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Interchange Instabilities in Lightning Discharges

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Triggered lightning experiments have been recently carried out in the International Center for Triggered and Natural Lightning in Cachoeira Paulista, SP, Brazil. Small rockets carrying a thin copper wire connected to a launching platform artificially triggered nine flashes. This setup is similar to previous experiments[1] , but a fast CCD camera was used for the first time to obtain detailed images during the continuous current phase (pause) of lightning discharges. Figure 1 shows a sequence of pictures (taken at 1000 frames per second) that starts with the first stroke of a series of two subsequent return strokes and proceeds with the slow cooling stage of the discharge channel, which is maintained by a continuous current that flows during the pause between successive strokes. The peak measured current during this triggered lightning is 45 kA and the visible channel has an estimated radius of 0.5 m. The last frame shows the start of the second return stroke, which ends on the tip of a Franklin lightning rod that can be seen on the right side of the sequence of pictures. The beaded appearance usually registered during the decay of triggered lightning discharges is clearly seen in this sequence.

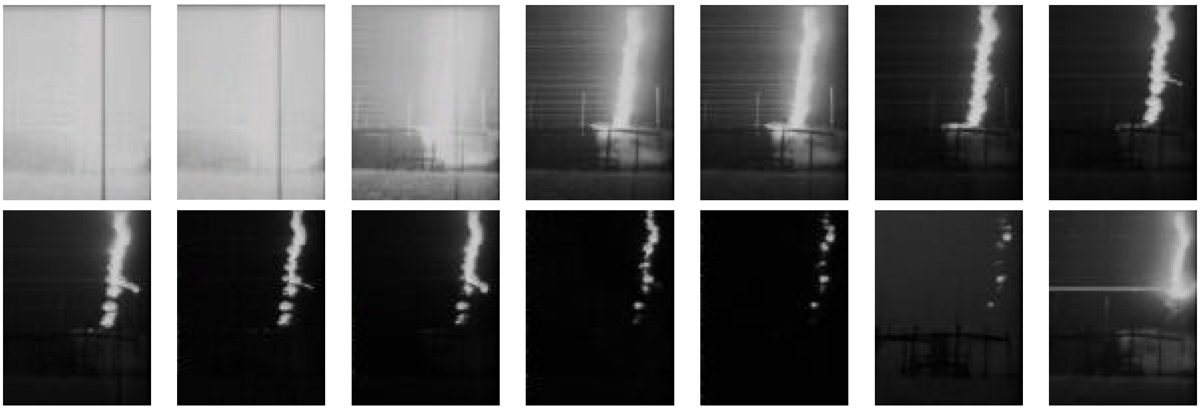

Figure 1: Beaded structure in a lightning discharge shown at 1 ms time intervals.

Typically, the current in the main return stroke attains the median value of 30-40 kA in 1-3 μs and decays with a time constant 30-60 μs to a continuous current value of the order of 100 A during pauses between successive strokes[2] . Figure 2 illustrates the current waveform $I(t)$ assumed for the discharge shown in Fig. 1, according with standard models, at two time scales, 0-200 μs and 200-1000 μs.

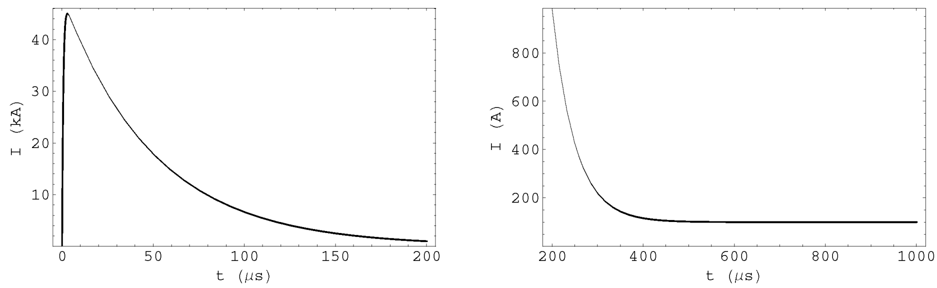

Figure 2: Current waveform for a lightning discharge with the peak value 45 kA.

