv_\perp}{\omega})\frac{\partial f_o}{\partial v_\parallel}\right]\frac{E_+}{\sqrt{2}}\, e^{i\phi} +$$

$$+ \frac{i\,e}{m_e(\omega - kv_\parallel + \omega_{ce})}\left[(1 - \frac{kv_\parallel}{\omega})\frac{\partial f_o}{\partial v_\perp} + (\frac{kv_\perp}{\omega})\frac{\partial f_o}{\partial v_\parallel}\right]\frac{E_-}{\sqrt{2}}\, e^{-i\phi} +$$

$$+ \frac{i\,e}{m_e(\omega - kv_\parallel)}\,(\frac{\partial f_o}{\partial v_\parallel})\, E_\parallel \tag{2.35}$$

2.3 - Perturbation current density

Since the space-time dependence of the electromagnetic fields are of the form $\exp(i\,\underset{\sim}{k}\cdot\underset{\sim}{r} - i\,\omega\, t)$, we expect the current density to behave also as

$$\underset{\sim}{J}(\underset{\sim}{r}, t) = \underset{\sim}{J}\exp(i\,\underset{\sim}{k}\cdot\underset{\sim}{r} - i\,\omega\, t) \tag{2.36}$$

where the phasor amplitude of the current density is given by

$$\underset{\sim}{J} = -e \int_V \underset{\sim}{v} \, f_1 \, (\underset{\sim}{v}) \, d^3v$$

where the integration is to be performed over all of velocity space. It is also convenient to separate $\underset{\sim}{J}$ into two, oppositely directed, circularly polarized components, and a longitudinal component along $\underset{\sim}{B}_o$. For this purpose, we express the electron velocity in a form analogous to (2.16),

$$\underset{\sim}{v} = v_+ \frac{(\underset{\sim}{x} + i\hat{\underset{\sim}{y}})}{\sqrt{2}} + v_- \frac{(\hat{\underset{\sim}{x}} - i\hat{\underset{\sim}{y}})}{\sqrt{2}} + \hat{z} \, v_{\parallel} \tag{2.38}$$

where

$$v_{\pm} = \frac{1}{\sqrt{2}} (v_x \mp i \, v_y) \tag{2.39}$$

Thus, with this representation for $\underset{\sim}{v}$, we obtain the following corresponding components for $\underset{\sim}{J}$,

$$J_+ = -e \int_V v_+ \, f_1 \, (\underset{\sim}{v}) \, d^3v \tag{2.40}$$

$$J_- = -e \int_V v_- \, f_1 \, (\underset{\sim}{v}) \, d^3v \tag{2.41}$$

$$J_{\parallel} = - e \int_V v_{\parallel} f_1 (\underset{\sim}{v}) d^3v \qquad (2.42)$$

According to Eqs. (2.27), (2.30), (2.32) and (2.34),we can replace $f_1 (\underset{\sim}{v})$ by

$$f_1 (\underset{\sim}{v}) = f_{1+} (v_{\parallel}, v_{\perp}) e^{i\phi} + f_{1-} (v_{\parallel}, v_{\perp}) e^{-i\phi} + f_{1\parallel} (v_{\parallel}, v_{\perp}) \qquad (2.43)$$

Further, in view of (2.9), we also have

$$v_+ = \frac{1}{\sqrt{2}} v_{\perp} e^{-i\phi} \qquad v_- = \frac{1}{\sqrt{2}} v_{\perp} e^{+i\phi} \qquad (2.44)$$

so that Eqs. (2.40), (2.41) and (2.42) become

$$J_+ = - \frac{e}{\sqrt{2}} \int_V v_{\perp} e^{-i\phi} \left[f_{1+} (v_{\parallel}, v_{\perp}) e^{i\phi} + f_{1-}(v_{\parallel}, v_{\perp}) e^{-i\phi} + \right.$$

$$\left. + f_{1\parallel} (v_{\parallel}, v_{\perp}) \right] d^3v \qquad (2.45)$$

$$J_- = - \frac{e}{\sqrt{2}} \int_V v_{\perp} e^{+i\phi} \left[f_{1+} (v_{\parallel}, v_{\perp}) e^{i\phi} + f_{1-} (v_{\parallel}, v_{\perp}) e^{-i\phi} + \right.$$

$$\left. + f_{1\parallel} (v_{\parallel}, v_{\perp}) \right] d^3v \qquad (2.46)$$

$$J_{\parallel} = - e \int_V v_{\parallel} \left[f_{1+} (v_{\parallel}, v_{\perp}) e^{i\phi} + f_{1-} (v_{\parallel}, v_{\perp}) e^{-i\phi} + f_{1\parallel} (v_{\parallel}, v_{\perp}) \right] d^3v \qquad (2.47)$$

In cylindrical coordinates we have $d^3v = v_{\perp}\, dv_{\perp}\, dv_{\parallel}\, d\phi$. Evaluating the integrals with respect to ϕ, from $\phi = 0$ to $\phi = 2\pi$, yields the following simple results

$$J_+ = - e \pi \sqrt{2} \int_0^{\infty} v_{\perp}^2\, dv_{\perp} \int_{-\infty}^{+\infty} f_{1+} (v_{\parallel}, v_{\perp})\, dv_{\parallel} \qquad (2.48)$$

$$J_- = - e \pi \sqrt{2} \int_0^{\infty} v_{\perp}^2\, dv_{\perp} \int_{-\infty}^{+\infty} f_{1-} (v_{\parallel}, v_{\perp})\, dv_{\parallel} \qquad (2.49)$$

$$J_{\parallel} = - e \pi\, 2 \int_0^{\infty} v_{\perp}\, dv_{\perp} \int_{-\infty}^{+\infty} v_{\parallel} f_{1\parallel} (v_{\parallel}, v_{\perp})\, dv_{\parallel} \qquad (2.50)$$

since

$$\int_0^{2\pi} e^{\pm in\phi}\, d\phi = 0 \quad ; \quad \text{for} \quad n = 1, 2, 3, \ldots \qquad (2.51)$$

$$= 2\pi \quad ; \quad \text{for} \quad n = 0$$

From Eqs. (2.31), (2.33) and (2.34), together with Eqs. (2.23), (2.24) and (2.25), we see that J_+, J_- and $J_{\parallel}$ depend, respectively, on *only*

E_+, E_- and $E_{\parallel}$. This result justifies the use of the method of decomposition of the field vectors into the sum of two, oppositely directed, circularly polarized components in the plane normal to $\underset{\sim}{B}_o$, and a longitudinal component along $\underset{\sim}{B}_o$.

2.4 - Separation into the various modes

From Maxwell equations, and for the special case in which all field vectors vary as exp $(i\,\underset{\sim}{k}\cdot\underset{\sim}{r} - i\,\omega\,t)$, with $\underset{\sim}{k} = k\hat{\underset{\sim}{z}}$, we have

$$k\hat{\underset{\sim}{z}} \times \underset{\sim}{E} = \omega\,\underset{\sim}{B} \tag{2.52}$$

$$i\,k\,\hat{\underset{\sim}{z}} \times \underset{\sim}{B} = \mu_o\,\underset{\sim}{J} - \frac{i\omega}{c^2}\,\underset{\sim}{E} \tag{2.53}$$

Noting that $\hat{\underset{\sim}{z}} \times \underset{\sim}{E} = \hat{\underset{\sim}{y}}\,E_x - \hat{\underset{\sim}{x}}\,E_y$, Eqs. (2.52) and (2.53) can be rewritten in component form as

$$\omega\,B_x = -k\,E_y \tag{2.54}$$

$$\omega\,B_y = k\,E_x \tag{2.55}$$

$$\omega\,B_{\parallel} = 0 \tag{2.56}$$

and

$$-i\,k\,B_y = \mu_o\,J_x - \frac{i\omega}{c^2}\,E_x \tag{2.57}$$

$$i\,k\,B_x = \mu_o\,J_y - \frac{i\omega}{c^2}\,E_y \tag{2.58}$$

$$0 = \mu_o J_\parallel - \frac{i\omega}{c^2} E_\parallel \tag{2.59}$$

Now we define, as in (2.17),

$$B_\pm = \frac{1}{\sqrt{2}} (B_x \mp i B_y) \tag{2.60}$$

Multiplying (2.54) by $1/\sqrt{2}$, and (2.55) by $\mp i/\sqrt{2}$, and adding the resulting expressions, yields

$$B_\pm = \mp i \frac{k}{\omega} E_\pm \tag{2.61}$$

Note that the signs are coupled, that is, *either* upper signs or lower signs are to be used. Similarly, combining Eqs. (2.57) and (2.58), and noting that

$$J_\pm = \frac{1}{\sqrt{2}} (J_x \mp i J_y) \tag{2.62}$$

we obtain

$$\mp k B_\pm = - \mu_o J_\pm + \frac{i\omega}{c^2} E_\pm \tag{2.63}$$

From these equations it is clear that the total electromagnetic field can be separated into four *independent* groups, involving the following quantities:

1. $J_{\parallel}$, $E_{\parallel}$ [Eq. (2.59)]

2. $B_{\parallel}$ [Eq. (2.56)]

3. J_-, E_-, B_- [Eqs. (2.61) and (2.63), lower signs]

4. J_+, E_+, B_+ [Eqs. (2.61) and (2.63), upper signs]

Note that J_+, J_- and $J_{\parallel}$ depend, respectively, only on E_+, E_- and $E_{\parallel}$. The first group contains an electric field and an electric current in the direction of $\underset{\sim}{k}$ which, in this section, is also the direction of $\underset{\sim}{B}_0$. Further, there is no associated magnetic field. Therefore, it represents the electrostatic *longitudinal plasma wave*. The second group does not constitute a mode of propagation but only shows, through (2.56), that for a wave propagating parallel to $\underset{\sim}{B}_0$ the time--varying magnetic field in the parallel direction is zero. The third and fourth groups represent, respectively, the *left circularly polarized* and the *right circularly polarized transverse electromagnetic waves*. Thus, we can separately analyse the characteristics of the longitudinal plasma wave and the two circularly polarized transverse electromagnetic waves.

2.5 - Longitudinal plasma wave

To deduce the dispersion relation for the longitudinal plasma wave propagating along the magnetostatic field $\underset{\sim}{B}_0$, we substitute $J_{\parallel}$, from Eq. (2.50), into Eq. (2.59), obtaining

$$E_{\parallel} = + \frac{i\, 2\pi\, e}{\varepsilon_0\, \omega} \int_0^{\infty} v_{\perp}\, dv_{\perp} \int_{-\infty}^{+\infty} v_{\parallel}\, f_{1\parallel}\, (v_{\parallel}, v_{\perp})\, dv_{\parallel} \tag{2.64}$$

From Eqs. (2.34) and (2.25) we can replace $f_{1\parallel}\,(v_{\parallel}, v_{\perp})$ in Eq. (2.64), to obtain the following *dispersion relation*

$$1 = - \frac{\omega_{pe}^2}{n_0\, \omega}\, 2\pi \int_0^{\infty} v_{\perp}\, dv_{\perp} \int_{-\infty}^{+\infty} \frac{v_{\parallel}\, (\partial f_0/\partial v_{\parallel})}{(\omega - k\, v_{\parallel})}\, dv_{\parallel} \tag{2.65}$$

This dispersion relation can be conveniently recasted as

$$1 = - \frac{\omega_{pe}^2}{n_0\, \omega} \int_v \frac{v_{\parallel}\, (\partial f_0/\partial v_{\parallel})}{(\omega - k v_{\parallel})}\, d^3v \tag{2.66}$$

since in cylindrical coordinates $d^3v = v_{\perp}\, dv_{\perp}\, dv_{\parallel}\, d\phi$ and $\int_0^{2\pi} d\phi = 2\pi$.

This equation is identical to the dispersion equation (18.4.2), deduced for longitudinal waves in an isotropic plasma, except for the fact that the directions x and z are interchanged (here $\underset{\sim}{k} \parallel \underset{\sim}{B}_0 \parallel \hat{\underset{\sim}{z}}$). Thus, the characteristic behavior of the longitudinal plasma wave, for propagation along the magnetostatic field, is identical to the case of the plasma with no external magnetostatic field. The magnetostatic field, therefore, has no influence on the longitudinal plasma wave. This result is due to the fact that the magnetostatic field exerts no force on the charged particles moving in the direction parallel to it, and therefore it does not influence the distribution of the electrons in the longitudinal direction. It is the perturbation in the distribution of the velocities of the electrons in the longitudinal direction that accounts for the characteristics of the longitudinal plasma wave. Recall that the longitudinal plasma wave separates out as an independent mode of propagation.

