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16.Summary/Notes

This is the twentieth chapter, in a series of twenty two, written on the fundamentals of plasma physics. It describes the dynamics of binary collisions and introduces the concepts of differential scattering cross section, total scattering cross section and momentum transfer cross section. These cross sections are explicitly calculated for the cases of the hard sphere model and of the Coulomb interaction potential. The effect of screening of the Coulomb potential, due to the Debye shielding, are incorporated on the cross sections for the Coulomb potential in the final section of the chapter.

17.Remarks

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CHAPTER 20

PARTICLE INTERACTIONS IN PLASMAS

1. INTRODUCTION

The fundamental properties of a plasma depend upon the interactions of the plasma particles with the force fields existing inside it. These fields may be externally applied, or they can be internal fields associated with the nature and motion of the particles themselves. These mutual particle interactions are called collisions. In this chapter the words *collision* and *interaction* are used synonymously. The notion of a collision as a physical contact between bodies loses its utility in the microscopic world. In the atomic level a *collision* between particles must be regarded as an *interaction* between the fields of force associated with each of the interacting particles.

Collisional phenomena can be broadly divided into two categories: *elastic* and *inelastic*. In *elastic collisions* there is conservation of mass, momentum and energy in such a way that there are no changes in the internal states of the particles involved, and there is neither creation nor annihilation of particles. In *inelastic collisions* the internal states of some or all of the particles involved are changed, and particles may be created, as well as destroyed. In inelastic collisions a charged particle may *recombine*

with another to form a neutral particle; it can *attach* itself with a neutral particle to form a heavier charged particle; the energy state of an electron in an atom may be raised and electrons can be removed from their atoms resulting in *ionization*.

In plasmas there is an important distinction to be made between interactions involving electrically charged particles and interactions involving charged and neutral particles. Electrically charged particles interact with one another according to Coulomb's law. This Coulomb interaction, in view of its $1/r^2$ dependence, is a *long-range interaction,* so that the field of one particle interacts simultaneously with a large number of other particles. Therefore, it involves *multiple* interactions. In contrast, the fields associated with neutral particles are significantly strong only within the electronic shells of the particles. Thus, they are *short-range fields* and a neutral particle only occasionally interacts with another particle, and very rarely it interacts simultaneously with more than one particle. Therefore, these short-range fields result primarily in *binary* interactions.

The multiple particle Coulomb interaction, however, can be thought of as a number of simultaneous binary interactions. In fact, one way of dealing with multiple interactions is to consider that a series of consecutive small-angle binary interactions describes the

situation. The multiple interactions, which result from the Coulomb force, are of essential importance in understanding the behavior of plasmas and underlines the validity of describing a plasma as the fourth state of matter. Nevertheless, binary collisions adequately describe plasma phenomena in the case of *weakly ionized plasmas*. In fact, we use the term weakly ionized plasma to mean a plasma in which multiple particle interactions can be ignored. In these plasmas the electrons tend to dominate the situation, since they respond quickly to the influence of electric and magnetic fields, in view of their low inertia.

In this chapter we deal with the collision processes that are of importance in plasmas, from the point of view of classical dynamics. The results are valid to a good approximation, even though the internal structure of the particles is ignored. More important, however, the procedures to be developed are useful whether the mechanics is classical or quantum.

2. BINARY COLLISIONS

Consider an elastic collision between two particles of mass m and m_1, having velocities $\underset{\sim}{v}$ and $\underset{\sim}{v}_1$ before collision, and $\underset{\sim}{v}'$ and $\underset{\sim}{v}'_1$ after collision. This binary interaction is illustrated in Fig. 1, as seen from the laboratory system. In what follows, the variables indicated with a *prime* are *after* - collision variables.
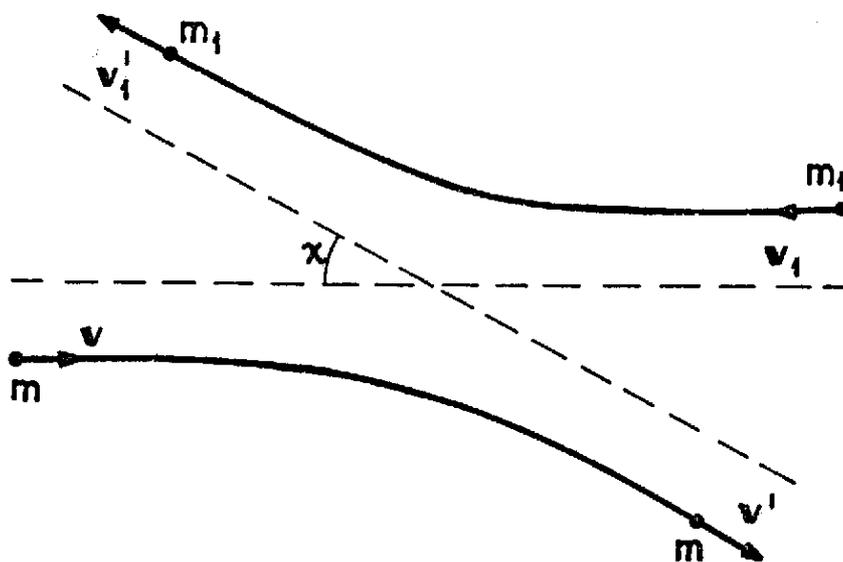
Fig. 1 - Binary interaction of two particles of mass m
and m_1, with velocities $\underline{v}$ and $\underline{v}_1$, respectively,
as viewed from the laboratory system.

It is convenient to adopt a coordinate system in which the particle having mass m is at rest, and the particle having mass m_1 approaches with the *relative velocity*

$$\underline{g} = \underline{v}_1 - \underline{v} \tag{2.1}$$

After collision, the relative velocity is

$$\underline{g}' = \underline{v}'_1 - \underline{v}' \tag{2.2}$$

The geometry of the interaction is shown in Fig. 2. The *impact parameter*, defined as the minimum distance of approach if there were no interaction, is denoted by b, the *scattering angle* by χ, and the

orientation of the *orbital plane* (or *collision plane*), with respect to some given direction in a plane normal to the orbital plane, is denoted by ε.
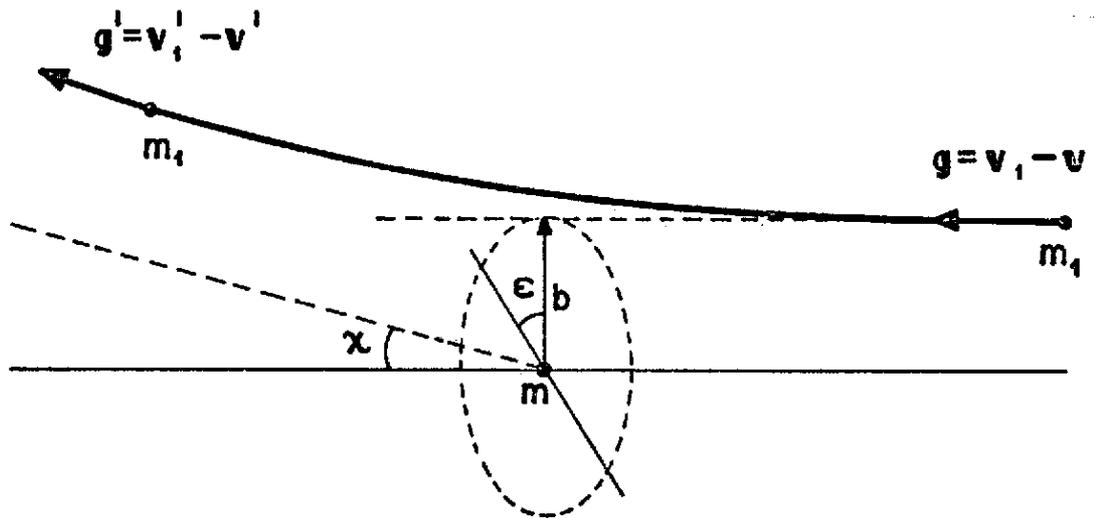
Fig. 2 - Geometry of a collision between a particle of mass m and velocity $\underset{\sim}{v}$, and a particle of mass m_1 and velocity $\underset{\sim}{v}_1$, viewed from a coordinate system in which the first particle is at rest.