	$t = 5\,\mu$s	$t = 300\,\mu$s	$t = 1\,$ms
I (A)	>30,000	$\gtrsim$100	$\cong$100
T (K)	40,000	10,000	8,000
p (atm)	>15	$\lesssim$1	$\cong$1
a (cm)	<1	9	8
	expansion	**contraction**	

Table 1: Typical parameters in the evolution of atmospheric discharges.

Table 1 gives succinct results of a numerical simulation[3] that correspond to typical parameters in the evolution of atmospheric discharges. After the initial explosive expansion, the simulation shows a slow contraction of the channel boundary of radius a during the stage of formation of the continuous current channel. This situation is favorable to the occurrence of the Rayleigh-Taylor instability (the pressure and density gradients are oppositely directed) with turbulent fluctuations concentrated in a thin layer at the channel boundary, where the density gradient is largest[3] .

The purpose of the present paper is to investigate the role of interchange (sausage and kink) instabilities in the formation of the beaded structure of lightning discharges. It will be shown that the periodicity in space of the structure can be explained in terms of interchange instabilities in a cylindrical discharge with anomalous viscosity. The viscosity contributes with a surface tension that generates the beaded configuration.

Considering the discharge as a perfectly conductive viscous fluid the single-fluid equations of motion are written as follows:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\left(\rho\overrightarrow{u}\right) = 0 \qquad \text{mass conservation}$$

$$\rho\frac{d\overrightarrow{u}}{dt} = -\nabla p + \overrightarrow{j}\times\overrightarrow{B} + \eta\nabla^2\overrightarrow{u} + \frac{1}{3}\eta\nabla\left(\nabla\cdot\overrightarrow{u}\right) \qquad \text{Navier-Stokes equation}$$

$$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0 \qquad \text{equation of state}$$

$$\nabla\times\overrightarrow{B} = \mu_0\overrightarrow{j} \qquad \text{Ampère's law}$$

$$\frac{\partial\overrightarrow{B}}{\partial t} = \nabla\times\left(\overrightarrow{u}\times\overrightarrow{B}\right) \qquad \text{Faraday and Ohm's law } (\sigma\to\infty)$$

The stability analysis is carried out for the simplest equilibrium where the inside of the channel is both current and magnetic field free. The external azimuthal magnetic field B_θ of the pinch is produced by a surface current $K = I/\left(2\pi a\right)$ flowing in the z direction. The equilibrium profiles are

$$B_\theta = \frac{\mu_0 I}{2\pi r}U\left(r-a\right) \qquad j_z = \frac{I}{2\pi a}\delta\left(r-a\right) \qquad p = p_e + \frac{\mu_0 I}{8\pi^2 a^2}\left[1 - U\left(r-a\right)\right]$$

where p_e is the external gas pressure, and $U\left(r\right)$ and $\delta\left(r\right)$ are the unit step and Dirac delta functions, respectively.

For a small perturbation in the fluid velocity, given in terms of the Lagrangian displacement $\overrightarrow{\xi}$ by $\delta\overrightarrow{u} = \partial\overrightarrow{\xi}/\partial t = i\omega\overrightarrow{\xi}$, the linearized equation of motion becomes

$$\begin{aligned}-\rho\omega^2\overrightarrow{\xi} &= \nabla\left(\overrightarrow{\xi}\cdot\nabla p + \gamma p\nabla\cdot\overrightarrow{\xi}\right) + \frac{1}{\mu_0}\left(\nabla\times\overrightarrow{Q}\right)\times\overrightarrow{B} + \frac{1}{\mu_0}\left(\nabla\times\overrightarrow{B}\right)\times\overrightarrow{Q} \\ &\quad + i\omega\eta\left[\frac{4}{3}\nabla\left(\nabla\cdot\overrightarrow{\xi}\right) - \nabla\times\left(\nabla\times\overrightarrow{\xi}\right)\right]\end{aligned}$$