2.6 - Transverse electromagnetic waves

Consider now the two circularly polarized transverse waves ($\underset{\sim}{E}$ normal to the direction of propagation). To deduce the dispersion relation for both waves, we first eliminate $B_{\pm}$ from Eqs. (2.61) and (2.63), and express $J_{\pm}$ in terms of $E_{\pm}$ as

$$J_{\pm} = \frac{i\,\varepsilon_o}{\omega}\,(\omega^2 - k^2 c^2)\,E_{\pm} \tag{2.67}$$

Substituting $J_{\pm}$, from Eqs. (2.48) and (2.49), with $f_{1+}(v_{\parallel}, v_{\perp})$ and $f_{1-}(v_{\parallel}, v_{\perp})$ given by Eqs. (2.31) and (2.33), respectively, yields

$$-e\,\pi\,\sqrt{2}\,\omega_{ce}\int_0^{+\infty} v_{\perp}^2\,dv_{\perp}\int_{-\infty}^{+\infty}\frac{F_{\pm}(v_{\parallel}, v_{\perp})}{(\omega - kv_{\parallel} \pm \omega_{ce})}\,dv_{\parallel} = \frac{\varepsilon_o}{\omega}\,(\omega^2 - k^2 c^2)\,E_{\pm} \tag{2.68}$$

If Eqs. (2.23) and (2.24) are used to replace $F_{\pm}(v_{\parallel}, v_{\perp})$, we find the following *dispersion relation* for the transverse electromagnetic waves ($E_{\pm} \neq 0$)

$$k^2 c^2 = \omega^2 + \omega_{pe}^2 \frac{\pi}{n_o} \int_0^\infty v_\perp^2 \, dv_\perp \int_{-\infty}^{+\infty} \frac{(\omega - kv_\parallel)(\partial f_o/\partial v_\perp) + kv_\perp \, (\partial f_o/\partial v_\parallel)}{(\omega - kv_\parallel \mp \omega_{ce})} \, dv_\parallel \tag{2.69}$$

where the upper sign refers to the right circularly polarized wave, and the lower sign to the left circularly polarized wave.

An alternative form of this equation can be obtained by integrating the right-hand side by parts. First, integrating over $v_\parallel$ by parts, we have

$$\int_0^\infty v_\perp^2 \, dv_\perp \int_{-\infty}^{+\infty} \frac{kv_\perp \, (\partial f_o/\partial v_\parallel)}{(\omega - kv_\parallel \mp \omega_{ce})} \, dv_\parallel =$$

$$= - \int_0^\infty \int_{-\infty}^{+\infty} \frac{(k^2 \, v_\perp^2) \, f_o}{(\omega - kv_\parallel \mp \omega_{ce})^2} \, v_\perp \, dv_\perp \, dv_\parallel \tag{2.70}$$

and integrating over $v_\perp$ by parts, we have

$$\int_{-\infty}^{+\infty} \frac{(\omega - kv_\parallel) \, dv_\parallel}{(\omega - kv_\parallel \mp \omega_{ce})} \int_0^\infty v_\perp^2 (\partial f_o/\partial v_\perp) dv_\perp =$$

$$= - 2 \int_0^\infty \int_{-\infty}^{+\infty} \frac{(\omega - kv_\parallel) \, f_o}{(\omega - kv_\parallel \mp \omega_{ce})} \, v_\perp \, dv_\perp \, dv_\parallel \tag{2.71}$$

Since $2\pi = \int_0^{2\pi} d\phi$ and $d^3v = v_\perp \, dv_\perp \, dv_\parallel \, d\phi$, we obtain the following alternative form of the dispersion relation (2.69)

$$k^2 c^2 = \omega^2 - \frac{\omega_{pe}^2}{n_o} \int\int_v \left[\frac{(\omega - kv_\parallel)}{(\omega - kv_\parallel \mp \omega_{ce})} + \right.$$

$$\left. + \frac{(k^2 v_\perp^2)/2}{(\omega - kv_\parallel \mp \omega_{ce})^2} \right] f_o \, d^3v \qquad (2.72)$$

Let us investigate first the plasma behavior for the case of an *isotropic* equilibrium distribution function. Thus, we choose f_o (v) to be the Maxwell-Boltzmann distribution function (18.4.22). Note that, in this case, the vector $\nabla_v f_o$ (v) is parallel to $\mathbf{v}$, so that the magnetic force term $[\mathbf{v} \times \mathbf{B} (\mathbf{r}, t)] \cdot \nabla_v f_o$ (v), in the linearized Vlasov equation (2.4), vanishes. Consequently, for an *isotropic* equilibrium distribution function, the magnetic field $\mathbf{B}$ ($\mathbf{r}$, t) of the *wave* has no influence on the plasma behavior in the linear approximation. Also, it is easy to verify that, in the isotropic case, all factors in the numerator of the integrands in Eqs. (2.69) and (2.72), which contain the propagation coefficient k, vanish. The dispersion equation (2.69) then reduces to

$$k^2 c^2 = \omega^2 + \frac{\omega_{pe}^2 \, \pi}{n_o} \int_0^\infty v_\perp^2 \, dv_\perp \int_{-\infty}^{+\infty} \frac{\omega \, (\partial f_o / \partial v_\perp)}{(\omega - kv_\parallel \mp \omega_{ce})} \, dv_\parallel \qquad (2.73)$$

or, equivalently,

$$k^2 c^2 = \omega^2 + \frac{\omega_{pe}^2 \omega}{2n_o} \int_v \frac{v_\perp (\partial f_o/\partial v_\perp)}{(\omega \mp \omega_{ce}) - kv_\parallel} d^3v \tag{2.74}$$

The alternative form of this equation, corresponding to (2.72) for the isotropic case, is

$$k^2 c^2 = \omega^2 - \frac{\omega_{pe}^2 \omega}{n_o} \int_v \frac{f_o}{(\omega \mp \omega_{ce}) - kv_\parallel} d^3v \tag{2.75}$$

Substituting f_o (v) from (18.4.22) and performing the integration over $v_\perp$ and ϕ, the dispersion relation (2.75) becomes

$$k^2 c^2 = \omega^2 - \omega_{pe}^2 \omega \left(\frac{m_e}{2\pi k_B T_e}\right)^{1/2} \int_{-\infty}^{+\infty} \frac{\exp\left(-m_e v_\parallel^2/2k_B T_e \right)}{(\omega \mp \omega_{ce}) - kv_\parallel} dv_\parallel \tag{2.76}$$

where the upper and the lower signs correspond to the right and the left circularly polarized waves, respectively.

At this point it is convenient to introduce the following dimensionless parameters

$$\alpha_\pm = \frac{(\omega \mp \omega_{ce})/k_\pm}{(2 k_B T_e/m_e)^{1/2}} \tag{2.77}$$

$$\beta_\pm = \frac{\omega/k_\pm}{(2k_B T_e/m_e)^{1/2}} \tag{2.78}$$

The subscripts + and - are used in k to denote that it corresponds either to the right or to the left circularly polarized wave, respectively. Thus, $\beta_\pm$ represents the phase velocity of the wave normalized to the most probable speed of the electrons $(2\, k_B\, T_e/m_e)^{1/2}$. Setting, as in (18.4.25),

$$q = \frac{v_\parallel}{(2\, k_B\, T_e/\, m_e)^{1/2}} \tag{2.79}$$

the dispersion relation (2.76) can be rewritten in the following simplified form

$$k_\pm^2\, c^2 = \omega^2 + \omega_{pe}^2\, \beta_\pm\, I\, (\alpha_\pm) \tag{2.80}$$

where $I\,(\alpha_\pm)$ denotes the integral

$$I\,(\alpha_\pm) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{\exp\,(-q^2)}{(q - \alpha_\pm)}\, dq \tag{2.81}$$

This integral is the same as that defined by (18.4.32), with s = 1, and has been calculated in section 4, of Chapter 18. Hence, with the help of (18.4.32) and (18.4.45), (2.80) can be rewritten as

$$k_{\pm}^2 c^2 = \omega^2 + i \sqrt{\pi}\ \omega_{pe}^2\ \beta_{\pm} \exp(-\alpha_{\pm}^2) - 2\ \omega_{pe}^2\ \beta_{\pm} \int_0^{\alpha_{\pm}} \exp(W^2 - \alpha_{\pm}^2)\ dW \tag{2.82}$$

This is the dispersion equation for the *right* (upper sign) and the *left* (lower sign) *circularly polarized transverse electromagnetic waves* propagating along the magnetostatic field in a hot plasma, whose equilibrium state is characterized by the isotropic Maxwell-Boltzmann distribution function.

2.7 - Temporal damping of the transverse electromagnetic waves

A careful examination of Eq. (2.82) reveals that, for $k_{\pm}$ real, ω has a negative imaginary part, indicating that the amplitude of the waves are damped with time.

To establish if this temporal damping is significant or not, let us evaluate the asymptotic series expansion of the integral in (2.82) for the case when $|\alpha_{\pm}| >> 1$. For this purpose, we expand the integral in (2.82) in inverse powers of $\alpha_{\pm}$. According to Eq. (18.4.51), it is found that, as the first approximation (retaining only the leading term),

$$\int_0^{\alpha_{\pm}} \exp(W^2 - \alpha_{\pm}^2)\ dW = \frac{1}{2\alpha_{\pm}} \tag{2.83}$$

With this result, and making use of the definitions (2.77) and (2.78), the dispersion equation (2.82) simplifies to

$$k_{\pm}^2 \, c^2 = \omega^2 - \omega_{pe}^2 \frac{\omega}{(\omega \mp \omega_{ce})} + i \sqrt{\pi} \, \omega_{pe}^2 \, \beta_{\pm} \exp\left(-\alpha_{\pm}^2\right) \tag{2.84}$$

Furthermore, for $|\alpha_{\pm}| \gg 1$ the exponential damping term may be omitted in the first approximation, so that Eq. (2.84) becomes

$$k_{\pm}^2 \, c^2 = \omega^2 - \omega_{pe}^2 \frac{\omega}{(\omega \mp \omega_{ce})} \tag{2.85}$$

This dispersion equation corresponds to the results obtained using the *cold plasma* model, with the upper sign for the right circularly polarized wave and the lower sign for the left circularly polarized wave. Consequently, it follows that the results of the cold plasma model are valid only if $|\alpha_{\pm}| \gg 1$.

In the case of the *left circularly polarized wave,* for a given real propagation coefficient, k_-, we find, from Eq. (2.85), that ω is real and satisfies the condition

$$\omega > -\frac{\omega_{ce}}{2} + \sqrt{\left(\frac{\omega_{ce}}{2}\right)^2 + \omega_{pe}^2} \tag{2.86}$$

The phase velocity (ω/k_-) of the left circularly polarized wave is greater than the velocity of light, c, for all k_- and, therefore, β_- is a large number of the order of the ratio of c to the thermal velocity of the electrons. Since $\alpha_-/\beta_- = (\omega + \omega_{ce})/\omega$ is positive and greater than unit, it follows that $\alpha_+ \gg 1$ for all k_-. Consequently, the Landau damping of the left circularly polarized wave propagating along the magnetostatic field in a hot plasma is always negligible. This result was also obtained for the case of transverse electromagnetic waves in a hot isotropic plasma. Further, as far as the characteristics of the left circularly polarized waves are concerned, the cold plasma model is a very good approximation for all real propagation coefficients.

In the case of the *right circularly polarized wave,* for a given real propagation coefficient, k_+, it is seen, from Eq. (2.85), that ω is real and satisfies the conditions

$$0 < \omega < \omega_{ce} \qquad (2.87)$$

$$\omega > \frac{\omega_{ce}}{2} + \sqrt{\left(\frac{\omega_{ce}}{2}\right)^2 + \omega_{pe}^2} \qquad (2.88)$$

An important feature associated with the right circularly polarized wave is the existence of two natural frequency ranges of propagation, whereas for the left circularly polarized wave there is only one natural frequency range of propagation. However, the results for ω, in the range specified in (2.87), do not strictly hold for frequencies of the order of the ion plasma frequency and lower, since at these frequencies the motion of the ions cannot be neglected. For this reason we omit, in the following discussion, the very low frequency region ($\omega < \omega_{ci}$) of (2.87). In the frequency range (2.88), it is found that the phase velocity (ω/k_+) of the right circularly polarized wave is always greater than the velocity of light c, whereas in the frequency range (2.87) the phase velocity (ω/k_+) is less than, but of the order of c, except in the close neighborhood of ω_{ce}. Therefore, we see that β_+ is a large number and, since $|\alpha_+/\beta_+| = |(\omega - \omega_{ce})/\omega|$ is of the order of unity, we conclude that $|\alpha_+| >> 1$, except for ω close to ω_{ce}. Thus, the temporal damping of the right circularly polarized wave is also negligibly small and the cold plasma model is a very good approximation for ω not close to ω_{ce}.

2.8 - Cyclotron damping of the right circularly polarized transverse wave

For ω in the close neighborhood of ω_{ce}, the phase velocity (ω/k_+) of the right circularly polarized wave is of the order of the thermal velocity of the electrons or lower, so that $\beta_+ \leqslant 1$. Consequently, since $|\alpha_+/\beta_+| = |(\omega - \omega_{ce})/\omega|$ is much less than unity, it follows that $|\alpha_+| << 1$. This implies that the asymptotic series expansion in inverse powers of $\alpha_\pm$, Eq. (2.83), valid for $|\alpha_\pm| >> 1$, is not applicable for $\omega \approx \omega_{ce}$.