The velocity of the *center of mass* of the colliding particles, before collision, is defined by

$$\underset{\sim}{c}_0 = \frac{m\underset{\sim}{v} + m_1 \underset{\sim}{v}_1}{m + m_1} \tag{2.3}$$

and, after collision, by

$$\underset{\sim}{c}'_0 = \frac{m\underset{\sim}{v}' + m_1 \underset{\sim}{v}'_1}{m + m_1} \tag{2.4}$$

We can express the initial velocities in terms of c_0 and g. From Eqs. (2.3) and (2.1), which define c_0 and g, respectively, we find

$$\underset{\sim}{v} = \underset{\sim}{c}_0 - (\mu/m)\,\underset{\sim}{g} \tag{2.5}$$

$$\underset{\sim}{v}_1 = \underset{\sim}{c}_0 + (\mu/m_1)\,\underset{\sim}{g} \tag{2.6}$$

where μ denotes the reduced mass, defined by

$$\mu = \frac{m\,m_1}{m + m_1} \tag{2.7}$$

Similarly, from (2.4) and (2.2) we obtain, for the final velocities,

$$\underset{\sim}{v}' = \underset{\sim}{c}_0' - (\mu/m)\,\underset{\sim}{g}' \tag{2.8}$$

$$\underset{\sim}{v}_1' = \underset{\sim}{c}_0' + (\mu/m_1)\,\underset{\sim}{g}' \tag{2.9}$$

From the law of *conservation of momentum* for the collision event, we have

$$m\,\underset{\sim}{v} + m_1\,\underset{\sim}{v}_1 = m\underset{\sim}{v}' + m_1\,\underset{\sim}{v}_1' \tag{2.10}$$

or, using Eqs. (2.3) and (2.4),

$$(m + m_1) \; \underset{\sim}{c}_0 \; = \; (m + m_1) \; \underset{\sim}{c}_0' \tag{2.11}$$

Thus,

$$\underset{\sim}{c}_0 = \underset{\sim}{c}_0' \tag{2.12}$$

that is, the velocity of the center of mass is the same before and after the interaction event.

From the law of *conservation of energy* for elastic collisions, we have

$$\frac{1}{2} \; (mv^2 + m_1 \; v_1^2) = \frac{1}{2} \left[m \; (v')^2 + m_1 \; (v_1')^2 \right] \tag{2.13}$$

and using Eqs. (2.5), (2.6), (2.8) and (2.9) we find, by direct calculation,

$$\frac{1}{2} \; (mv^2 + m_1 \; v_1^2) = \frac{1}{2} \; (m + m_1) \; c_0^2 + \frac{1}{2} \; \mu g^2 \tag{2.14}$$

$$\frac{1}{2} \left[m \; (v')^2 + m_1 \; (v_1')^2 \right] = \frac{1}{2} \; (m + m_1) \; (c_0')^2 + \frac{1}{2} \; \mu \; (g')^2 \tag{2.15}$$

Now, since $\underset{\sim}{c}_0 = \underset{\sim}{c}_0'$, we conclude that

$$g = g' \tag{2.16}$$

Thus, the *magnitude*, but not the direction of the relative velocity
is conserved in a binary elastic collision. Eqs. (2.14) and (2.15)
show that the total instantaneous kinetic energy of the two-particle
system is equivalent to that associated with the motion of the center
of mass plus the motion of one particle relative to the other, but
using the reduced mass.

The angle between $\underline{g}$ and $\underline{g}'$ is called the *scattering
angle*, or *deflection angle*, and is denoted by χ. To relate the relative
velocity vectors $\underline{g}$ and $\underline{g}'$, we can choose, for instance, a Cartesian
coordinate system with the z-axis along $\underline{g}$, as shown in Fig. 3.
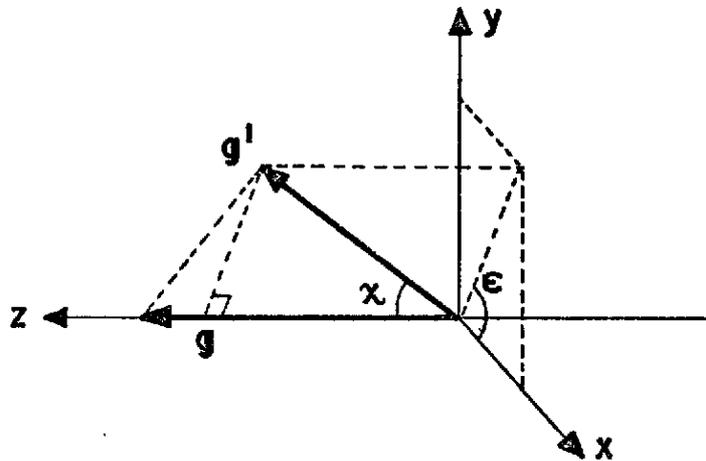
Fig. 3 - Relationship between the relative velocities $\underline{g}$
and $\underline{g}'$, in a Cartesian coordinate system for
which $\underline{g}$ = g $\hat{z}$. The angle ε defines the relative
orientation of the plane containing the
particle trajectory.

Thus, we have

$$g_x = g_y = 0 \qquad\qquad (2.17)$$

$$g_z = g = g' \qquad\qquad (2.18)$$

$$g'_x = g \sin (\chi) \cos (\varepsilon) \qquad\qquad (2.19)$$

$$g'_y = g \sin (\chi) \sin (\varepsilon) \qquad\qquad (2.20)$$

$$g'_z = g \cos (\chi) \qquad\qquad (2.21)$$

where ε defines the relative orientation of the *collision plane*. Therefore, knowing the initial velocites and the scattering angle χ we can determine the after-collision velocities. The opposite is also true, that is, if we know the final velocities and the scattering angle we can find the initial velocities.

It is of interest to consider the *inverse collision* (see Fig. 1, of Chapter 7), in which a particle with initial velocity $\underline{v}'$ collides with another particle having initial velocity $\underline{v}'_1$, the velocities *after* collision being $\underline{v}$ and $\underline{v}_1$, respectively. For the inverse collision the scattering angle χ is the same as that for the direct collision, since the impact parameter b, the interparticle force law and the relative speed g are all the same.

The scattering angle is the only quantity appearing in the analysis presented in this section that depends on the details of the collision process. For interparticle force laws which depend only on the distance between the interacting particles, χ depends

on the following quantities:

(a) Interparticle force law;

(b) Magnitude of the relative velocity, g;

(c) The value of the impact parameter, b.

Therefore, in order to determine χ, we must analyse the classical dynamics of binary collisions.

3. DYNAMICS OF BINARY COLLISIONS

The dynamics of a binary collision is governed by the interparticle force law. For each impact parameter, b, there will be associated a given scattering angle, χ, the relation being dependent on the interparticle force law. This information is contained in the *differential cross section*, which is defined in section 5.

Consider the collision of two particles of masses m and m_1, viewed from a system of reference in which the first particle is at rest. Let $\underset{\sim}{r}$ be the position vector of the particle of mass m_1 with respect to that of mass m (Fig. 4). The force of interaction between the two particles is assumed to be a central force, which acts along the straight line joining the two particles, that is,

$$\underset{\sim}{F} (\underset{\sim}{r}) = F (r) \; \hat{\underset{\sim}{r}} \tag{3.1}$$

This force is related to the potential energy, ϕ $(\underset{\sim}{r})$, of the

interaction by the condition

$$\underset{\sim}{F}\,(\underset{\sim}{r}) = -\underset{\sim}{\nabla}\,\phi\,(\underset{\sim}{r})$$

$$= -\frac{\partial\phi\,(r)}{\partial r}\,\hat{\underset{\sim}{r}} \tag{3.2}$$
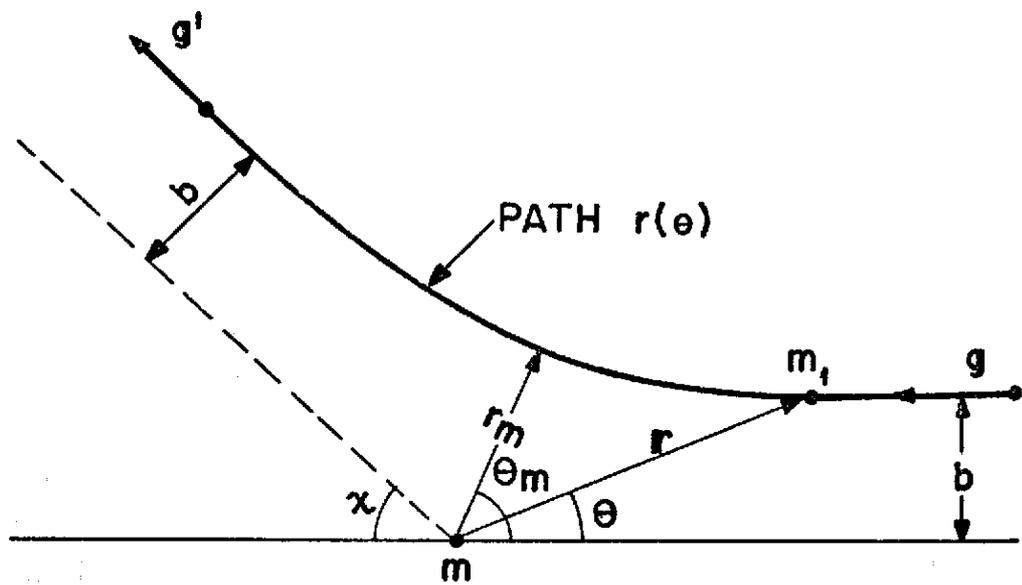
Fig. 4 - Path r (θ) of the particle of mass m_1, relative to the particle of mass m.