where $\overrightarrow{Q} = \nabla\times\left(\overrightarrow{\xi}\times\overrightarrow{B}\right)$. The air outside the discharge is neutral and can be assumed inviscid for the ambient temperature. The magnetic field perturbation outside is solenoidal and irrotational, and can be derived from a scalar potential. The air inside is current and magnetic field free for the assumed equilibrium (in any case, the Alfvén velocity is much smaller than the sound speed and the magnetic perturbations can be neglected inside the ionized channel). Viscosity effects are relevant inside the discharge because of the high temperature and the transient turbulent fluctuations.

If $\widehat{n}$ denotes the unit normal to the interface and $\langle X \rangle$ the increment of X across the interface, the boundary conditions are:

$$\widehat{n} \cdot \overrightarrow{B} = 0 \qquad \text{required in the presence of } \overrightarrow{K}$$

$$\widehat{n} \cdot \langle \overrightarrow{u} \rangle = 0 \qquad \text{fluid continuity}$$

$$\widehat{n} \times \left\langle \eta \left(\nabla \overrightarrow{u} \right) \cdot \widehat{n} + \frac{\eta}{2} \left(\nabla \times \overrightarrow{u} \right) \times \widehat{n} \right\rangle = 0 \qquad \text{stress continuity}$$

$$\left\langle p - 2\eta \widehat{n} \cdot \left(\nabla \overrightarrow{u} \right) \cdot \widehat{n} + \frac{2}{3} \eta \nabla \cdot \overrightarrow{u} + \frac{B^2}{2\mu_0} \right\rangle = 0 \qquad \text{pressure balance}$$

Taking into account the perturbation of the normal vector

$$\delta \widehat{n} = - \left(\nabla \overrightarrow{\xi} \right) \cdot \widehat{n} + \widehat{n} \left[\widehat{n} \cdot \left(\nabla \overrightarrow{\xi} \right) \cdot \widehat{n} \right]$$

the pressure balance condition for the perturbations becomes

$$\left\langle - \left(\gamma p - \frac{2}{3} i \omega \eta \right) \nabla \cdot \overrightarrow{\xi} - 2 i \omega \eta \widehat{n} \cdot \left(\nabla \overrightarrow{\xi} \right) \cdot \widehat{n} + \frac{\overrightarrow{B} \cdot \delta \overrightarrow{B}}{\mu_0} + \left(\overrightarrow{\xi} \cdot \nabla \right) \frac{B^2}{2\mu_0} \right\rangle = 0$$

Now, to find the normal modes one must solve a boundary layer problem (Prandtl, 1905) by the method of matched asymptotic expansions in the inverse Reynolds number $\eta_i / \left(\rho_i c_i a \right)$, where η_i, ρ_i and c_i denote the viscosity, mass density and sound speed inside the discharge, respectively. However, for low-frequency perturbations (small growth rate of the instability) one can neglect terms of order $\eta_i^1 \omega^2$ with respect to $\eta_i^0 \omega^2$ and $\eta_i^1 \omega^1$ (vorticity effects and small corrections to the sound speed are neglected in this approximation), and show that the pressure balance equation can be satisfied by substituting the internal inviscid solution $\overrightarrow{\xi}_{i,0}$ in the above expression, yielding

$$-\gamma_i p_i \nabla \cdot \overrightarrow{\xi}_{i,0} - 2 i \omega \eta_i \widehat{n} \cdot \left(\nabla \overrightarrow{\xi}_{i,0} \right) \cdot \widehat{n} \cong -\gamma_e p_e \nabla \cdot \overrightarrow{\xi}_e + \frac{\overrightarrow{B} \cdot \delta \overrightarrow{B}}{\mu_0} + \left(\overrightarrow{\xi}_e \cdot \nabla \right) \frac{B^2}{2\mu_0}$$