As a first approximation to the dispersion relation (2.82) for the limiting case of $|\alpha_+| << 1$, we can set α_+ equal to zero in Eq. (2.82), to obtain

$$\left(\frac{k_+ c}{\omega}\right)^3 - \left(\frac{k_+ c}{\omega}\right) = i \frac{c\, \pi^{1/2}}{(2 k_B T_e/m_e)^{1/2}} \left(\frac{\omega_{pe}}{\omega_{ce}}\right)^2 \qquad (2.89)$$

The second term in the left hand side of (2.89) can be omitted in a first approximation, as compared to the first term, since $(\omega/k_+)/c \ll 1$. Hence, (2.89) simplifies to

$$\left(\frac{\omega}{k_+ c}\right)^3 = -i \frac{(2k_B T_e / m_e)^{1/2}}{c\, \pi^{1/2}} \left(\frac{\omega_{ce}}{\omega_{pe}}\right)^2 \tag{2.90}$$

Solving this equation explicitly for ω, gives

$$\omega = \omega_r + i\,\omega_i \tag{2.91}$$

where

$$\omega_r = \frac{\sqrt{3}}{2} k_+ \left[\frac{(2k_B T_e / m_e)^{1/2}}{\pi^{1/2}} c^2 \left(\frac{\omega_{ce}}{\omega_{pe}}\right)^2 \right]^{1/3} \tag{2.92}$$

$$\omega_i = -\frac{1}{2} k_+ \left[\frac{(2k_B T_e / m_e)^{1/2}}{\pi^{1/2}} c^2 \left(\frac{\omega_{ce}}{\omega_{pe}}\right)^2 \right]^{1/3} \tag{2.93}$$

Since ω has a negative imaginary part, it follows that the right circularly polarized wave, which is initially set to propagate along the magnetostatic field, is *temporally damped* for ω_r close to ω_{ce}. This temporal damping is usually called *cyclotron damping* and is similar to the Landau damping of the longitudinal plasma wave.

The cyclotron damping, however, differs from the Landau damping in some aspects. The most important one is the fact that the acceleration is perpendicular to the drift motion of the particles and, since the perpendicular electric acceleration does not, in the first approximation, modify the parallel drift velocity, there is no tendency toward trapping. Therefore, trapping is insignificant in cyclotron damping. The charged particles moving along lines of force will feel the oscillations of the perpendicular electric field at a frequency which differs from the plasma rest-frame frequency by the Doppler shift. Since the electrons rotate about $\underset{\sim}{B}_o$ in the same direction as the electric field of the right circularly polarized wave (Fig. 2), some of them will feel the oscillations at their own cyclotron frequency and they will absorb energy from the field. As a consequence of this wave-particle interaction at the resonance

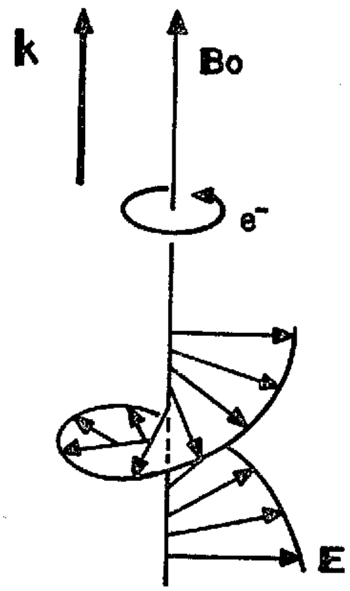

Fig. 2 - Illustrating the resonance which occurs at $\omega = \omega_{ce}$ between the electrons and the electric field of the right circularly polarized wave propagating along $\underset{\sim}{B}_o$.

frequency $\omega = \omega_{ce}$, the electrons absorb energy from the wave electric field, causing the plasma wave to damp out with time. In the absence of resonant particles, there is no energy exchange between the electric field and the particles, and hence ω is real.

As a final point note that, in the limiting case of $\omega_{ce} \to 0$, that is, in the absence of the magnetostatic field, we have $\alpha_{\pm} = \beta_{\pm} = C$ and Eq. (2.82) becomes identical to the dispersion relation (18.5.8) for transverse waves in an isotropic plasma.

2.9 - Instabilities in the right circularly polarized transverse wave

We have seen that for an *isotropic* equilibrium distribution function the resonance at ω_{ce}, between the electrons and the right circularly polarized wave, leads to a temporal damping of the wave amplitude. However, depending on the characteristics of the distribution function, resonance can also lead to instabilities (which are associated with a *positive* imaginary part of ω)

Recall that for the case of an isotropic velocity distribution function the magnetic field of the wave has no effect on the plasma behavior in the linear approximation, since $\nabla_{\underset{\sim}{v}} f_0(v)$ is parallel to $\underset{\sim}{v}$ and, consequently, the magnetic force term $[\underset{\sim}{v} \times \underset{\sim}{B}(\underset{\sim}{r}, t)] \cdot \underset{\sim}{\nabla}_v f_0(v)$ vanishes. However, when the condition of velocity isotropy is dropped, the effects that arise from the magnetic field associated with the wave become important and may lead to instabilities. Although the wave magnetic field itself does not exchange energy with the particles, it exerts a force in the parallel (z) direction on the particles, which destroys the isotropy of the velocity distribution function in the plane perpendicular to $\underset{\sim}{B}_0$. This effect can lead to instabilities depending on the particle distribution function.

For the purpose of demonstrating such an instability, consider the following simple *anisotropic* equilibrium distribution function

$$f_0(v_\parallel, v_\perp) = \delta(v_\parallel)\, f_0(v_\perp) \tag{2.94}$$

which represents cold electrons in the parallel (z) direction, but with a Maxwellian velocity distribution function in the plane normal to $\underset{\sim}{B}_0$. Inserting (2.94) into the dispersion relation (2.72) for the right circularly polarized wave (upper sign), gives

$$k^2 c^2 = \omega^2 - \frac{\omega_{pe}^2}{n_0} \left[\int_{-\infty}^{+\infty} \frac{(\omega - kv_\parallel)\, \delta(v_\parallel)}{(\omega - kv_\parallel - \omega_{ce})}\, dv_\parallel \int_0^\infty f_0(v_\perp)\, dv_\perp \int_0^{2\pi} d\phi \, + \right.$$

$$\left. + \int_{-\infty}^{+\infty} \frac{\delta(v_\parallel)}{(\omega - kv_\parallel - \omega_{ce})^2}\, dv_\parallel \int_0^\infty \frac{1}{2} k^2 v_\perp^2 f_0(v_\perp)\, v_\perp\, dv_\perp \int_0^{2\pi} d\phi \right] \tag{2.95}$$

Using the following property of the Dirac delta function

$$\int_{-\infty}^{+\infty} f\,(x)\ \delta\ (x - x_o)\ dx = f\ (x_o), \tag{2.96}$$

substituting $f_o\ (v_\perp)$ by

$$f_o\ (v_\perp) = n_o\ (\frac{m_e}{2\pi k_B\ T_e})\ \exp\ (-\ \frac{m_e\ v_\perp^2}{2k_B\ T_e}) \tag{2.97}$$

and performing the integrals, yields

$$k^2\ c^2 = \omega^2 - \omega_{pe}^2 \left[\frac{\omega}{(\omega - \omega_{ce})} + \frac{(k^2/2)\ (2k_B\, T_e\,/\,m_e)}{(\omega - \omega_{ce})^2} \right] \tag{2.98}$$

This equation can be rearranged in the form

$$k^2 = \frac{\omega^2\ (\omega - \omega_{ce})^2 - \omega_{pe}^2\ \omega\ (\omega - \omega_{ce})}{c^2\ (\omega - \omega_{ce})^2 + (\omega_{pe}^2/2)(2\ k_B\, T_e\,/\ m_e)} \tag{2.99}$$

It is a simple matter to verify that, for large values of k^2, ω becomes complex. Thus, in the limit of $k^2 \to \infty$, the denominator of (2.99) vanishes, and we obtain

$$\omega^2\ - 2\ \omega_{ce}\ \omega + \omega_{ce}^2\ + \frac{\omega_{pe}^2\ (2k_B\ T_e\,/\,m_e)}{2\ c^2} = 0 \tag{2.100}$$

The solution of this second degree equation is

$$\omega = \omega_{ce} \pm i \, \frac{\omega_{pe} \, (2k_B \, T_e / m_e)^{1/2}}{\sqrt{2} \, c} \tag{2.101}$$

which shows that *growing modes* (instabilities) can occur for $\omega_r \simeq \omega_{ce}$.

Choosing an anisotropic equilibrium distribution function with some velocity spread along the parallel (z) direction, instead of (2.94), we expect this instability to diminish, while turning into damping for an isotropic distribution function. The analysis of this statement is left as an exercise for the reader.

3. WAVE PROPAGATION ACROSS THE MAGNETOSTATIC FIELD IN A HOT PLASMA

We consider now the problem of wave propagation in a direction perpendicular to an externally applied uniform magnetostatic field, $\underset{\sim}{B}_0$. As before, we choose the z-axis along the magnetostatic field, that is, $\underset{\sim}{B}_0 = B_0 \, \hat{\underset{\sim}{z}}$. The propagation coefficient, $\underset{\sim}{k}$, is normal to $\underset{\sim}{B}_0$ and along the x-axis, $\underset{\sim}{k} = k \, \hat{\underset{\sim}{x}}$ (Fig. 3), with k considered to be real. All field quantities are assumed to vary harmonically in space and time, with the phase factor exp $(i \, \underset{\sim}{k} \cdot \underset{\sim}{r} - i \, \omega t)$. As in the previous cases, we take

$$f(\underset{\sim}{r}, \underset{\sim}{v}, t) = f_0(v_\parallel, v_\perp) + f_1(\underset{\sim}{r}, \underset{\sim}{v}, t) \qquad (|f_1| \ll f_0) \tag{3.1}$$

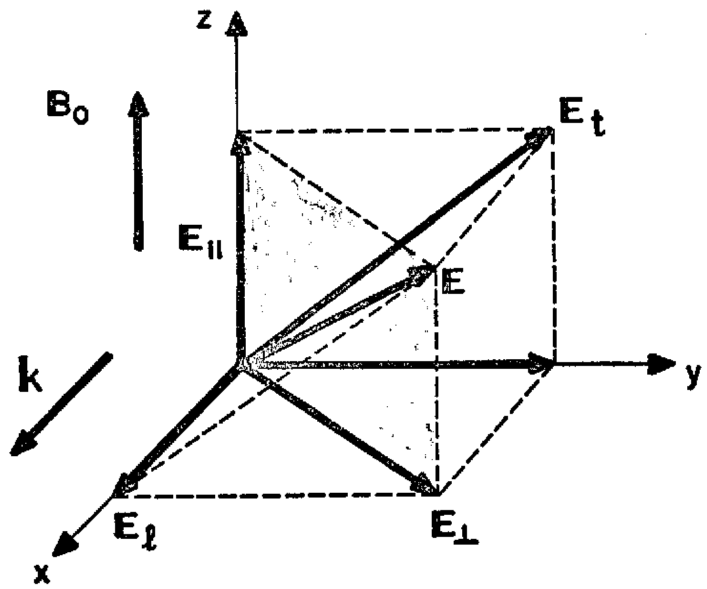

Fig. 3 - Decomposition of the wave electric field vector into components parallel and perpendicular to $\underset{\sim}{B}_0$, or in components longitudinal and transverse with respect to $\underset{\sim}{k}$.

where f_0 $(v_{\parallel}, v_{\perp})$ is the equilibrium distribution function of the electrons under the presence of the magnetostatic field, satisfying Eq. (2.1), $v_{\parallel} = v_z$ is the velocity component of the electrons in the direction parallel to $\underset{\sim}{B}_0$, and $v_{\perp}$ the velocity component of the electrons in the plane (x - y) normal to $\underset{\sim}{B}_0$. For the perturbation distribution function we have

$$f_1 (\underset{\sim}{r}, \underset{\sim}{v}, t) = f_1 (\underset{\sim}{v}) \exp (i kx - i \omega t) \tag{3.2}$$

and for the *wave* electric and magnetic fields

$$\underset{\sim}{E} (\underset{\sim}{r}, t) = \underset{\sim}{E} \exp (ikx - i\omega t) \tag{3.3}$$

$$\underset{\sim}{B}\ (\underset{\sim}{r},\ t) = \underset{\sim}{B}\ \exp\ (ikx - i\omega t) \tag{3.4}$$

where $f_1\ (\underset{\sim}{v})$, $\underset{\sim}{E}$ and $\underset{\sim}{B}$ are the phasor amplitudes.

As in the previous section, the purpose is to deduce the dispersion equation giving the functional relationship between k and ω, and, from an analysis of the dispersion relation, determine the intrinsic behavior of the plasma for the case under consideration.