For a central force the torque $\underset{\sim}{N} = \underset{\sim}{r} \times \underset{\sim}{F}(\underset{\sim}{r})$ vanishes, because $\underset{\sim}{F}\,(\underset{\sim}{r})$ is parallel to $\underset{\sim}{r}$. Since the torque is the time rate of change of the angular momentum, $\underset{\sim}{L}$,

$$\underset{\sim}{N} = \frac{d\underset{\sim}{L}}{dt} \tag{3.3}$$

we conclude that the angular momentum is a constant of the motion. Furthermore, since $\underline{L} = \underline{r} \times \underline{p}$, we see that $\underline{r}$ is always normal to the constant direction of $\underline{L}$ in space, and the motion lies therefore in a plane.

Using polar coordinates (r, θ) and noting that the unit vectors $\hat{\underline{r}}$ and $\hat{\underline{\theta}}$ depend on θ (Fig. 5), we have for the instantaneous relative velocity,

$$\frac{d\underline{r}}{dt} = \frac{dr}{dt} \, \hat{\underline{r}} + r \, \frac{d\hat{\underline{r}}}{dt}$$

$$= \frac{dr}{dt} \, \hat{\underline{r}} + r \, \frac{d\hat{\underline{r}}}{d\theta} \, \frac{d\theta}{dt} \tag{3.4}$$

Since it can be shown that $d\hat{\underline{r}}/d\theta = \hat{\underline{\theta}}$, we obtain
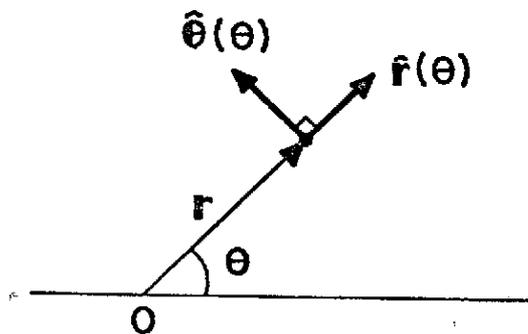
Fig. 5 - Polar coordinates (r, θ), showing that the directions of the unit vectors $\hat{\underline{r}}$ and $\hat{\underline{\theta}}$ depend on θ.

$$\frac{d\underset{\sim}{r}}{dt} = \frac{dr}{dt} \hat{\underset{\sim}{r}} + r \frac{d\Theta}{dt} \hat{\underset{\sim}{\theta}} \tag{3.5}$$

or, using a dot over the variable to denote the time derivative,

$$\dot{\underset{\sim}{r}} = \dot{r} \hat{\underset{\sim}{r}} + r \dot{\theta} \hat{\underset{\sim}{\theta}} \tag{3.6}$$

The trajectory of the particle in the equivalent one body problem can be easily found by using the laws of conservation of *energy* and *angular momentum*. The kinetic energy of the relative motion is given by

$$K = \frac{1}{2} \mu \dot{\underset{\sim}{r}} \cdot \dot{\underset{\sim}{r}} = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\theta}^2) \tag{3.7}$$

where μ is the reduced mass, defined by $m m_1/(m + m_1)$. From the law of conservation of energy we can equate the kinetic plus potential energy, at any point, to the initial kinetic energy ($\frac{1}{2} \mu g^2$), since the initial potential energy is zero. Thus, we have

$$\frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\theta}^2) + \phi (r) = \frac{1}{2} \mu g^2 \tag{3.8}$$

The angular momentum relative to the origin is given by

$$\underset{\sim}{L} = \underset{\sim}{r} \times (\mu \dot{\underset{\sim}{r}}) = \mu r^2 \dot{\theta} (\hat{\underset{\sim}{r}} \times \hat{\underset{\sim}{\theta}}) \tag{3.9}$$

Setting the angular momentum, at any point, equal to its initial value, $b\mu g$ ($\hat{r} \times \hat{\theta}$), we get

$$r^2 \dot{\theta} = bg \tag{3.10}$$

From Eqs. (3.8) and (3.1) we can easily obtain a differential equation for the orbit r (θ). First, we write

$$\frac{dr}{dt} = \frac{dr}{d\theta} \frac{d\theta}{dt} \tag{3.11}$$

and use Eqs. (3.10) and (3.8) to eliminate $d\theta/dt$ and dr/dt. The resulting differential equation for the trajectory is found to be

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{r^4}{b^2} \left[1 - \frac{b^2}{r^2} - \frac{2\phi \ (r)}{\mu g^2}\right] \tag{3.12}$$

Rearranging (3.12) yields the following result

$$d\theta = \pm \frac{b}{r^2} \left[1 - \frac{b^2}{r^2} - \frac{2\phi \ (r)}{\mu g^2}\right]^{-1/2} dr \tag{3.13}$$

The choice of sign must be made on physical grounds. The coordinates of the position of the particle, when it is at the distance of closest approach, are denoted by r_m and θ_m (Fig. 4). This position is called the *vertex* of the trajectory, and the line connecting the origin to the vertex is called the *apse line*. Thus, θ_m specifies the orientation of the apse line. The plus sign in Eq. (3.13) must be

used when θ is greater than θ_m, since for $\theta > \theta_m$ we see that r increases with θ. On the other hand, for $\theta < \theta_m$ we see that r decreases as θ increases, so that the minus sign in Eq. (3.13) must be used when θ is less than θ_m. This also shows that the trajectory is *symmetrical* about the apse line.

The *distance of closest approach*, r_m, can be obtained from Eq. (3.12) by noting that $dr/d\theta = 0$ when $r = r_m$. Thus, we have

$$1 - \frac{b^2}{r_m^2} - \frac{2\phi\ (r_m)}{\mu g^2} = 0 \tag{3.14}$$

or

$$r_m = b\left[1 - \frac{2\phi\ (r_m)}{\mu g^2}\right]^{-1/2} \tag{3.15}$$

To compute the *scattering angle*, χ, we first note, from Fig. 4, that

$$\chi = \pi - 2\theta_m \tag{3.16}$$

In order to determine θ_m we integrate Eq. (3.13) from θ_m to some other angle θ, obtaining

$$\theta - \theta_m = \pm \int_{r_m}^{r} \frac{b}{r'^2}\left[1 - \frac{b^2}{r'^2} - \frac{2\phi\ (r')}{\mu g^2}\right]^{-1/2} dr' \tag{3.17}$$

where the plus sign is to be used when $\theta > \theta_m$ and the minus sign when $\theta < \theta_m$. When $r \to \infty$ we have $\theta_{(-)} \to 0$, while $\theta_{(+)} \to 2\theta_m$, so that Eq. (3.17) gives, for the orientation of the apse line,

$$\theta_m = \int_{r_m}^{\infty} \frac{b}{r^2} \left[1 - \frac{b^2}{r^2} - \frac{2\phi(r)}{\mu g^2}\right]^{-1/2} dr \tag{3.18}$$

The scattering angle is therefore given by

$$\chi(b, g) = \pi - 2 \int_{r_m}^{\infty} \frac{b}{r^2} \left[1 - \frac{b^2}{r^2} - \frac{2\phi(r)}{\mu g^2}\right]^{-1/2} dr \tag{3.19}$$

To compute χ from this equation we must know the impact parameter b, the magnitude of the initial relative velocity g, and the interparticle potential energy $\phi(r)$.

4. EVALUATION OF χ FOR SOME SPECIAL CASES

In this section we present two examples of the use of Eq. (3.19) to determine the scattering angle in terms of the impact parameter b, and the initial relative velocity g. First, we consider the collision between two perfectly elastic hard spheres and afterwards the case of the Coulomb interaction.