3.1 - Solution of the linearized Vlasov equation

From the linearized Vlasov equation (2.4), replacing the differential operators $\partial/\partial t$ and $\underset{\sim}{\nabla}$ by $- i\omega$ and $ik\hat{\underset{\sim}{x}}$, respectively, and making use of relation (2.10), we obtain

$$\frac{df_1\ (\underset{\sim}{v})}{d\phi} - i\ \frac{(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v})}{\omega_{ce}}\ f_1\ (\underset{\sim}{v}) = \frac{e}{m_e\ \omega_{ce}}\ (\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B})\cdot\ \underset{\sim}{\nabla}_v\ f_o\ (v_{\parallel},\ v_{\perp}) \tag{3.5}$$

where, now, $\underset{\sim}{k}\cdot\underset{\sim}{v} = kv_x = kv_{\perp}\cos\phi$. From Maxwell equation $\underset{\sim}{k} \times \underset{\sim}{E} = \omega\,\underset{\sim}{B}$ we have

$$\underset{\sim}{B} = (k/\omega)\ (E_y\ \hat{\underset{\sim}{z}} - E_z\ \hat{\underset{\sim}{y}}) \tag{3.6}$$

Using this expression for $\underset{\sim}{B}$, we get

$$\underset{\sim}{v} \times \underset{\sim}{B} = (k/\omega) \left[(v_y E_y + v_z E_z)\, \hat{\underset{\sim}{x}} - v_x E_y \hat{\underset{\sim}{y}} - v_x E_z \hat{\underset{\sim}{z}} \right] \tag{3.7}$$

Noting that

$$\frac{\partial f_o}{\partial v_x} = \cos\phi \frac{\partial f_o}{\partial v_\perp} \tag{3.8}$$

$$\frac{\partial f_o}{\partial v_y} = \sin\phi \frac{\partial f_o}{\partial v_\perp} \tag{3.9}$$

$$\frac{\partial f_o}{\partial v_z} = \frac{\partial f_o}{\partial v_\parallel} \tag{3.10}$$

we obtain

$$(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) \cdot \underset{\sim}{\nabla}_v f_o = \left[E_x + (\frac{k}{\omega}) (v_y E_y + v_z E_z) \right] \cos\phi \frac{\partial f_o}{\partial v_\perp} +$$

$$+ (1 - \frac{kv_x}{\omega}) E_y \sin\phi \frac{\partial f_o}{\partial v_\perp} + (1 - \frac{kv_x}{\omega}) E_z \frac{\partial f_o}{\partial v_z} = \frac{\partial f_o}{\partial v_\perp} (\cos\phi\, E_x +$$

$$+ \sin\phi\, E_y) + \left[(\frac{k}{\omega}) (v_z \frac{\partial f_o}{\partial v_\perp} - v_\perp \frac{\partial f_o}{\partial v_z}) \cos\phi + \frac{\partial f_o}{\partial v_z} \right] E_z \tag{3.11}$$

The linearized Vlasov equation (3.5) becomes, therefore,

$$\frac{df_1(\underset{\sim}{v})}{d\phi} - i\,\frac{(\omega - kv_\perp \cos\phi)}{\omega_{ce}}\, f_1(\underset{\sim}{v}) = \frac{e}{m_e\,\omega_{ce}} \left\{ \frac{\partial f_o}{\partial v_\perp}(\cos\phi\, E_x + \sin\phi\, E_y) + \right.$$

$$\left. + \left[\left(\frac{k}{\omega}\right)\left(v_z \frac{\partial f_o}{\partial v_\perp} - v_\perp \frac{\partial f_o}{\partial v_z}\right)\cos\phi + \frac{\partial f_o}{\partial v_z} \right] E_z \right\} \qquad (3.12)$$

The integrating factor for this first order differential equation is found to be

$$h(\phi) = \exp\left[- \int_0^\phi i\,\frac{(\omega - kv_\perp \cos\phi)}{\omega_{ce}}\, d\phi \right]$$

$$= \exp\left[-i\left(\frac{\omega}{\omega_{ce}}\right)\phi + i\left(\frac{kv_\perp}{\omega_{ce}}\right)\sin\phi \right] \qquad (3.13)$$

Multiplying both sides of Eq. (3.12) by the integrating factor (3.13), gives

$$\frac{d}{d\phi}\left\{ f_1(\underset{\sim}{v}) \exp\left[-i\left(\frac{\omega}{\omega_{ce}}\right)\phi + i\left(\frac{kv_\perp}{\omega_{ce}}\right)\sin\phi \right] \right\} =$$

$$= \frac{e}{m_e\,\omega_{ce}} \left\{ \frac{\partial f_o}{\partial v_\perp}(\cos\phi\, E_x + \sin\phi\, E_y) + \left[\left(\frac{k}{\omega}\right)\left(v_z \frac{\partial f_o}{\partial v_\perp} - \right.\right.\right.$$

$$\left.\left.\left. - v_\perp \frac{\partial f_o}{\partial v_z}\right)\cos\phi + \frac{\partial f_o}{\partial v_z}\right] E_z \right\} \exp\left[-i\left(\frac{\omega}{\omega_{ce}}\right)\phi + i\left(\frac{kv_\perp}{\omega_{ce}}\right)\sin\phi \right]$$

$$(3.14)$$

The solution for $f_1(\underset{\sim}{v})$ is obtained by integrating this equation over ϕ,

$$f_1(\underset{\sim}{v}) = \frac{e}{m_e \omega_{ce}} \exp\left[i \left(\frac{\omega}{\omega_{ce}}\right)\phi - i\left(\frac{kv_\perp}{\omega_{ce}}\right)\sin\phi \right] \int_{-\infty}^{\phi} \left\{ \frac{\partial f_o}{\partial v_\perp}(\cos\phi'' E_x + \right.$$

$$\left. + \sin\phi'' E_y) + \left[\left(\frac{k}{\omega}\right)\left(v_z \frac{\partial f_o}{\partial v_\perp} - v_\perp \frac{\partial f_o}{\partial v_z}\right)\cos\phi'' + \frac{\partial f_o}{\partial v_z} \right] E_z \right\} \cdot$$

$$\exp\left[- i\left(\frac{\omega}{\omega_{ce}}\right)\phi'' + i\left(\frac{kv_\perp}{\omega_{ce}}\right)\sin\phi'' \right] d\phi'' \qquad (3.15)$$

If the variable of integration is changed to $\phi' = \phi - \phi''$, Eq. (3.15) becomes

$$f_1(\underset{\sim}{v}) = \frac{e}{m_e \omega_{ce}} \exp\left[-i\left(\frac{kv_\perp}{\omega_{ce}}\right)\sin\phi \right] \int_0^{\infty} \left\{ \frac{\partial f_o}{\partial v_\perp}\left[\cos(\phi - \phi') E_x + \right.\right.$$

$$\left.\left. + \sin(\phi - \phi') E_y \right] + \left[\left(\frac{k}{\omega}\right)\left(v_z \frac{\partial f_o}{\partial v_\perp} - v_\perp \frac{\partial f_o}{\partial v_z}\right)\cos(\phi - \phi') + \frac{\partial f_o}{\partial v_z} \right] E_z \right\} \cdot$$

$$\exp\left[+ i\left(\frac{\omega}{\omega_{ce}}\right)\phi' + i\left(\frac{kv_\perp}{\omega_{ce}}\right)\sin(\phi - \phi') \right] d\phi' \qquad (3.16)$$

Note that ϕ occurs only as the argument of periodic functions of period 2π, which is an agreement with the physical requirement that $f_1(\underset{\sim}{v})$ be a single valued function of ϕ.

3.2 - Current density and the conductivity tensor

The current density is given by

$$\underset{\sim}{J}\ (\underset{\sim}{r},\ t) = \underset{\sim}{J} \exp\ (i\,kx - i\omega t) \tag{3.17}$$

where the phasor amplitude, $\underset{\sim}{J}$, is

$$\underset{\sim}{J} = -e \int_V f_1\ (\underset{\sim}{v})\ \underset{\sim}{v}\ d^3v \tag{3.18}$$

or

$$J = -e \int_0^{\infty} v_\perp\, dv_\perp \int_0^{2\pi} d\phi \int_{-\infty}^{+\infty} dv_z\ f_1\ (\underset{\sim}{v})\ (v_\perp \cos\phi\ \hat{\underset{\sim}{x}} +$$

$$+ v_\perp \sin\phi\ \hat{\underset{\sim}{y}} + v_z\ \hat{\underset{\sim}{z}}) \tag{3.19}$$

For the purpose of calculating the components of $\underset{\sim}{J}$, it is appropriate to express $\underset{\sim}{J}$ as

$$\underset{\sim}{J} = \underset{\approx}{\sigma} \cdot \underset{\sim}{E} \tag{3.20}$$

or, in explicit form,

$$J_x = \sigma_{xx} E_x + \sigma_{xy} E_y + \sigma_{xz} E_z \tag{3.21}$$

$$J_y = \sigma_{yx} E_x + \sigma_{yy} E_y + \sigma_{yz} E_z \tag{3.22}$$

$$J_z = \sigma_{zx} E_x + \sigma_{zy} E_y + \sigma_{zz} E_z \qquad (3.23)$$

where $\underset{\approx}{\sigma}$ is the conductivity tensor, whose components can be arranged in matrix form as

$$\underset{\approx}{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \qquad (3.24)$$

If $f_1(\underset{\sim}{v})$, from Eq. (3.16), is substituted into Eq. (3.19), and the resulting expression is compared with Eqs. (3.21) to (3.23), we identify the components of the conductivity tensor as

$$\sigma_{xx} = -\frac{e^2}{m_e \omega_{ce}} \int_0^\infty v_\perp^2 \, dv_\perp \int_0^{2\pi} \cos\phi \, d\phi \int_{-\infty}^{+\infty} dv_z \exp\left[-i\left(\frac{kv_\perp}{\omega_{ce}}\right)\sin\phi\right] \cdot$$

$$\cdot \int_0^\infty d\phi' \left(\frac{\partial f_o}{\partial v_\perp}\right) \cos(\phi - \phi') \exp\left[g_1(\phi')\right] \qquad (3.25)$$

$$\sigma_{xy} = -\frac{e^2}{m_e \omega_{ce}} \int_0^\infty v_\perp^2 \, dv_\perp \int_0^{2\pi} \cos\phi \, d\phi \int_{-\infty}^{+\infty} dv_z \exp\left[-i\left(\frac{kv_\perp}{\omega_{ce}}\right)\sin\phi\right] \cdot$$

$$\cdot \int_0^\infty d\phi' \left(\frac{\partial f_o}{\partial v_\perp}\right) \sin(\phi - \phi') \exp\left[g_1(\phi')\right] \qquad (3.26)$$

$$\sigma_{xz} = - \frac{e^2}{m_e\, \omega_{ce}} \int_0^{\infty} v_\perp^2\, dv_\perp \int_0^{2\pi} \cos\phi\, d\phi \int_{-\infty}^{+\infty} dv_z \exp\left[- i \left(\frac{kv_\perp}{\omega_{ce}}\right) \sin\phi\right] \cdot$$

$$\cdot \int_0^{\infty} d\phi' \left[\left(\frac{k}{\omega}\right) \left(v_z \frac{\partial f_o}{\partial v_\perp} - v_\perp \frac{\partial f_o}{\partial v_z}\right) \cos(\phi - \phi') + \frac{\partial f_o}{\partial v_z} \right] \cdot$$

$$\exp\left[g_1 (\phi') \right] \tag{3.27}$$

$$\sigma_{yx} = \frac{e^2}{m_e\, \omega_{ce}} \int_0^{\infty} v_\perp^2\, dv_\perp \int_0^{2\pi} \sin\phi\, d\phi \int_{-\infty}^{+\infty} dv_z \exp\left[- i \left(\frac{kv_\perp}{\omega_{ce}}\right) \sin\phi\right] \cdot$$

$$\cdot \int_0^{\infty} d\phi' \left(\frac{\partial f_o}{\partial v_\perp}\right) \cos(\phi - \phi') \exp\left[g_1 (\phi') \right] \tag{3.28}$$

$$\sigma_{yy} = - \frac{e^2}{m_e\, \omega_{ce}} \int_0^{\infty} v_\perp^2\, dv_\perp \int_0^{2\pi} \sin\phi\, d\phi \int_{-\infty}^{+\infty} dv_z \exp\left[-i \left(\frac{kv_\perp}{\omega_{ce}}\right) \sin\phi\right] \cdot$$

$$\cdot \int_0^{\infty} d\phi' \left(\frac{\partial f_o}{\partial v_\perp}\right) \sin(\phi - \phi') \exp\left[g_1 (\phi') \right] \tag{3.29}$$

$$\sigma_{yz} = - \frac{e^2}{m_e\, \omega_{ce}} \int_0^{\infty} v_\perp^2\, dv_\perp \int_0^{2\pi} \sin\phi\, d\phi \int_{-\infty}^{+\infty} dv_z \exp\left[-i \left(\frac{kv_\perp}{\omega_{ce}}\right) \sin\phi\right] \cdot$$

$$\cdot \int_0^\infty d\phi' \left[\left(\frac{k}{\omega}\right) \left(v_z \frac{\partial f_o}{\partial v_\perp} - v_\perp \frac{\partial f_o}{\partial v_z}\right) \cos(\phi - \phi') + \frac{\partial f_o}{\partial v_z} \right] \cdot$$

$$\exp \left[g_1 (\phi') \right] \tag{3.30}$$

$$\sigma_{zx} = - \frac{e^2}{m_e \omega_{ce}} \int_0^\infty v_\perp \, dv_\perp \int_0^{2\pi} d\phi \int_{-\infty}^{+\infty} v_z \, dv_z \exp \left[- i \left(\frac{k v_\perp}{\omega_{ce}}\right) \sin \phi \right] \cdot$$

$$\cdot \int_0^\infty d\phi' \left(\frac{\partial f_o}{\partial v_\perp}\right) \cos(\phi - \phi') \exp \left[g_1 (\phi') \right] \tag{3.31}$$

$$\sigma_{zy} = - \frac{e^2}{m_e \omega_{ce}} \int_0^\infty v_\perp \, dv_\perp \int_0^{2\pi} d\phi \int_{-\infty}^{+\infty} v_z \, dv_z \exp \left[- i \left(\frac{k v_\perp}{\omega_{ce}}\right) \sin \phi \right] \cdot$$

$$\cdot \int_0^\infty d\phi' \left(\frac{\partial f_o}{\partial v_\perp}\right) \sin(\phi - \phi') \exp \left[g_1 (\phi') \right] \tag{3.32}$$

$$\sigma_{zz} = - \frac{e^2}{m_e \omega_{ce}} \int_0^\infty v_\perp \, dv_\perp \int_0^{2\pi} v_z \, dv_z \exp \left[- i \left(\frac{k v_\perp}{\omega_{ce}}\right) \sin \phi \right] \cdot$$

$$\cdot \int_0^\infty d\phi' \left[\left(\frac{k}{\omega}\right) \left(v_z \frac{\partial f_o}{\partial v_\perp} - v_\perp \frac{\partial f_o}{\partial v_z}\right) \cos(\phi - \phi') + \frac{\partial f_o}{\partial v_z} \right] \cdot$$

$$\exp\left[g_1(\phi')\right] \tag{3.33}$$

where we have used the notation

$$g_1(\phi') = i\left(\frac{\omega}{\omega_{ce}}\right)\phi' + i\left(\frac{kv_\perp}{\omega_{ce}}\right)\sin(\phi - \phi') \tag{3.34}$$