4.1 - Two perfectly elastic hard spheres

Consider the collision between two perfectly elastic hard spheres of radii R_1 and R_2 (Fig. 6). The potential energy of interaction is given by

$$\phi \ (r) \ = \ \begin{cases} 0 \quad \text{for} \quad r > R_1 + R_2 \\ \\ \infty \quad \text{for} \quad r < R_1 + R_2 \end{cases} \tag{4.1}$$

For $b > R_1 + R_2$ there is no interaction and we must have $r_m = b$, whereas for $b < R_1 + R_2$ the particles collide and we have $r_m = R_1 + R_2$. In either case, however, since the spheres are impenetrable, we have $r \geqslant R_1 + R_2$, so that Eq. (3.19) becomes
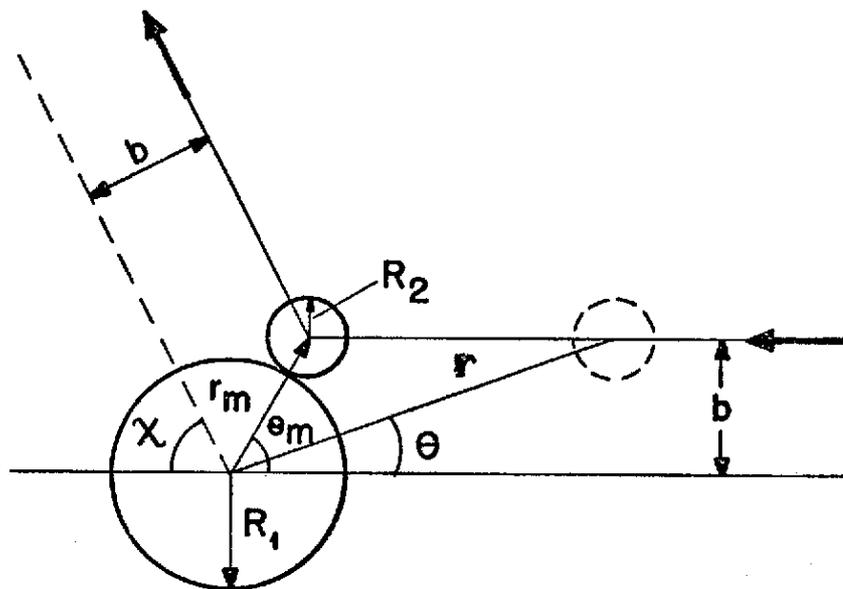
Fig. 6 - Collision between two perfectly elastic impenetrable spheres.

$$\chi = \pi - 2 \int_{r_m}^{\infty} \frac{b}{r^2} \left(1 - \frac{b^2}{r^2}\right)^{-1/2} dr \qquad (4.2)$$

To solve this integral it is convenient to define a new variable by $y = b/r$ and write Eq. (4.2) in the form

$$\chi = \pi - 2 \int_{0}^{b/r_m} (1 - y^2)^{-1/2} dy \qquad (4.3)$$

which gives

$$\chi = \pi - 2 \sin^{-1} (b/r_m) \qquad (4.4)$$

Therefore, we find that

$$\chi = \begin{cases} \pi - 2 \sin^{-1} \left[b/(R_1 + R_2) \right] & \text{for } b \leqslant R_1 + R_2 \\ 0 & \text{for } b \geqslant R_1 + R_2 \end{cases} \qquad (4.5)$$

4.2 - Coulomb interaction potential

Let us consider now the important case of the Coulomb potential field, whose potential energy of interaction is given by

$$\phi (r) = \frac{1}{4\pi \epsilon_0} \frac{q \, q_1}{r} \qquad (4.6)$$

where q and q_1 denote the electric charge of the particles of mass m and m_1, respectively. Substituting (4.6) into (3.19), gives

$$\chi(b, g) = \pi - 2 \int_{r_m}^{\infty} \frac{b}{r^2} \left(1 - \frac{b^2}{r^2} - \frac{2 q q_1}{\mu g^2 4\pi\epsilon_o r}\right)^{-1/2} dr \qquad (4.7)$$

The distance of closest approach, r_m, is obtained from Eqs. (3.15) and (4.6), and is found to be

$$r_m = \frac{b^2}{- b_o + \sqrt{b_o^2 + b^2}} \qquad (4.8)$$

where, for convenience, we have introduced the notation

$$b_o = \frac{q q_1}{4\pi\epsilon_o \mu g^2} \qquad (4.9)$$

Making the change of variable $y = 1/r$ and inserting the value for b_o, given by (4.9), into (4.7), gives for the deflection angle

$$\chi(b, g) = \pi - 2b \int_{o}^{1/r_m} (1 - 2b_o y - b^2 y^2)^{-1/2} dy \qquad (4.10)$$

The integral appearing here is of the standard form

$$\int \left(\alpha + \beta x + \gamma x^2 \right)^{-1/2} dx = \frac{1}{\sqrt{-\gamma}} \sin^{-1} \left[\frac{(-2\gamma x - \beta)}{(\beta^2 - 4\alpha\gamma)^{1/2}} \right] \tag{4.11}$$

where, in our case, $\alpha = 1$, $\beta = -2b_0$, and $\gamma = -b^2$. Applying the limits of integration, with r_m as given by (4.8), yields for the deflection angle

$$\chi(b, g) = 2 \sin^{-1} \left[\frac{b_0}{(b_0^2 + b^2)^{1/2}} \right] \tag{4.12}$$

This equation for $\chi(b, g)$ can be written in the alternative form

$$\tan\left(\frac{\chi}{2}\right) = \frac{b_0}{b} \tag{4.13}$$

Note that for $\chi = \pi/2$ we have $b = b_0$, that is, b_0 is the value of the impact parameter for a 90^0 deflection angle. If the sign of the two charged particles are the same, then b_0 and χ will both the positive. On the other hand, if the sign of the two charged particles are opposite, then b_0 and χ will be negative. The two situations are illustrated in Fig. 7, for a deflection angle of 90^0. It is also noted, from Eq. (4.13), that $\chi = \pi$ for $b = 0$, χ decreases as b increases, and $\chi = 0$ only in the limit of $b \to \infty$. Therefore, scattering occurs for all (finite) values of the impact parameter b, so that there is no cut-off value for b.
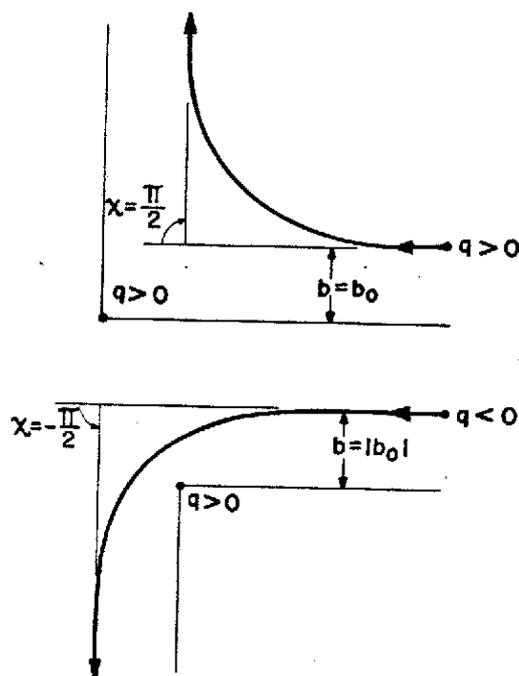
Fig. 7 - Scattering in a Coulomb potential field for a deflection angle, χ, of 90°.

5. CROSS SECTIONS

So far we have considered specifically only the interaction between two particles. Cross sections are usually defined in terms of a beam of identical particles incident on a center of force (target particle). Therefore, let us imagine a steady beam of identical particles of mass m_1, uniformly spread out in space, incident with velocity $\underset{\sim}{g} = \underset{\sim}{v_1} - \underset{\sim}{v}$ upon the center of force provided by the particle of mass m, in its rest frame of reference. For simplicity, the incident beam of particles is assumed to be monoenergetic, so that all the particles in the beam have the same initial relative velocity, $\underset{\sim}{g}$, with respect to the scattering particle. Since the particles in the incident beam are supposed to be identical, the interaction potential is the same for all the particles in the beam.

The particles incident with an impact parameter b are scattered through some deflection angle χ, whereas the particles incident with an impact parameter b + db will be scattered through the deflection angle χ + $d\chi$ (Fig. 8). The number of particles scattered per second, between the angles χ and χ + $d\chi$, depends on the incident particle current density (particle flux), $\underline{\Gamma}$, that is, on the number of particles in the incident beam crossing a unit area normal to the beam per unit time.