3.3 - Evaluation of the integrals

In simplifying the expression for σ_{xx}, it is advantageous to calculate first the integral with respect to ϕ'. From Eq. (3.25) consider, therefore, the integral

$$I_1 = \int_0^\infty d\phi' \cos(\phi - \phi') \quad \exp\left[g_1(\phi')\right] \tag{3.35}$$

Differentiating Eq. (3.34) with respect to ϕ', we find

$$\cos(\phi - \phi') = \left(\frac{\omega}{kv_\perp}\right) + i\left(\frac{\omega_{ce}}{kv_\perp}\right)\frac{dg_1(\phi')}{d\phi'} \tag{3.36}$$

Thus, Eq. (3.35) becomes

$$I_1 = \left(\frac{\omega}{kv_\perp}\right)\int_0^\infty d\phi' \exp\left[g_1(\phi')\right] + i\left(\frac{\omega_{ce}}{kv_\perp}\right)\int_0^\infty d\left\{\exp\left[g_1(\phi')\right]\right\} \tag{3.37}$$

since d $\left\{ \exp \left[g_1 (\phi') \right] \right\} = \exp \left[g_1 (\phi') \right] dg_1 (\phi')/d\phi'$. Therefore,

$$I_1 = (\frac{\omega}{kv_\perp}) \int_0^\infty d\phi' \exp \left[g_1 (\phi') \right] - i (\frac{\omega_{ce}}{kv_\perp}) \exp \left[i (\frac{kv_\perp}{\omega_{ce}}) \sin \phi \right] \tag{3.38}$$

In order to evaluate the integral in Eq. (3.38), let

$$\xi = \frac{kv_\perp}{\omega_{ce}} \tag{3.39}$$

and express the term $\exp \left[g_1(\phi') \right]$ in an infinite series expansion in terms of the Bessel functions, $J_n (\xi)$,

$$\exp \left[i (\frac{\omega}{\omega_{ce}}) \phi' \right] \exp \left[i\xi \sin (\phi - \phi') \right] =$$

$$\exp \left[i (\frac{\omega}{\omega_{ce}}) \phi' \right] \sum_{n=-\infty}^{+\infty} J_n (\xi) \exp \left[in (\phi - \phi') \right] =$$

$$= \sum_{n=-\infty}^{+\infty} J_n (\xi) \exp (in\phi) \quad \exp \left[i (\frac{\omega}{\omega_{ce}} - n) \phi' \right] \tag{3.40}$$

where $J_n (\xi)$ is the Bessel function of the first kind of order n, and where the factor $\exp \left[i \xi \sin (\phi - \phi') \right]$ is identified with the so-called generating function of the Bessel functions. Substituting (3.40) into (3.38), gives

$$I_1 = -\frac{i}{\xi}\exp(i\,\xi\sin\phi) + \frac{\omega}{kv_\perp}\sum_{n=-\infty}^{+\infty} J_n(\xi)\exp(i\,n\,\phi)\int_0^\infty d\phi'$$

$$\exp\left[i\left(\frac{\omega}{\omega_{ce}} - n\right)\phi'\right] \tag{3.41}$$

$$= -\frac{i}{\xi}\exp(i\,\xi\sin\phi) + \frac{i\omega}{kv_\perp}\sum_{n=-\infty}^{+\infty} J_n(\xi)\exp(in\phi)\,\frac{1}{(\omega/\omega_{ce} - n)}$$

(3.42)

As the next step in evaluating σ_{xx}, we calculate the integral with respect to ϕ. Substituting (3.42) into the expression (3.25) for σ_{xx}, we find the integral with respect to ϕ to be

$$I_2 = \int_0^{2\pi} d\phi\cos\phi\left[-\frac{i}{\xi} + \frac{i\omega}{kv_\perp}\sum_{n=-\infty}^{+\infty}\frac{J_n(\xi)\exp(in\phi - i\,\xi\sin\phi)}{(\omega/\omega_{ce} - n)}\right]$$

(3.43)

The first term within the square brackets in this equation integrates to zero. For the remaining terms note first that we can write

$$\cos\phi = \frac{n}{\xi} + \frac{i}{\xi}\frac{d}{d\phi}(i\,n\,\phi - i\,\xi\sin\phi) \tag{3.44}$$

so that the integral I_2 becomes

$$I_2 = \frac{i\omega}{kv_\perp} \sum_{n=-\infty}^{+\infty} \frac{J_n(\xi)}{(\omega/\omega_{ce} - n)} \left\{ \frac{n}{\xi} \int_0^{2\pi} d\phi \exp(in\phi - i\xi \sin\phi) + \right.$$

$$\left. + \frac{i}{\xi} \int_0^{2\pi} d \left[\exp(i n\phi - i\phi \sin\phi) \right] \right\} \tag{3.45}$$

The second integral within brackets in this equation vanishes and the first integral can be expressed in terms of a Bessel function, according to the relation,

$$\int_0^{2\pi} d\phi \exp(i n\phi - i\xi \sin\phi) = 2\pi J_n(\xi) \tag{3.46}$$

which is known as the Bessel integral. Therefore, (3.45) becomes

$$I_2 = \frac{2\pi i\omega}{\xi^2 \omega_{ce}} \sum_{n=-\infty}^{+\infty} \frac{n J_n^2(\xi)}{(\omega/\omega_{ce} - n)} \tag{3.47}$$

This result can be written in a slightly different form by noting that

$$I_2 = \frac{2\pi i}{\xi^2} \sum_{n=-\infty}^{+\infty} J_n^2(\xi) \frac{n(\omega/\omega_{ce} - n + n)}{(\omega/\omega_{ce} - n)}$$

$$= \frac{2\pi i}{\xi^2} \sum_{n=-\infty}^{+\infty} \left[n\, J_n^2\,(\xi) + \frac{n^2\, J_n^2\,(\xi)}{(\omega/\omega_{ce} - n)} \right] \tag{3.48}$$

Now, since $J_{-n}\,(\xi) = (-1)^n\, J_n\,(\xi)$, we have

$$\sum_{n=-\infty}^{+\infty} n\, J_n^2\,(\xi) = 0 \tag{3.49}$$

and the integral (3.48) simplifies to

$$I_2 = \frac{2\pi i}{\xi^2} \sum_{n=-\infty}^{+\infty} \frac{n^2\, J_n^2\,(\xi)}{(\omega/\omega_{ce} - n)} \tag{3.50}$$

From Eqs. (3.25), (3.35) and (3.43), we see that the expression for σ_{xx} can be written as

$$\sigma_{xx} = -\frac{e^2}{m_e\, \omega_{ce}} \int_0^{\infty} v_\perp^2\, dv_\perp \int_{-\infty}^{+\infty} dv_z\, \frac{\partial f_o}{\partial v_\perp}\, I_2 \tag{3.51}$$

Thus, the substitution of (3.50) into (3.51), yields

$$\sigma_{xx} = -\frac{2\pi i\, e^2}{m_e\, \omega_{ce}} \int_0^{\infty} v_\perp^2\, dv_\perp \int_{-\infty}^{+\infty} dv_z\, \left(\frac{\omega_{ce}}{kv_\perp}\right)^2 \frac{\partial f_o}{\partial v_\perp} \sum_{n=-\infty}^{+\infty} \frac{n^2\, J_n^2\,(kv_\perp/\omega_{ce})}{(\omega/\omega_{ce} - n)} \tag{3.52}$$

This expression for σ_{xx} is valid for any cylindrically symmetric equilibrium distribution function f_o ($v_\parallel$, $v_\perp$). However, in what follows, all the details will be restricted to the the case in which the equilibrium state is characterized by the isotropic Maxwell-Boltzmann distribution function f_o (v), for simplicity. Thus, we take

$$f_o (v) = n_o \left(\frac{m_e}{2\pi k_B T_e}\right)^{3/2} \exp\left[-\frac{m_e (v_\perp^2 + v_z^2)}{2k_B T_e}\right] \qquad (3.53)$$

for the evaluation of the integrals over $v_\perp$ and v_z in Eq. (3.52). In order to perform the integral over $v_\perp$, it is convenient to introduce the following parameter

$$\tilde{\nu} = \frac{k_B T_e k^2}{m_e \omega_{ce}^2} \qquad (3.54)$$

Performing the differentiation $\partial f_o/\partial v_\perp$ and using Eq. (3.54), the expression (3.52) for σ_{xx} simplifies to

$$\sigma_{xx} = \frac{i n_o e^2}{m_e \omega_{ce} \tilde{\nu}^2} \sum_{n=-\infty}^{+\infty} \frac{n^2}{(\omega/\omega_{ce} - n)} \int_0^\infty \xi \, d\xi \, J_n^2 (\xi) \exp(-\xi^2/2\tilde{\nu}) \qquad (3.55)$$

From the theory of Bessel functions we have the following Weber's second exponential integral

$$\int_0^\infty \exp(-p^2 t^2) J_n (at) J_n (bt) t \, dt =$$

$$= \frac{1}{2p^2} \exp\left(-\frac{a^2 + b^2}{4p^2}\right) I_n\,(ab/2p^2) \qquad (3.56)$$

where $I_n\,(x)$ is the Bessel function of the second kind, which is related to the Bessel function of the first kind with an imaginary argument, $J_n\,(ix)$, by

$$I_n\,(x) = (-i)^n\, J_n\,(ix) \qquad (3.57)$$

Substituting (3.56) into (3.55), yields

$$\sigma_{xx} = \frac{i\, n_o\, e^2}{m_e\, \omega_{ce}} \quad \frac{e^{-\tilde{\nu}}}{\tilde{\nu}} \sum_{n=-\infty}^{+\infty} \frac{n^2\, I_n\,(\tilde{\nu})}{(\omega/\omega_{ce} - n)} \qquad (3.58)$$

The components σ_{xz}, σ_{yz}, σ_{zx} and σ_{zy} of the conductivity tensor vanish, since the integrands in Eqs. (3.27), (3.30), (3.31) and (3.32) are found to be odd functions of v_z. Thus, performing the integrations with respect to v_z first, we find

$$\sigma_{xz} = \sigma_{yz} = \sigma_{zx} = \sigma_{zy} = 0 \qquad (3.59)$$

The component σ_{zz} of the conductivity tensor, for the case of the isotropic Maxwell-Boltzmann distribution function, simplifies to

$$\sigma_{zz} = -\frac{e^2}{m_e\, \omega_{ce}} \int_0^{\infty} v_\perp\, dv_\perp \int_{-\infty}^{+\infty} d\phi \int_{-\infty}^{+\infty} v_z\, dv_z\, \exp\,(-i\, \xi \sin\phi)\ \cdot$$

$$\cdot \int_0^\infty d\phi' \frac{\partial f_o}{\partial v_z} \exp\left[g_1 (\phi')\right] \tag{3.60}$$

The integrals appearing here are evaluated as in the case of σ_{xx}, yielding the result

$$\sigma_{zz} = \frac{i\, n_o\, e^2}{m_e\, \omega_{ce}}\, e^{-\tilde{\nu}} \sum_{n=-\infty}^{+\infty} \frac{I_n\,(\tilde{\nu})}{(\omega/\omega_{ce} - n)} \tag{3.61}$$

The components σ_{xy}, σ_{yx} and σ_{yy} of the conductivity tensor will not be needed here, in order to investigate the characteristics of waves propagating across the magnetostatic field in a hot plasma. Therefore, explicit expressions for these components of $\underset{\approx}{\sigma}$ will not be presented.