5.1 - Differential scattering cross section

Let dN/dt denote the number of particles scattered per unit time into the differential element of solid angle $d\Omega$, oriented at (χ, ε), as shown in Fig. 8. The *differential elastic scattering cross section*, $\sigma(\chi, \varepsilon)$ (also referred to, in the literature, as the *angular distribution function*), is defined as the number of particles scattered per unit time, per unit incident flux and per unit solid angle oriented at (χ, ε). Thus, according to this definition we have

$$\frac{dN}{dt} = \sigma (\chi, \varepsilon) \; \Gamma \; d\Omega \tag{5.1}$$

The number of particles incident per unit time, with impact parameter between b and b + db, and with the orbital plane oriented between ε and ε + $d\varepsilon$, is Γ bdb $d\varepsilon$. These same particles are scattered per unit time into the differential element of solid angle $d\Omega$ contained between χ and χ + $d\chi$ and between ε and ε + $d\varepsilon$. Thus, we have

$$\frac{dN}{dt} = \Gamma \; b \; db \; d\varepsilon \tag{5.2}$$
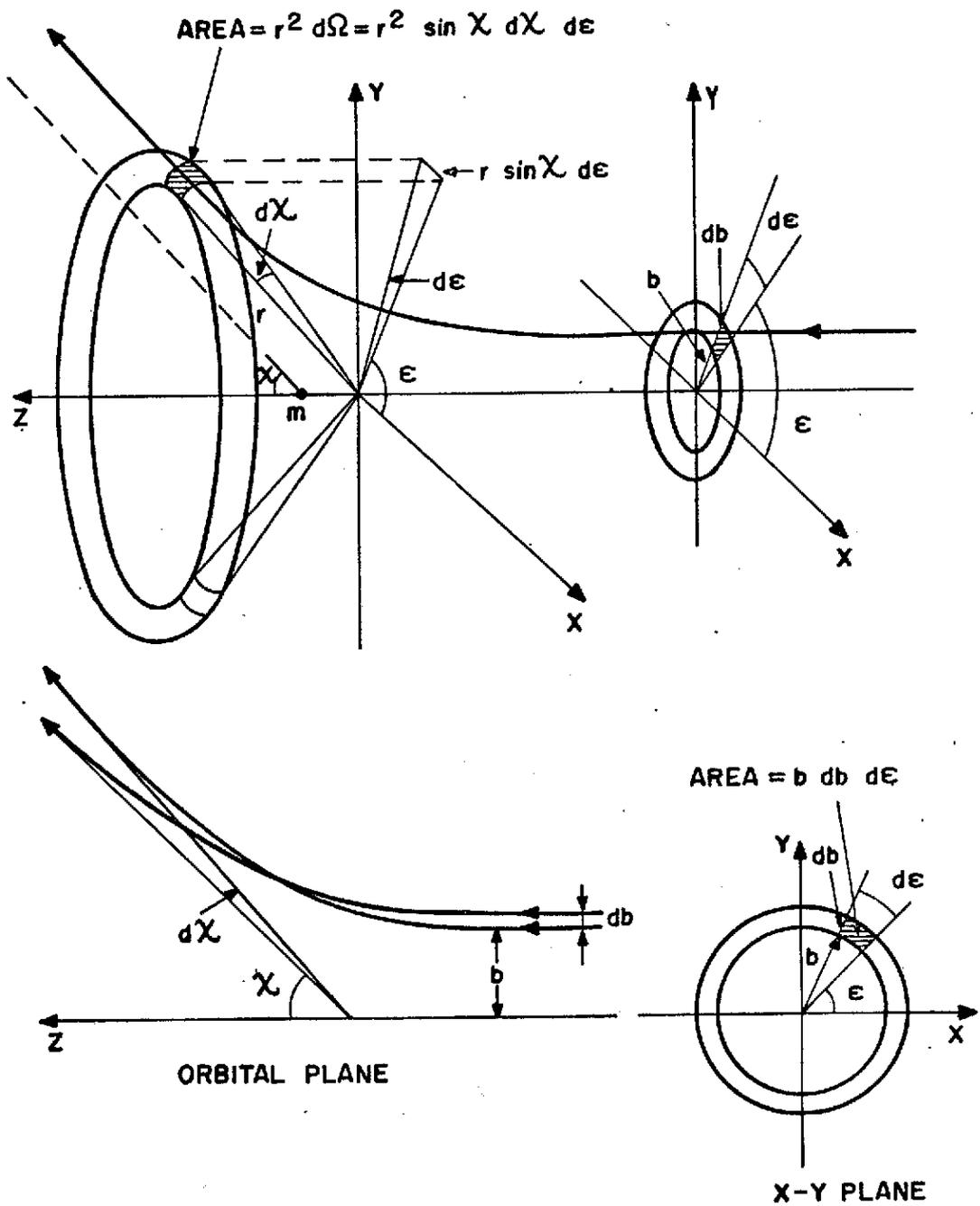
Fig. 8 - Scattering in a central field of force.

Comparing Eqs. (5.1) and (5.2) we see that, according to the definition of the differential scattering cross section, σ (χ, ε), we have

$$\sigma \ (\chi, \ \varepsilon) \ d\Omega = b \ db \ d\varepsilon \tag{5.3}$$

Since $d\Omega = \sin \chi \ d\chi \ d\varepsilon$, this equation can also be written as

$$\sigma \ (\chi, \ \varepsilon) \ \sin \chi \ d\chi = b \ db \tag{5.4}$$

Solving for the differential scattering cross section, we obtain

$$\sigma \ (\chi, \ \varepsilon) = \frac{b}{\sin \chi} \left| \frac{db}{d\chi} \right| \tag{5.5}$$

The absolute value of $db/d\chi$ is used here, because χ normally decreases when b increases, and the differential scattering cross section σ (χ, ε) is inherently a positive quantity, since it is associated with the number of particles being scattered. The quantity $db/d\chi$ can be obtained from Eq. (3.19), which gives χ (b, g) once the potential energy function, ϕ (r), is known.

The differential cross section has the dimensions of area and may be interpreted in a geometrical way such that the number of particles scattered into the solid angle element $d\Omega$ per second is equal to the number of particles crossing an area equal to σ (χ, ε) $d\Omega$ (or b db $d\varepsilon$) in the incident beam per second.

The form of σ (χ, ε) depends on the interparticle force law and, if this force law is known, σ (χ, ε) can be calculated. However, a quantum mechanical calculation must be employed for this purpose, since the quantum wave packets of the colliding particles necessarily overlap and the problem is no longer a classical one. For a collection of atoms or molecules to be regarded classically, with each particle having a rather well-defined position and momentum, it is necessary that the particles be localized wave packets whose extensions are small compared to the average interparticle distance. For a classical treatment the average de Broglie wavelength of each particle must, therefore, be much smaller than the average interparticle separation.

The differential cross section is also a directly measurable quantity and can be obtained experimentally. For our purposes, we will consider the differential cross section, σ (χ, ε), which contains the nature of the collisional interaction, as a known quantity.

5.2 - Total scattering cross section

The *total scattering cross section*, σ_t, is defined as the number of particles scattered per unit time and per unit incident flux, in *all directions* from the scattering center. It is obtained by integrating σ (χ, ε) dΩ over the entire solid angle,

$$\sigma_t = \int_\Omega \sigma(\chi, \varepsilon) \, d\Omega = \int_0^{2\pi} d\varepsilon \int_0^\pi \sigma(\chi, \varepsilon) \sin \chi \, d\chi \qquad (5.6)$$

Both $\sigma(\chi, \varepsilon)$ and σ_t depend on the magnitude of the relative particle velocity, g.

In the special case when the interaction potential is *isotropic*, that is, when the differential scattering cross section is independent of ε, we can immediately perform the integral over ε in (5.6), to get

$$\sigma_t = 2\pi \int_0^\pi \sigma(\chi) \sin \chi \, d\chi \qquad (5.7)$$

This is the case, for example, of the Coulomb interaction potential.

5.3 - Momentum transfer cross section

A cross section can be defined for various processes of interaction. It will be seen later that the transfer of momentum, during a collision, is the basic microscopic event in the transport phenomena of diffusion and mobility. Hence, it is appropriate to define a cross section for the rate of transfer of momentum, σ_m, as the total momentum transferred per unit time to the scattering center, per unit incident momentum flux (momentum per normal unit area, per unit time),

$$\sigma_m = \frac{(\text{momentum transferred per second})}{(\text{incident momentum flux})} \tag{5.8}$$

The momentum of a particle in the beam, before interaction, is μg, where μ is the reduced mass and g is the initial relative velocity. The incident momentum flux is therefore $\Gamma\mu g$. After interaction, the momentum of a particle in the beam, in the direction of incidence, and which is scattered at an angle χ, is $\mu g \cos\chi$. Therefore, the momentum transferred by this particle to the scattering center is $\mu g \, (1 - \cos\chi)$. The total momentum transferred per second to the scattering center, by all the particles scattered in all directions in space, is given by

$$\Gamma \, \mu g \int_\Omega (1 - \cos\chi) \, \sigma \, (\chi, \, \varepsilon) \, d\Omega \tag{5.9}$$

Recall that $\sigma \, (\chi, \, \varepsilon)$ can be considered as an angular distribution function. Since the total incident momentum flux is $\Gamma \, \mu g$, we obtain for the *momentum transfer cross section*,

$$\sigma_m = \int_\Omega (1 - \cos\chi) \, \sigma \, (\chi, \, \varepsilon) \, d\Omega \tag{5.10}$$

For the special case of an *isotropic* interaction potential, and noting that $d\Omega = \sin\chi \, d\chi \, d\varepsilon$, we can perform the integral over ε, in Eq. (5.10), obtaining

$$\sigma_m = 2\pi \int_0^\pi (1 - \cos \chi) \; \sigma (\chi) \; \sin \chi \; d\chi \qquad (5.11)$$

Since $\sigma (\chi)$ is an angular distribution function, it can be used as a weight function to calculate the *mean value* of any function $F (\chi)$ of the scattering angle. The contribution to the total value of $F (\chi)$, resulting from the particles scattered into $d\Omega$, is $F (\chi) \; \sigma (\chi) \; d\Omega$. Since the total number of particles scattered is $\int \sigma (\chi) \; d\Omega$, it follows that the *mean value* of $F (\chi)$, averaged over all values of χ, is given by

$$< F (\chi) > \; = \frac{\displaystyle\int_\Omega F (\chi) \; \sigma (\chi) \; d\Omega}{\displaystyle\int_\Omega \sigma (\chi) \; d\Omega} \qquad (5.12)$$

which may be written as

$$< F (\chi) > \; = \frac{2\pi}{\sigma_t} \int_0^\pi F (\chi) \; \sigma (\chi) \; \sin \chi \; d\chi \qquad (5.13)$$

According to this definition of mean values we see that Eq. (5.11) can be written in the form

$$\sigma_m = \sigma_t < 1 - \cos \chi > \qquad (5.14)$$

Thus, the cross section for momentum transfer is a weighted cross section in which scattering angles of zero degrees do not count at all, scatterings of 90° count as one, and scatterings of 180° count as two. This weighting is proportional to the amount of momentum transferred from the incident beam to the scattering center.

6. CROSS SECTIONS FOR THE HARD SPHERE MODEL

6.1 - Differential scattering cross section

To calculate $\sigma(\chi, \varepsilon)$, as given in Eq. (5.5), we first obtain from (4.5), for $b \leq R_1 + R_2$,

$$b = (R_1 + R_2) \cos (\chi/2) \tag{6.1}$$

and

$$\left| \frac{db}{d\chi} \right| = \frac{1}{2} (R_1 + R_2) \sin (\chi/2) \tag{6.2}$$

Substituting these last two expressions into (5.5), yields for the differential scattering cross section

$$\sigma = (R_1 + R_2)^2/4 \qquad (6.3)$$

6.2 - Total scattering cross section

Integrating (6.3) over the whole solid angle, we obtain

$$\sigma_t = 2\pi \int_0^\pi \frac{(R_1 + R_2)^2}{4} \sin \chi \, d\chi = \pi \, (R_1 + R_2)^2 \qquad (6.4)$$

Two special simple cases may be mentioned here. For the collision between an electron and a molecule of radius R, we have $\sigma = R^2/4$ and $\sigma_t = \pi R^2$. For molecules colliding with themselves, their diameter being D, we have $\sigma = D^2/4$ and $\sigma_t = \pi D^2$.

Note that in this case there is a cut-off value for the impact parameter b, beyond which collisions do not occur. It is the existence of this cut-off value for b that leads to a finite value for the total scattering cross section σ_t. This conclusion is made clear in the next section.

6.3 - Momentum transfer cross section

From Eqs. (6.3) and (5.11) we obtain

$$\sigma_m = 2\pi \int_0^\pi \frac{(R_1 + R_2)^2}{4}(1 - \cos \chi)\, \sin \chi\; d\chi \qquad (6.5)$$

$$= \frac{\pi\,(R_1 + R_2)^2}{2}\left(\int_0^\pi \sin \chi\, d\chi - \int_0^\pi \cos \chi\, \sin \chi\, d\chi\right)$$

Performing the integrals, yields

$$\sigma_m = \pi\,(R_1 + R_2)^2 \qquad (6.6)$$

The average value of momentum loss per particle is found, from Eq. (5.13), to be given by

$$< \mu g\,(1 - \cos \chi) > \; = \frac{2\pi}{\sigma_t} \int_0^\pi \mu g\,(1 - \cos \chi)\,\sigma\,(\chi)\,\sin \chi\; d\chi \qquad (6.7)$$

Using Eq. (5.11), we get

$$< \mu g\,(1 - \cos \chi) > \; = \mu g\;\sigma_m/\sigma_t \qquad (6.8)$$

Thus, from Eqs. (6.4), (6.7) and (6.8) we deduce that the average value of momentum loss, per particle, for the hard sphere model, is

$$< \mu g\,(1 - \cos \chi) > \; = \mu g \qquad (6.9)$$

For collisions between electrons and neutral particles, for example, in a weakly ionized plasma, the mass of the electron can be neglected as compared to the mass of the neutral particle, so that the reduced mass becomes equal to the electron mass. From Eq. (6.9) it is seen, in the first approximation, that the entire momentum of an electron is lost in a collision with a neutral particle. Assuming that the motion of the heavy particles can be ignored, and if ν is the collision frequency, that is, the number of collisions between electrons and neutral particles per second, then the rate of loss of momentum of an average electron is $\nu\, m_e\, \underset{\sim}{u}$, where $\underset{\sim}{u}$ denotes the electron velocity. However, in general an electron does not lose its entire momentum on a collision with a neutral particle, and also the perfectly elastic hard sphere model is not a very good representation for the interaction of an electron with a neutral particle. Consequently, rate of loss of momentum is written as, $\nu_{en}\, m_e\, \underset{\sim}{u}$, where ν_{en} is an *effective collision frequency* for momentum transfer between electrons and neutral particles. This term is used in the *Langevin equation*, introduced in Chapter 10, to represent the time rate of transfer of momentum due to collisions.

7. CROSS SECTIONS FOR THE COULOMB POTENTIAL FIELD

7.1 - Differential scattering cross section

Differentiating Eq. (4.13) we find

$$\left| \frac{db}{d\chi} \right| = \frac{b^2}{2b_0 \cos^2 (\chi/2)} \tag{7.1}$$

Thus, the differential scaterring cross section, given in Eq. (5.5), becomes

$$\sigma(\chi) = \frac{b^3}{2b_0 \sin (\chi) \cos^2 (\chi/2)} \tag{7.2}$$

Using (4.13) this equation can be rearranged as

$$\sigma(\chi) = \frac{b_0^2}{4 \sin^4(\chi/2)} \tag{7.3}$$

This equation is known as the *Rutherford scattering formula*. Since $2 \sin^2 (\chi/2) = (1 - \cos \chi)$, it can also be written as

$$\sigma \, (\chi) \;\; = \;\; \frac{b_0^2}{(1 - \cos \, \chi)^2} \tag{7.4}$$

The Rutherford scattering formula shows that the differential scattering cross section is equal to $b_0^2/4$ for the deflection angle $\chi = \pi$, increases monotonically as χ is decreased, and tends to infinity as χ tends to zero.