3.4 - Separation into the various modes

With the time and space dependence of the fields, as given by Eqs. (3.3) and (3.4), and expressing $\underset{\sim}{J}$ as $\underset{\approx}{\sigma} \cdot \underset{\sim}{E}$, Maxwell curl equations reduce to

$$k\underset{\sim}{\hat{x}} \times \underset{\sim}{E} = \omega \underset{\sim}{B} \tag{3.62}$$

$$ik\underset{\sim}{\hat{x}} \times \underset{\sim}{B} = (\mu_o \underset{\approx}{\sigma} - \frac{i\omega}{c^2} \underset{\approx}{1}) \cdot \underset{\sim}{E}$$

$$= - \frac{i\omega}{c^2} \underset{\approx}{\varepsilon} \cdot \underset{\sim}{E} \tag{3.63}$$

where $\underset{\approx}{1}$ denotes the unit dyad and

$$\underset{\approx}{\varepsilon} = \underset{\approx}{1} + \frac{i}{\omega\,\varepsilon_0}\,\underset{\approx}{\sigma} \tag{3.64}$$

is the relative permittivity dyad. In component form, Eqs. (3.62) and (3.63) become, respectively,

$$B_x = 0 \tag{3.65}$$

$$E_z = -(\omega/k)\,B_y \tag{3.66}$$

$$E_y = (\omega/k)\,B_z \tag{3.67}$$

and

$$-(\omega/kc^2)\,(\varepsilon_{xx}\,E_x + \varepsilon_{xy}\,E_y) = 0 \tag{3.68}$$

$$-(\omega/kc^2)\,(\varepsilon_{yx}\,E_x + \varepsilon_{yy}\,E_y) = -B_z \tag{3.69}$$

$$-(\omega/kc^2)\,E_{zz}\,E_z = B_y \tag{3.70}$$

From Eqs. (3.64), (3.58) and (3.61), it follows that

$$\varepsilon_{xx} = 1 - \frac{\omega_{pe}^2}{\omega\,\omega_{ce}}\,\frac{e^{-\tilde{\nu}}}{\tilde{\nu}}\sum_{n=-\infty}^{+\infty}\frac{n^2\,I_n\,(\tilde{\nu})}{(\omega/\omega_{ce} - n)} \tag{3.71}$$

$$\varepsilon_{zz} = 1 - \frac{\omega_{pe}^2}{\omega\,\omega_{ce}}\,e^{-\tilde{\nu}}\sum_{n=-\infty}^{+\infty}\frac{I_n\,(\tilde{\nu})}{(\omega/\omega_{ce} - n)} \tag{3.72}$$

and, from Eq. (3.59),

$$\varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_{zx} = \varepsilon_{zy} = 0 \tag{3.73}$$

The expressions for the other components of $\underset{\approx}{\varepsilon}$ will not be needed in what follows.

An analysis of Eqs. (3.65) to (3.70) shows that the waves are *transverse magnetic* (TM) with respect to the direction (x) of propagation, since $B_x = 0$. Also, we see that the remaining field components can be separated into two independent groups, involving the following variables each:

1. E_x, E_y, B_z [Eqs. (3.67), (3.68), (3.69)] (TM mode)
2. E_z, B_y [Eqs. (3.66) and (3.70)] (TEM mode)

The first group represents the TM (Transverse Magnetic) mode, since there is no component of the wave magnetic field in the direction of propagation (x). The second group represents the TEM (Transverse Electric-Magnetic) mode, since it has no component of either the electric or the magnetic field in the direction of propagation (Fig. 3). It is a degenerate case of the TM mode. Since the electric field is in the direction of $\underset{\sim}{B}_0$, the TEM mode is called (in magneto-ionic theory) the *ordinary wave*, and it is not affected by $\underset{\sim}{B}_0$.

3.5 - Dispersion relations

To deduce the dispersion equation for the TM mode, we first combine Eqs. (3.68) and (3.69) to eliminate E_x, obtaining

$$\frac{kc^2}{\omega} B_z = (\varepsilon_{yy} - \frac{\varepsilon_{xy}\,\varepsilon_{yx}}{\varepsilon_{xx}})\, E_y \tag{3.74}$$

Substituting E_y from (3.67), into (3.74), yields

$$(\frac{k^2\,c^2}{\omega^2} - \varepsilon_{yy} + \frac{\varepsilon_{xy}\,\varepsilon_{yx}}{\varepsilon_{xx}})\, B_z = 0 \tag{3.75}$$

For a nontrivial solution for B_z, and also for E_x and E_y, the term within parenthesis in Eq. (3.75) must vanish, which results in the following *dispersion relation* for the *TM mode*,

$$\frac{k^2\,c^2}{\omega^2} = \frac{\varepsilon_{xx}\,\varepsilon_{yy} - \varepsilon_{xy}\,\varepsilon_{yx}}{\varepsilon_{xx}} \tag{3.76}$$

To obtain the dispersion equation for the TEM mode, we substitute E_z from (3.66), into (3.70), to find

$$(\frac{k^2\,c^2}{\omega^2} - \varepsilon_{zz})\, B_y = 0 \tag{3.77}$$

For a nontrivial solution for B_y, and therefore E_z, we must require that

$$\frac{k^2\ c^2}{\omega^2} = \varepsilon_{zz} \tag{3.78}$$

which is the *dispersion relation* for the *TEM mode*.

3.6 - The quasistatic mode

Since the dispersion relation (3.76), for the TM mode, is very complicated, in what follows we shall analyse this dispersion relation only for the limiting case of kc/ω tending to infinity. This limiting condition of $kc/\omega \to \infty$ defines the resonance condition.

From Eq. (3.69) we see that, for finite values of E_x and E_y, B_z must be equal to zero in the limiting case of kc/ω equal to infinity. From Eq. (3.67) it follows, therefore, that E_y vanishes. Consequently, for a nontrivial solution for E_x, the dispersion relation for $kc/\omega \to \infty$ becomes

$$\varepsilon_{xx} = 0 \tag{3.79}$$

This equation is known as the dispersion relation for the *quasistatic wave* propagating across the magnetostatic field, since the magnetic field is negligible and the electric field is essentially in the direction of propagation. In the limit $kc/\omega \to \infty$, the longitudinal wave is already uncoupled from the transverse wave and the dispersion relation $\varepsilon_{xx} = 0$ refers to the *longitudinal* wave ($E_x \neq 0$). The TM mode, in the zero-temperature limit, corresponds to the *extraordinary* wave of magnetoionic theory. As a matter of fact, the dispersion relation (3.79) can be derived directly from the laws of electrostatics, instead of using Maxwell equations. Thus, since the magnetic field can be omitted at the outset, Eq. (3.79) is also called the dispersion relation for the *electrostatic wave*. Although (3.79) is strictly correct only for $kc/\omega = \infty$, it can be considered to be a reasonably good approximation for $kc/\omega >> 1$.

From Eq. (3.71) the explicit expression for the dispersion relation (3.79), for the quasistatic wave, is found to be

$$\frac{\omega_{pe}^2}{\omega\,\omega_{ce}} \frac{e^{-\tilde{\nu}}}{\tilde{\nu}} \sum_{n=-\infty}^{+\infty} \frac{n^2 I_n(\tilde{\nu})}{(\omega/\omega_{ce} - n)} = 1 \qquad (3.80)$$

Since $I_{-n}(\tilde{\nu}) = I_n(\tilde{\nu})$, we have,

$$\sum_{n=-\infty}^{+\infty} n\, I_n(\tilde{\nu}) = 0 \qquad (3.81)$$

so that multiplying (3.81) by $(\omega_{pe}^2/\omega\omega_{ce})\,(e^{-\tilde{\nu}}/\tilde{\nu})$ and adding it to (3.80), we find

$$\tilde{\nu}\, \frac{\omega_{ce}^2}{\omega_{pe}^2} = e^{-\tilde{\nu}} \sum_{n=-\infty}^{+\infty} \frac{n\, I_n(\tilde{\nu})}{(\omega/\omega_{ce} - n)} \qquad (3.82)$$

This equation was extensively investigated by Bernstein, who showed that it has solutions for both ω and k real. For this reason, these solutions are often called the *Bernstein modes*.

In order to show the *absence of complex solutions for* ω, let us first write the dispersion equation (3.82) in a more convenient form. Making use of the expansion

$$\exp(\tilde{\nu} \cos y) = \sum_{n=-\infty}^{+\infty} I_n(\tilde{\nu}) \exp(i n y) \tag{3.83}$$

and setting $y = 0$ in this expansion, we obtain

$$1 = e^{-\tilde{\nu}} \sum_{n=-\infty}^{+\infty} I_n(\tilde{\nu}) \tag{3.84}$$

Adding Eqs. (3.84) and (3.82), gives

$$1 + \tilde{\nu} \frac{\omega_{ce}^2}{\omega_{pe}^2} = \omega e^{-\tilde{\nu}} \sum_{n=-\infty}^{+\infty} \frac{I_n(\tilde{\nu})}{(\omega - n\,\omega_{ce})} \tag{3.85}$$

From Eq. (3.54) it is seen that $\tilde{\nu}$ is real and positive, and therefore $I_n(\tilde{\nu})$ is also real and positive. Hence, writing the angular frequency as

$$\omega = \omega_r + i\,\omega_i \tag{3.86}$$

where ω_r and ω_i are the real and imaginary parts of ω, respectively, we can separate Eq. (3.85) into its real and imaginary parts, as follows:

$$\textit{real part:} \quad 1 + \tilde{\nu} \frac{\omega_{ce}^2}{\omega_{pe}^2} = e^{-\tilde{\nu}} \sum_{n=-\infty}^{+\infty} I_n(\tilde{\nu}) \left[1 + \frac{\omega_{ce} + n(\omega_r - n\omega_{ce})}{(\omega_r - n\omega_{ce})^2 + \omega_i^2} \right] \tag{3.87}$$

$$\textit{imaginary part:} \quad 0 = -\omega_i \; e^{-\tilde{\nu}} \sum_{n=-\infty}^{+\infty} I_n(\tilde{\nu}) \frac{n\,\omega_{ce}}{(\omega_r - n\omega_{ce})^2 + \omega_i^2} \tag{3.88}$$

It can be shown that Eq. (3.88) can be satisfied only if $\omega_i = 0$. This result means that the dispersion equation for the quasistatic wave has only real solutions for ω and, therefore, there is neither temporal damping nor instability of the quasistatic waves.

Next, we obtain explicit *real solutions for* ω, for two limiting cases. First, we consider the special case $\tilde{\nu} \ll 1$ which, as seen from Eq. (3.54), corresponds to the *zero-temperature limit*, and afterwards we analyse the case $\tilde{\nu} \gg 1$, which corresponds to the *high-temperature limit*.

For $\tilde{\nu} \ll 1$ (zero-temperature limit), we have $I_{\pm 1}(\tilde{\nu}) \simeq \tilde{\nu}/2$, while $I_{\pm n}(\tilde{\nu}) = 0(\tilde{\nu}^n)$. If ω/ω_{ce} is not close to n only the terms corresponding to $n = \pm 1$, in the infinite series on the right-hand side of (3.83), contribute significantly, whereas the other terms are small and can be neglected. Thus, Eq. (3.82) becomes for $\tilde{\nu} << 1$,

$$\tilde{\nu} \frac{\omega_{ce}^2}{\omega_{pe}^2} = - \frac{I_{-1}(\tilde{\nu})}{(\omega/\omega_{ce} + 1)} + \frac{I_1(\tilde{\nu})}{(\omega/\omega_{ce} - 1)} \tag{3.89}$$

which simplifies to

$$\omega = (\omega_{pe}^2 + \omega_{ce}^2)^{1/2} \tag{3.90}$$

This is known as the *upper hybrid resonant frequency.* This resonant frequency is also predicted in the cold plasma model treatment of waves propagating across a magnetostatic field. Thus, we find that the hot plasma theory confirms the results predicted by the cold plasma model in the zero-temperature limit.

In addition, the hot plasma theory establishes the existence of other resonant frequencies not predicted by the cold plasma model. The dispersion equation (3.82) can also be satisfied by taking $\omega = n\,\omega_{ce}$, for $n \geq 2$, and arranging such that only the n^{th} term contributes, which it will if $\omega/\omega_{ce} - n = 0\ (\tilde{\nu}^{n-1})$. Hence, in the zero-temperature limit ($\tilde{\nu} << 1$), the hot plasma theory predicts resonant frequencies at each harmonic of the electron cyclotron frequency,

$$\omega = n\,\omega_{ce} \qquad n \geq 2 \quad (\text{for } \tilde{\nu} << 1) \tag{3.91}$$

These resonant frequencies are not predicted by the cold plasma model.

For the high-temperature case ($\tilde{\nu} >> 1$), we have $e^{-\tilde{\nu}} \cdot I_n(\tilde{\nu}) = 0(\tilde{\nu}^{-1/2})$ and it is found that the dispersion relation (3.82) is satisfied for

$$\omega = n\,\omega_{ce} \qquad n \geq 1 \qquad (\text{for } \tilde{\nu} >> 1) \tag{3.92}$$

Therefore, in the limit $\tilde{\nu} >> 1$, the resonances occur at the fundamental, as well as at all the harmonics of the electron cyclotron frequency.