7.2 - Total scattering cross section

Since the differential scattering cross section increases rapidly to infinity as χ goes to zero, it turns out that the total scattering cross section σ_t becomes infinite. From Eqs. (5.7) and (7.4), we obtain

$$\sigma_t = 2\pi \int_{\chi_{min}}^{\pi} \sigma \, (\chi) \, \sin \chi \;\; d\chi \; = \; 2\pi \, b_0^2 \int_{\chi_{min}}^{\pi} \frac{\sin \, \chi}{(1 - \cos \, \chi)^2} \; d\chi \tag{7.5}$$

where $\chi_{min} = 0$. The lower limit has been written implicitly for reasons that will become apparent in what follows. Evaluating the integral in (7.5), yields

$$\sigma_t = \pi b_0^2 \left[\frac{1}{\sin^2 \, (\chi_{min}/2)} - 1 \right] \tag{7.6}$$

which clearly gives $\sigma_t = \infty$ for $\chi_{min} = 0$. The particles with very small deflection angles contribute to make σ_t infinite.

7.3 - Momentum transfer cross section

The substitution of (7.4) into (5.11) gives the following expression for the momentum transfer cross section

$$\sigma_m = 2\pi \int_{\chi_{min}}^{\pi} (1 - \cos \chi) \ \sigma (\chi) \sin \chi \ d\chi = 2\pi \ b_0^2 \int_{\chi_{min}}^{\pi} \frac{\sin \chi}{(1 - \cos \chi)} d\chi$$

$$(7.7)$$

where, again, $\chi_{min} = 0$. Evaluating the integral we find that

$$\sigma_m = 4\pi \ b_0^2 \ \ell n \left[\frac{1}{\sin (\chi_{min}/2)} \right] \qquad (7.8)$$

Setting $\chi_{min} = 0$ we also find that $\sigma_m = \infty$. Thus, the Coulomb potential gives infinite values for both σ_t and σ_m, the particles with very small deflections angles being responsible for this infinite result.

8. EFFECT OF SCREENING OF THE COULOMB POTENTIAL

The infinite results for σ_t and σ_m, obtained in the previous section, may be interpreted as due to the absence of a cut-off value for the impact parameter b. Note that small values of χ

correspond to large values of b, and that for $\chi_{min} = 0$ we must have, from Eq. (4.13), $b_{max} = \infty$. In order to obtain finite and meaningful values for σ_t and σ_m, it is necessary to modify the basis of the treatment of interactions between individual charged particles and introduce, on some plausible grounds, a cut-off value $b = b_c$ for the impact parameter.

From Eqs. (5.3) and (5.6) we have for the total scattering cross section (considering σ independent of ε)

$$\sigma_t = 2\pi \int_0^{b_c} b \ db \tag{8.1}$$

where a cut-off value $b = b_c$ has been introduced for the impact parameter. With this cut-off, σ_t for the Coulomb potential is found to be

$$\sigma_t = \pi \ b_c^2 \tag{8.2}$$

The introduction of a cut-off value for b corresponds to the assumption that, for the charged particles incident with b greater than b_c there will be no interaction, whereas for the charged particles incident with b less than b_c there will be a Coulomb type interaction with the target particle.

The deflections that yield scattering angles between $\pi/2$ and π, and which are associated with values of b between 0 and b_0,

are usually called *large-angle deflections*, or *close encounters*. If only the large-angle deflections are taken into account, we obtain

$$\sigma_{t,large} = \pi b_0^2 \qquad (\pi/2 < \chi < \pi) \qquad (8.3)$$

with b_0 as given by (4.9).

If the charged particle is located inside a plasma we know that it will be surrounded by a shielding cloud of particles of opposite sign. The scale length for an effective shielding of the charged particle under consideration is the *Debye length* defined by

$$\lambda_D = (\frac{\varepsilon_0 \, kT}{n_0 \, e^2})^{1/2} \qquad (8.4)$$

The sphere of radius λ_D, surrounding the charged particle under consideration, is called the *Debye sphere*. We have seen (Chapter 11) that the charged particles lying within the Debye sphere shield the Coulomb potential due to the charged particle under consideration, reducing significantly its effect on the particles lying outside its Debye sphere. Taking this screening effect into account, we find that the interaction potential energy is of the form

$$\phi \ (r) \ = \ \frac{q \ q_1}{4\pi\varepsilon_0 \ r} \ \exp \ (\ - r/\lambda_D) \qquad\qquad (8.5)$$

Thus, when $r \ll \lambda_D$ the Debye potential, as given by (8.5), is very nearly equal to the Coulomb potential, whereas when $r \gg \lambda_D$ the Debye potential is nearly equal to zero. The analysis required for calculating σ_t, using the Debye potential, is excessively complicated and it must be done numerically. However, an alternative simple approach can be used that leads to results in very good agreement with those evaluated numerically using the Debye potential. It consists in assuming that the interaction potential is exactly equal to the Coulomb potential for $r < \lambda_D$ and is equal to zero for $r > \lambda_D$, as illustrated in Fig. 9. Therefore, it is convenient and more legitimate to introduce the cut-off in the impact parameter at $b_c = \lambda_D$ and not at $b_c = b_o$. In general we have

$$\lambda_D \ \gg \ b_o \qquad\qquad (8.6)$$

It is usual to denominate the deflections corresponding to $b_o < b < \lambda_D$, leading to $\chi < \pi/2$, as *small-angle deflections*. The contribution to the total scattering cross section from the small-angle deflections is deduced to be given by

$$\sigma_{t,small} \ = \ 2\pi \int_{b_o}^{\lambda_D} b \ db \ = \ \pi \ (\lambda_D^2 - b_o^2) \qquad (\chi < \pi/2) \qquad\qquad (8.7)$$
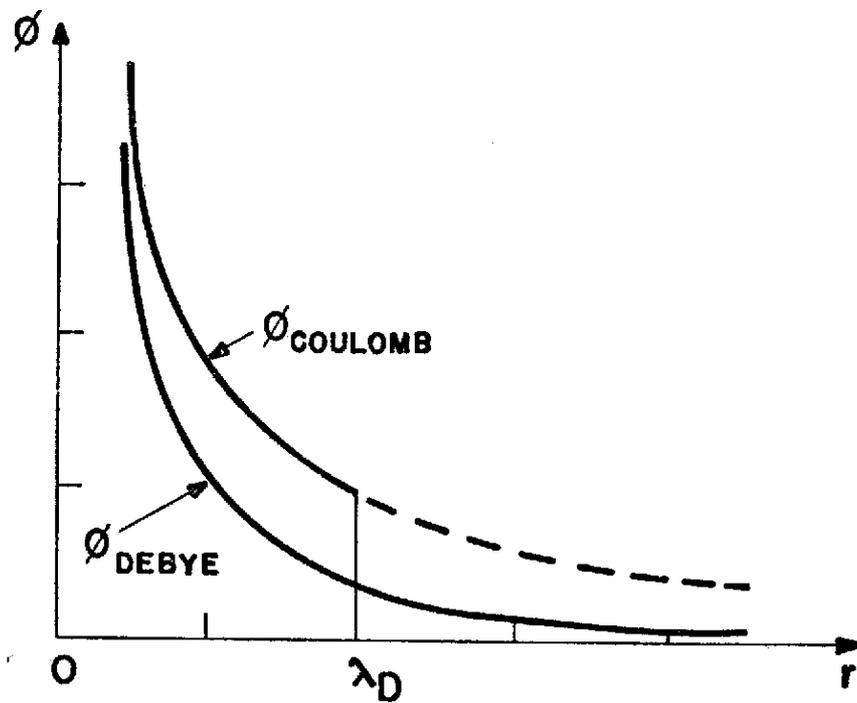
Fig. 9 - The approximation used to obtain $\sigma_t = \pi\lambda_D^2$ consists in assuming that for $r < \lambda_D$ the shielding effect is completely omitted and the particles interact according to the Coulomb potential, whereas for $r > \lambda_D$ the shielding of the target particle is assumed to be complete and there is no interaction.

Therefore, it follows from Eqs. (8.3) and (8.7) that

$$\frac{\sigma_{t, \text{small}}}{\sigma_{t, \text{large}}} = (\frac{\lambda_D}{b_o})^2 - 1 \approx (\frac{\lambda_D}{b_o})^2 \qquad (8.8)$$

since $\lambda_D \gg b_o$. This result shows that the large number of particles interacting mildly with the target particle, and therefore producing only small-angle deflections, are much more important than the small number of particles interacting strongly with the target particle and producing large-angle deflections. Therefore, if the impact parameter is cut-off at $b_c = \lambda_D$ we obtain, from (8.1), the following value for the total scattering cross section

$$\sigma_t = \pi \lambda_D^2 \qquad (8.9)$$

For the momentum transfer cross section, introducing the cut-off at $b_c = \lambda_D$, we obtain from Eq. (7.8)

$$\sigma_m = 2 \pi b_o^2 \, \ell n \left[1 + (\lambda_D/b_o)^2 \right] \qquad (8.10)$$

since from Eq. (4.12), letting $\chi = \chi_c$ for $b = b_c$, we have

$$\sin (\chi_c/2) = \left[1 + (b_c/b_o)^2 \right]^{-1/2} \qquad (8.11)$$

Using the notation

$$\Lambda = \lambda_D / b_o \qquad (8.12)$$

and noting that in general $\Lambda \gg 1$, Eq. (8.10) becomes

$$\sigma_m = 4\pi \, b_o^2 \, \ell n \, \Lambda \qquad (8.13)$$

It can be shown that for σ_m, as for the case of σ_t, the large number of particles producing small-angle deflections are much more important than the small number of particles producing large-angle deflections.