To obtain the resonant frequencies for intermediate values of $\tilde{\nu}$, Eq. (3.82) needs to be solved numerically. It is convenient, for numerical purposes, to rewrite (3.82) in the form

$$\tilde{\nu}\,\frac{\omega_{ce}^2}{\omega_{pe}^2} = F\,(\omega/\omega_{ce},\,\tilde{\nu}); \quad F\,(\omega/\omega_{ce},\,\tilde{\nu}) = 2\,e^{-\tilde{\nu}} \sum_{n=1}^{\infty} \frac{n^2\,I_n\,(\tilde{\nu})}{(\omega/\omega_{ce})^2 - n^2} \tag{3.93}$$

In Fig. 4 it is plotted the function $F\,(\omega/\omega_{ce}, \tilde{\nu})$ in terms of ω/ω_{ce}, for $\tilde{\nu} = 0.1$. The intersection points of this curve with the horizontal line corresponding to $\tilde{\nu}\,(\omega_{ce}^2/\omega_{pe}^2)$ give the resonant frequencies in the normalized form ω/ω_{ce}.

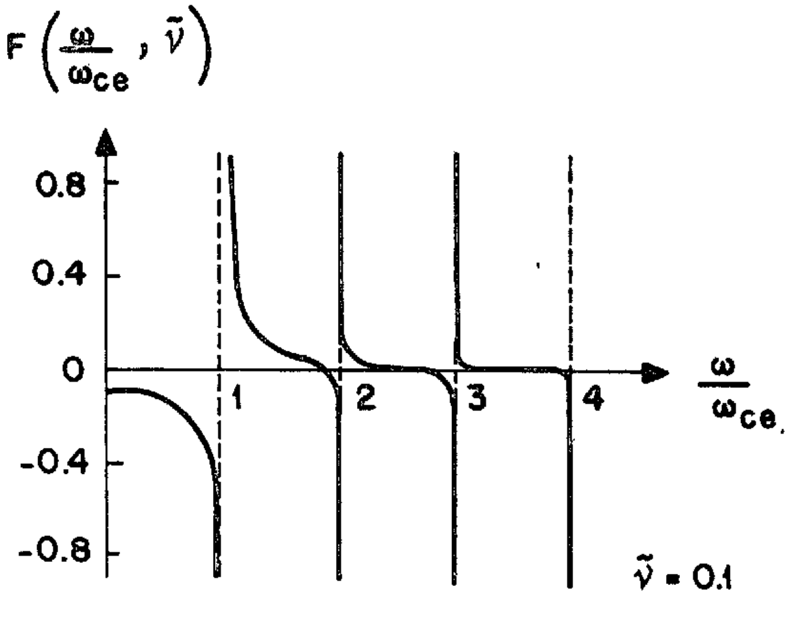

Fig. 4 - Dependence of the function F $(\omega/\omega_{ce}, \tilde{\nu})$, given by (3.93), in terms of ω/ω_{ce} for a fixed value of $\tilde{\nu}$ (here $\tilde{\nu}$ = 0.1), for the quasistatic waves.

In Fig. 5 it is shown the normalized resonant frequency, ω/ω_{ce}, as a function of $(\tilde{\nu})^{1/2}$, for a specified value of $(\omega_{ce}/\omega_{pe})$. Note, from this figure, that below each resonant frequency curve, corresponding to frequencies greater than the upper hybrid resonant frequency, there is a range of ω in which resonance does not occur for any value of $\tilde{\nu}$. Also, for $\tilde{\nu} << 1$ it is verified, from Fig. 5, that the first harmonic of the electron cyclotron frequency is not a solution of the dispersion equation (3.82).

An important difference between the quasistatic waves treated here, and the longitudinal plasma waves analyzed previously, is the absence of Landau damping for the quasistatic waves. The treatment of propagation of quasistatic waves at an arbitrary direction with respect to $\underset{\sim}{B}_0$ is left as an exercise for the student.

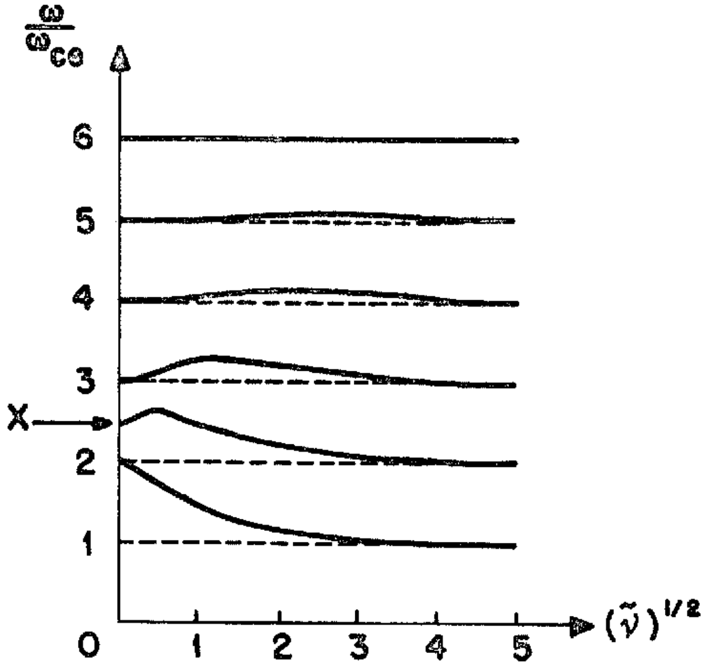

Fig. 5 - Curves of resonant frequencies for the quasistatic waves propagating across the magnetostatic field, as a function of $(\tilde{\nu})^{1/2}$, when $(\omega_{ce}/\omega_{pe})^2 = 0.2$. The resonant frequency, denoted by X, is the normalized upper hybrid frequency, $X = (\omega_{pe}^2 + \omega_{ce}^2)^{1/2} / \omega_{ce}$.

3.7 - <u>The TEM mode</u>

From Eqs. (3.78) and (3.72), the dispersion relation for the TEM mode propagating across the magnetostatic field in a hot plasma is given explicitly by

$$\frac{k^2 c^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega\, \omega_{ce}} \; e^{-\tilde{\nu}} \sum_{n=-\infty}^{+\infty} \frac{I_n(\tilde{\nu})}{(\omega/\omega_{ce} - n)} \tag{3.94}$$

This equation has to be analyzed numerically. However, some useful results can be obtained directly, without resorting to numerical work, for some special limiting cases.

For the limiting case $\tilde{\nu} << 1$, only the term corresponding to n = 0 is significant, while all other terms are small and can be neglected. Therefore, for $\tilde{\nu} << 1$, Eq. (3.94) simplifies to

$$\frac{k^2 c^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega^2} \tag{3.95}$$

where we have used the relation $I_0(0) = 1$. This result is the same as the dispersion relation for the TEM (ordinary) mode deduced from the cold plasma model. Thus we find that, in the limit of zero--temperature, the hot plasma theory agrees with the results of the cold plasma model, for the characteristics of the TEM mode propagating across the magnetostatic field.

For the limiting case $\tilde{\nu} >> 1$, we have $e^{-\tilde{\nu}} I_n(\tilde{\nu}) = 0(\tilde{\nu}^{-1/2})$ and Eq. (3.94) reduces to

$$\frac{kc}{\omega} = 1 \tag{3.96}$$

which is identical to the dispersion relation for electromagnetic waves propagating in free space. Note that the condition $\tilde{\nu} >> 1$, together with Eqs. (3.54) and (3.96), is equivalent to

$$\omega >> \omega_{ce} \left(\frac{m_e c^2}{k T_e}\right)^{1/2} \tag{3.97}$$

showing that the frequency must be very high. Hence, for $\tilde{\nu} >> 1$, or equivalently, for very high frequencies, the results of the hot plasma theory are also in agreement with those predicted by the cold plasma model.

Furthermore, according to the hot plasma theory, the TEM mode has resonances at the electron cyclotron frequency and all its harmonics, since (3.94) shows that $kc/\omega = \infty$ for

$$\omega = n\, \omega_{ce}; \qquad n \geqslant 1 \tag{3.98}$$

where n is an integer. The cold plasma model does not predict the existence of these harmonic resonances.

4. SUMMARY

4.1 - Propagation along $\underset{\sim}{B}_0$ in hot magnetoplasmas

(a) Longitudinal mode:

The dispersion relation is (with $\underset{\sim}{B}_0 = B_0\, \hat{\underset{\sim}{z}}$; $\underset{\sim}{k} = k\, \hat{\underset{\sim}{z}}$)

$$1 = - \frac{\omega_{pe}^2}{n_0\, \omega} \int_V \frac{v_\parallel}{(\omega - kv_\parallel)} \frac{\partial f_0\, (v_\perp, v_\parallel)}{\partial v_\parallel}\, d^3v \tag{2.66}$$

which is the same result obtained for the isotropic plasma.

(b) Transverse modes:

The dispersion relation, for the two transverse modes, is

$$k_{\pm}^2 c^2 - \omega^2 = \omega_{pe}^2 \frac{\pi}{n_o} \int_0^{\infty} v_{\perp}^2 \, dv_{\perp} \int_{-\infty}^{+\infty} \frac{(\omega - kv_{\parallel})(\partial f_o/\partial v_{\perp}) + kv_{\perp}(\partial f_o/\partial v_{\parallel})}{(\omega - kv_{\parallel} \mp \omega_{ce})} dv_{\parallel} \quad (2.69)$$

The *upper* sign gives the *right* circularly polarized wave, and the *lower* sign gives the *left* circularly polarized wave. An alternative form for this dispersion relation is

$$k_{\pm}^2 c^2 - \omega^2 = -\frac{\omega_{pe}^2}{n_o} \int_v \left[\frac{(\omega - kv_{\parallel})}{(\omega - kv_{\parallel} \mp \omega_{ce})} + \frac{k^2 v_{\perp}^2 / 2}{(\omega - kv_{\parallel} \mp \omega_{ce})^2} \right] f_o(v_{\perp}, v_{\parallel}) \, d^3v \quad (2.72)$$

When f_o is the isotropic Maxwell-Boltzmann distribution function,

$$k_{\pm}^2 c^2 - \omega^2 = \omega_{pe}^2 \left[i\sqrt{\pi}\, \beta_{\pm} (-\alpha_{\pm}^2) - 2\beta_{\pm} \int_0^{\alpha_{\pm}} \exp(W^2 - \alpha_{\pm}^2) \, dW \right] \quad (2.82)$$

In the limits of cold and warm plasma models,

$$k_{\pm}^2 c^2 = \omega^2 - \omega_{pe}^2 \frac{\omega}{(\omega \mp \omega_{ce})} \quad (2.85)$$

The Landau (temporal) damping is negligible, since $v_{ph} \gtrsim c$. But the right circularly polarized wave has temporal damping (cyclotron damping) for $\omega_r = \omega_{ce}$. The cyclotron damping constant (with $\omega = \omega_r + i\omega_i$), is

$$\omega_i = -\frac{1}{2} k_+ \left[\frac{(2k_B T_e/m_e)^{1/2}}{\pi^{1/2}} c^2 \left(\frac{\omega_{ce}}{\omega_{pe}}\right)^2 \right]^{1/3} \qquad (2.93)$$

4.2 - Propagation across $\underset{\sim}{B}_o$ in hot magnetoplasmas

(a) Transverse magnetic (TM) modes:

The longitudinal and transverse modes are coupled. The dispersion relation (with $\underset{\sim}{B}_o = B_o \hat{\underset{\sim}{z}}$; $\underset{\sim}{k} = k \hat{\underset{\sim}{x}}$), when f_o is the isotropic Maxwell-Boltzmann distribution function, is

$$\frac{k^2 c^2}{\omega^2} = \frac{\varepsilon_{xx}\,\varepsilon_{yy} - \varepsilon_{xy}\,\varepsilon_{yx}}{\varepsilon_{xx}} \qquad (3.76)$$

The TM mode corresponds to the extraordinary wave in magnetoionic theory (cold plasma).

In the limit $kc/\omega \to \infty$ (resonance condition),

$$\varepsilon_{xx} = 1 - \frac{\omega_{pe}^2}{\omega\,\omega_{ce}} \frac{e^{-\tilde{\nu}}}{\tilde{\nu}} \sum_{n=-\infty}^{+\infty} \frac{n^2 I_n(\tilde{\nu})}{(\omega/\omega_{ce} - n)} = 0 \qquad (3.79)$$

which is called the dispersion relation for the quasistatic mode (longitudinal mode; $E_x \neq 0$). In the limit $kc/\omega \to \infty$ the two TM modes are uncoupled, and the equation $\varepsilon_{xx} = 0$ applies to the longitudinal mode. The resonances are given by

$$\omega = (\omega_{pe}^2 + \omega_{ce}^2)^{1/2} \text{ ; cold plasma result} \tag{3.90}$$

$$\left.\begin{aligned} &\omega = n\,\omega_{ce}; \quad n \geq 2 \quad \text{for } \tilde{\nu} \ll 1 && (3.91) \\ &\omega = n\,\omega_{ce}; \quad n \geq 1 \quad \text{for } \tilde{\nu} \gg 1 && (3.92) \end{aligned}\right\} \text{Bernstein modes}$$

(b) <u>Transverse electric magnetic (TEM) mode</u>:

It corresponds to the ordinary mode in magnetoionic theory (cold plasma). The dispersion relation, when f_o is the Maxwell-Boltzmann distribution function, is

$$\frac{k^2\,c^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega\,\omega_{ce}}\, e^{-\tilde{\nu}} \sum_{n=-\infty}^{+\infty} \frac{I_n(\tilde{\nu})}{(\omega/\omega_{ce} - n)} \tag{3.94}$$

In the limit of cold plasma model ($\tilde{\nu} \ll 1$),

$$k^2\,c^2 = \omega^2 - \omega_{pe}^2 \tag{3.95}$$

In hot plasma theory the resonances are given by

$$\omega = n\,\omega_{ce}; \qquad n \geq 1 \tag{3.98}$$

PROBLEMS

19.1 - Show that the first and second terms in the right-hand side of (2.16) represent, respectively, right and left circularly polarized wave fields.