The function $\ell n \, \Lambda$ varies very slowly over a large range of variation of the parameters on which Λ depends. For most laboratory plasmas $\ell n \, \Lambda$ lies between 10 and 20. In order to calculate Λ it is usual to make some approximations. For this purpose consider the interaction between an electron gas (charge $q = -e$) and a singly charged ion gas ($q_1 = e$; $Z = 1$). Let n_0 denote the number density of both electrons and ions in the gas, which we assume to constitute a plasma, and let the temperature of the electrons and ions be equal to T. If we further assume that the electron and ion velocities have a Maxwellian equilibrium distribution function with no drift velocity, then we find, by direct calculation,

$$< g^2 > = \frac{1}{n_o^2} \int_v \int_{v_1} f_e \, f_{i1} \, (\underline{v}_1 - \underline{v})^2 \, d^3v \, d^3v_1 = \frac{1}{n_o} \int_v f_e \, (\frac{3kT}{m_i} + v^2) \, d^3v$$

$$= 3kT/\mu \qquad (8.14)$$

where k is Boltzmann's constant and μ is the reduced mass. Replacing g^2, in Eq. (4.9), by its average value, we find (for $q_1 = -q = e$)

$$b_o = \frac{e^2}{4\pi\varepsilon_o \, \mu <g^2>} = \frac{e^2}{12\,\pi\,\varepsilon_o \, kT} \tag{8.15}$$

Substituting this result into the expression (8.12), with λ_D as given by (8.4), gives

$$\Lambda = \frac{12\,\pi\,\varepsilon_o \, kT}{e^2} \, \lambda_D = 12\,\pi\,n_o \, \lambda_D^3 = 9 \, N_D \tag{8.16}$$

where N_D is the number of electrons in a Debye sphere. Table 1 presents values for $\ln \Lambda$, for various values of the electron number density, n_e, and the electron temperature, T.

TABLE 1

VALUES OF $\ln \Lambda$, FOR $Z = 1$

T (°K)	ELECTRON DENSITY n_e (cm^{-3})								
	1	10^3	10^6	10^9	10^{12}	10^{15}	10^{18}	10^{21}	10^{24}
10^2	16.3	12.8	9.43	5.97					
10^3	19.7	16.3	12.8	9.43	5.97				
10^4	23.2	19.7	16.3	12.8	9.43	5.97			
10^5	26.7	23.2	19.7	16.3	12.8	9.43	5.97		
10^6	29.7	26.3	22.8	19.3	15.9	12.4	8.96	5.54	
10^7	32.0	28.5	25.1	21.6	18.1	14.7	11.2	7.85	4.39
10^8	43.3	30.9	27.4	24.0	20.5	17.0	13.6	10.1	6.69

20.1 - For a differential scattering cross section with an angular dependence given by

$$\sigma(\chi) = \frac{\sigma_o}{2} (3 \cos^2 \chi - 1)$$

where σ_o is a constant, calculate the total cross section and the momentum transfer cross section.

20.2 - Consider a collision between two particles of mass m and m_1, in which the particle of mass m_1 is initially at rest. Denote the scattering angle in the *center of mass* coordinate system by χ, and in the *laboratory* coordinate system (as seen by an observer at rest) by χ_L.

(a) Show that

$$\tan \chi_L = \frac{\sin \chi}{\cos \chi + m/m_1}$$

(b) Show that the relationship between the differential scattering cross section in the laboratory system, $\sigma_L(\chi_L)$, and in the center of mass coordinate system, $\sigma(\chi)$, is given by

$$\sigma_L(\chi_L) = \sigma(\chi) \frac{\left[1 + 2(m/m_1) \cos \chi + (m/m_1)^2 \right]^{3/2}}{\left[1 + (m/m_1) \cos \chi \right]}$$

Note that when $m_1 = \infty$ we have $\chi_L = \chi$ and $\sigma_L(\chi_L) = \sigma(\chi)$.

(c) Verify that when $m = m_1$ we obtain $\chi_L = \chi/2$ and

$$\sigma_L(\chi_L) = 4 \cos (\chi/2) \sigma(\chi).$$

20.3 - Consider two particles whose interaction is governed by the following retangular-well potential,

$$\phi(r) = 0 \quad \text{for} \quad r > a$$

$$\phi(r) = -\phi_0 \quad \text{for} \quad r \leq a$$

(a) Calculate the differential scattering cross section $\sigma(\chi)$, and show that it is given by (for $b < a$)

$$\sigma(\chi) = \frac{p^2 a^2 \left[p \cos (\chi/2) - 1 \right] \left[p - \cos (\chi/2) \right]}{4 \cos (\chi/2) \left[1 - 2p \cos(\chi/2) + p^2 \right]^2}$$

where

$$p = (1 + 2\phi_0/\mu g^2)^{1/2}$$

(b) Show that the total scattering cross section is given by

$$\sigma_t = 2\pi \int_0^a b \, db = \pi a^2$$

20.4 - Consider a general inverse-power interpaticle force of the form

$$F(r) = \frac{K}{r^p}$$

where K is a constant and p is a positive integer number.

(a) Determine expressions for the scattering angle χ, the differential scattering cross section $\sigma(\chi, \epsilon)$, the total scattering cross section σ_t and the momentum transfer cross section σ_m.

(b) Calculate χ, $\sigma(\chi, \epsilon), \sigma_t$ and σ_m for the case of Maxwell molecules, for which p = 5.

20.5 - From the expression for σ_m, obtained in part (a) of Problem 20.4, verify that for p = 2 the momentum transfer cross section is given by

$$\sigma_m = 2\pi \left(\frac{K}{\mu} \right)^2 A_1(2) \frac{1}{g^4}$$

where $A_1(2)$ is given by (with $\ell = 1$ and p = 2)

$$A_\ell(p) = \int_0^\infty (1 - \cos^\ell \chi) \, v_o \, dv_o$$

$$v_o = b \left(\frac{\mu g^2}{K} \right)^{1/(p-1)}$$

Consequently, the velocity-dependent collision frequency, defined by

$$\nu_r(g) = n \sigma_m g$$

varies as g^{-3}. This inverse dependence on g accounts for the *electron runaway effect*. (In the presence of a sufficiently large electric field $\underline{E}$, some electrons will gain enough energy between collisions so as to decrease their cross section and collision frequency, which in turn allow them to pick up more energy from the field and decrease their cross section and collision frequency even further. If $\underline{E}$ is large enough, the collision frequency will fall so fast that these electrons will form an accelerated beam of runaway electrons).

20.6 - Show that for the case of Coulomb interactions (p = 2) we have

$$A_1(2) = \left(\frac{\mu g^2}{K} \right)^2 b_o^2 \, \ell n \, (1 + \Lambda^2)$$

$$A_2(2) = \left(\frac{\mu g^2}{K} \right)^2 2 b_o^2 \left[\ell n \, (1 + \Lambda^2) - \frac{\Lambda^2}{(1 + \Lambda^2)} \right]$$

where $b_o = q q_1 / (4 \pi \epsilon_o \mu g^2)$, $A_\ell(p)$ is as defined in Problem 20.5, and $\Lambda = \lambda_D / b_o$. For $\Lambda \gg 1$ verify that

$$A_1(2) = \left[\frac{\mu g^2}{K} \right]^2 2 b_o^2 \ln \Lambda = 2 \ln \Lambda$$

$$A_2(2) = \left[\frac{\mu g^2}{K} \right]^2 2 b_o^2 (2 \ln \Lambda - 1) = 2 (2 \ln \Lambda - 1)$$

Note that, since $K = qq_1/(4 \pi \varepsilon_o)$ for Coulomb interactions, we have $(\mu g^2/K)^2 b_o^2 = 1$.