19.2 - Derive expression (3.61) for σ_{zz}, starting from Eq. (3.60).

19.3 - Consider plane wave disturbances propagating along the magnetostatic field $\underset{\sim}{B}_o$ in a hot electron gas, whose equilibrium distribution function is homogeneous and isotropic. In spherical coordinates in velocity space (v, Θ, ϕ) with $\underset{\sim}{B}_o = B_o \hat{\underset{\sim}{z}}$ and $\underset{\sim}{k} \parallel \underset{\sim}{B}_o$ (Fig. P 19.1), show that the linearized Vlasov equation reduces to

$$i\omega_{ce} \frac{df_1(\underset{\sim}{v})}{d\phi} + (\omega - \underset{\sim}{k} \cdot \underset{\sim}{v})\, f_1\, (\underset{\sim}{v}) = \frac{ie}{m_e}\, \underset{\sim}{E} \cdot \underset{\sim}{\nabla}_v\, f_o(v)$$

Verify that this differential equation has the formal solution

$$f_1\, (\underset{\sim}{v}) = \frac{e}{m_e\, \omega_{ce}} \int_{-\infty}^{\phi} \underset{\sim}{E} \cdot \underset{\sim}{\nabla}_{v'}\, f_o(v)\, \exp\left[\frac{i}{-\omega_{ce}} (\omega - \underset{\sim}{k} \cdot \underset{\sim}{v})(\phi - \phi')\right] d\phi'$$

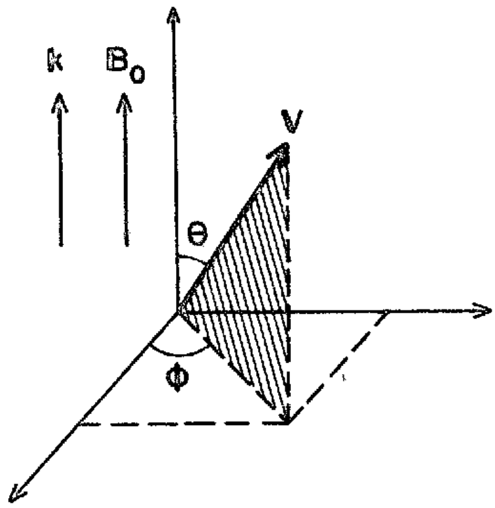

Fig. P 19.1

where $\underset{\sim}{v}'$ is the velocity vector with components (v, θ, ϕ'). Note that $\nabla_{\underset{\sim}{v}'} f_o(v) = (\underset{\sim}{v}'/v)\, d f_o(v)/dv$. Perform the integral in this expression for $f_1(\underset{\sim}{v})$ to obtain

$$f_1(\underset{\sim}{v}) = - \frac{ie}{m_e \omega_{ce}} \left\{ \frac{1}{(A^2-1)} \left[E_x \left(A \frac{\partial f_o}{\partial v_x} - i \frac{\partial f_o}{\partial v_y} \right) + E_y \left(A \frac{\partial f_o}{\partial v_y} + i \frac{\partial f_o}{\partial v_x} \right) \right] + \right.$$

$$\left. + \frac{E_z}{A} \frac{\partial f_o}{\partial v_z} \right\}$$

where $A = -(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v})/\omega_{ce}$. From Maxwell equations obtain the relation

$$\underset{\sim}{E} - \left(\frac{kc}{\omega} \right)^2 \underset{\sim}{E}_t = \frac{ie}{\omega \varepsilon_o} \int_v \underset{\sim}{v}\, f_1(\underset{\sim}{v})\, d^3v$$

where $\underset{\sim}{E}_t = \underset{\sim}{E} - E_z \hat{\underset{\sim}{z}}$ is the transverse part of the electric field $\underset{\sim}{E}$. Using the expression for $f_1(\underset{\sim}{v})$ in this equation, show that we obtain a dispersion relation with three wave solutions: the usual Landau damped longitudinal waves, and the left and the right circularly polarized waves (with $E_x = \pm i\, E_y$).

19.4 - An electron gas, immersed in a uniform magnetostatic field $\underset{\sim}{B}_o$, is characterized by the following modified Maxwellian distribution function

$$f_o(v_\parallel,\ v_\perp) = n_o \left(\frac{m_e}{2\pi\, k_B\, T_\parallel}\right)^{1/2} \left(\frac{m_e}{2\pi\, k_B\, T_\perp}\right) \exp\left(-\frac{m_e v_\parallel^2}{2\, k_B\, T_\parallel} - \frac{m_e\, v_\perp^2}{2\, k_B\, T_\perp}\right)$$

Use this distribution function in the dispersion relation for the *right circularly polarized transverse wave* propagating along $\underset{\sim}{B}_o$ [Eq.(2.69)],

$$k^2 c^2 = \omega^2 + \frac{\omega_{pe}^2\, \pi}{n_o} \int_0^\infty v_\perp^2\, dv_\perp \int_{-\infty}^{+\infty} \frac{(\omega - kv_\parallel)(\partial f_o/\partial v_\perp) + kv_\perp(\partial f_o/\partial v_\parallel)}{(\omega - kv_\parallel - \omega_{ce})}\, dv_\parallel$$

and evaluate the integrals to obtain the following dispersion relation

$$k^2 c^2 = \omega^2 - \tau\,\omega_{pe}^2 - i\sqrt{\pi}\,\omega_{pe}^2\;\tau(\alpha - \beta)\,\exp(-\alpha^2) +$$

$$+\,2\,\omega_{pe}^2\;\tau\,(\alpha - \beta)\int_0^{\alpha}\exp(W^2 - \alpha^2)\,dW$$

where

$$\tau = 1 - (T_\perp / T_\parallel)$$

$$\alpha = \frac{(\omega - \omega_{ce})/k}{(2\,k_B\,T_\parallel/m_e)^{1/2}}$$

$$\beta = \frac{\omega/(k\,\tau)}{(2\,k_B\,T_\parallel/m_e)^{1/2}}$$

Analyse this dispersion relation to verify the existence or not of instabilities (positive imaginary part of ω) and/or damping (negative imaginary part of ω) of the wave amplitude considering the propagation coefficient $\underset{\sim}{k} = k\,\underset{\sim}{\hat{z}}$ to be real. Determine the cyclotron damping coefficient. Analyze the results considering the isotropic case for which $T_\perp = T_\parallel$.

19.5 - In Problem 19.4 suppose that in the equilibrium state the velocity distribution function of the electrons is given by

$$f_o(\underset{\sim}{v}) = n_o \left(\frac{m_e}{2\pi k_B T_e} \right)^{3/2} \exp\left\{ - \frac{m_e}{2 k_B T_e} \left[v_\perp^2 + (v_\parallel - u_o)^2 \right] \right\}$$

which corresponds to an isotropic distribution but with the electrons drifting with speed u_o along $\underset{\sim}{B}_o$. Show that, with this choice of $f_o(\underset{\sim}{v})$, the dispersion relation for the right circularly polarized wave reduces to

$$k^2 c^2 = \omega^2 - \frac{\omega_{pe}^2}{n_o} \int_v \frac{(\omega - k u_o)}{\omega - k v_\parallel - \omega_{ce}} f_o(\underset{\sim}{v})\, d^3v$$

For the limiting case of $T_e = 0$, find the form of the distribution function $f_o(\underset{\sim}{v})$ and show that the dispersion relation becomes

$$k^2 c^2 = \omega^2 - \frac{\omega_{pe}^2 (\omega - k u_o)}{(\omega - k u_o - \omega_{ce})}$$

19.6 - For an unbounded homogeneous electron gas, characterized by the following velocity distribution function

$$f_o(\underset{\sim}{v}) = n_o \frac{a_o}{\pi^2} (v^2 + a_o^2)^{-2}$$

where a_o is a constant, show that the dispersion relation for the right circularly polarized wave, propagating along the magnetostatic field $\underset{\sim}{B}_o = B_o \hat{\underset{\sim}{z}}$, is given by

$$k^2 c^2 = \omega^2 - \frac{\omega_{pe}^2 \; \omega}{\omega + i k a_o - \omega_{ce}}$$

From this result show that the cyclotron damping coefficient is given approximately by

$$\omega_i = - \frac{k}{2} \left[a_o \left(c \frac{\omega_{ce}}{\omega_{pe}} \right)^2 \right]^{1/3}$$

19.7 - (a) Show that, starting from the Vlasov equation and the laws of electrostatics, it is obtained the following dispersion relation for the *quasistatic wave* propagating at an arbitrary direction with respect to a magnetostatic field $\underset{\sim}{B}_o$ in a hot plasma:

$$\frac{\omega_{ce}^2 \; \tilde{\nu}}{\omega_{pe}^2 \; \sin^2 \theta} = \frac{k_B T_e}{m_e} \left(\frac{k}{\omega_{pe}} \right)^2 = - \exp(-\tilde{\nu}) \sum_{n=-\infty}^{+\infty} (1 + \tilde{\omega} H_n) \, I_n(\tilde{\nu})$$

where

$$\tilde{\nu} = \frac{k_B T_e}{m_e} \quad \frac{k^2 \sin^2 \Theta}{\omega_{ce}^2}$$

$$H_n = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} d\tilde{v}_z \frac{\exp(-\tilde{v}_z^2)}{(\tilde{v}_z + n\,\tilde{\omega}_{ce} - \tilde{\omega})}$$

$$\tilde{v}_z = \frac{v_z}{V_e} \quad ; \qquad V_e = \left(\frac{2\, k_B\, T_e}{m_e} \right)^{1/2}$$

$$\tilde{\omega} = \omega/(k \cos \Theta\ V_e)$$

$$\tilde{\omega}_{ce} = \omega_{ce}/(k \cos \Theta\ V_e)$$

I_n $(\tilde{\nu})$ is the Bessel function of the second kind, and Θ is the angle between $\underset{\sim}{k}$ and $\underset{\sim}{B}_o$, as indicated in Fig. P 19.2.

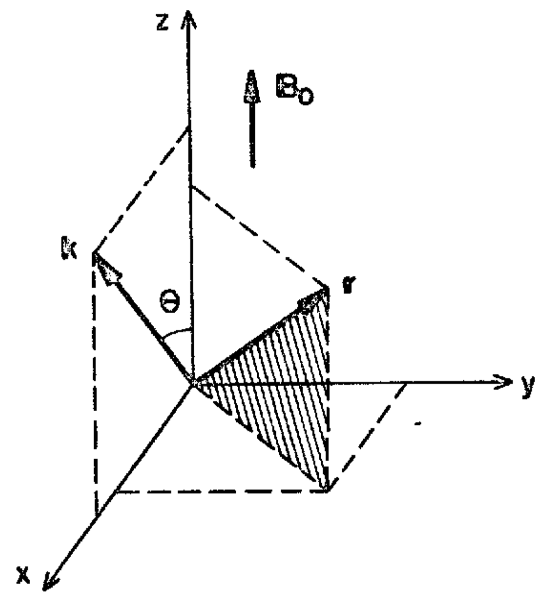

Fig. P 19.2

(b) Rewrite this dispersion relation in the form

$$1 + \frac{k_B T_e}{m_e} \left(\frac{k}{\omega_{pe}} \right)^2 = - i \omega \int_0^\infty dt \exp \Big\{ i \omega t -$$

$$- \left[1 - \cos(\omega_{ce} t)\right] k^2 V_e^2 \sin^2 \Theta / 2 \omega_{ce}^2 - k^2 V_e^2 t^2 \cos^2 \Theta / 4 \Big\}$$

(c) Simplify this expression for the case of a very weak magnetostatic field to obtain the following approximate expression for the frequency of oscillation

$$\omega^2 = \omega_{pe}^2 + \frac{3 k_B T_e}{m_e} k^2 + \omega_{ce}^2 \sin^2 \Theta$$

Compare this result with the cold and the warm plasma results for both the cases of $\underset{\sim}{k} \parallel \underset{\sim}{B}_o$ and $\underset{\sim}{k} \perp \underset{\sim}{B}_o$

19.8 - Deduce the dispersion relation for small amplitude waves propagating at an *arbitrary direction* with respect to an externally applied magnetostatic field $\underset{\sim}{B}_o = B_o \hat{\underset{\sim}{z}}$ in a hot plasma. Carry through the derivation as far as possible for an arbitrary value of the strength of the magnetostatic field. Then, particularize for the special case of a very weak magnetostatic field. For simplicity, assume the equilibrium distribution function to be the isotropic Maxwell-Boltzmann distribution (you may refer to the article "Waves in a Plasma in a Magnetic Field , by Ira B. Bernstein, *Physical Review,* 109 (1), 10-21, 1958).