which is calculated at the interface ($\overrightarrow{\xi}_e$ denotes the perturbation in the external fluid). It follows that the dispersion relation to lowest order in η_i is

$$\left(\frac{I_m \left(\kappa_i a \right)}{\kappa_i a I_m' \left(\kappa_i a \right)} - \frac{\left(\rho_e / \rho_i \right) K_m \left(\kappa_e a \right)}{\kappa_e a K_m' \left(\kappa_e a \right)} \right) \omega^2 - \frac{2 \eta_i \kappa_i I_m'' \left(\kappa_i a \right)}{\rho_i a I_m' \left(\kappa_i a \right)} i \omega + \left(1 + \frac{m^2 K_m (ka)}{ka K_m' (ka)} \right) \frac{v_A^2}{a^2} = 0$$

where $\kappa_i = \sqrt{k^2 - \omega^2 / c_i^2}$, $\kappa_e = \sqrt{k^2 - \omega^2 / c_e^2}$, m and k are the azimuthal and axial wave numbers, respectively, $v_A = \sqrt{B_\theta^2 (a) / \left(\mu_0 \rho_i \right)}$ is the Alfvén speed at the edge of the linear pinch, and I_m and K_m are the modified Bessel functions (ρ_e and $c_e = \sqrt{\gamma_e p_e / \rho_e}$ denote the mass density and sound speed in the external air, respectively).

For small values of ω the dispersion relation is essentially a second order equation. The first order term in ω, proportional to the viscosity, reduces the growth rate of the instability at short wavelengths and characterizes the spatial structure of the perturbed discharge. It is interesting to note that the viscosity of air at atmospheric pressure attains a maximum value $\eta_i \cong 2.1 \times 10^{-4}\,\mathrm{Pa}$ near $10{,}000\,\mathrm{K}$. However, this maximum value is insufficient to explain the structure observed in Fig. 1. It can be explained if one takes into account the anomalous character of the viscosity, associated with the transient turbulent fluctuations. In the following paragraph the magnitude of the effective viscosity coefficient $\eta_{turb} = \rho \nu_{turb}$ is estimated.

Using either dimensional analysis or mixing length arguments, the order of magnitude of the kinematic viscosity for turbulent motion characterized by the mass density ρ, the large scale velocity fluctuations Δu and the scale length ℓ is[4]

$$\nu_{turb} \sim \Delta u \ell$$

In terms of the Reynold's number $R = \Delta u \ell / \nu$ one has $\nu_{turb} \sim R\nu$. Now, the growth rate of the Rayleigh-Taylor instability for short wavelengths is given by $\gamma = \sqrt{g/L_\rho}$, where g is the "gravitational" acceleration and $L_\rho = \rho / |\nabla \rho|$ is the density scale-length. One can estimate $\ell \sim L_\rho$ and $\Delta u \sim \alpha c_i$, where $c_i = \sqrt{\gamma_i R_0 T_i / M_i}$ is the sound speed ($R_0 = 8.314 \times 10^3\,\mathrm{J\,K^{-1}\,Mole^{-1}}$, $\gamma_i \cong 1.4$, $M_i = 28.96\,\mathrm{kg\,Mole^{-1}}$) and α a numerical coefficient of the order of unity, resulting in a value of the Reynold's number $R = \alpha c_i L_\rho / \nu_i$ (possibly not fully developed turbulence). For a discharge at atmospheric pressure ($p_i \cong p_e = 1.013 \times 10^5\,\mathrm{Pa}$, $\rho_i = M_i p_i / (R_0 T_i)$) with internal temperature $T_i = 10,000\,\mathrm{K}$ one obtains $R \cong 3.4 \times 10^5 \alpha L_\rho$. From the simulation results[3] one can take $L_\rho / a \sim 10^{-2}$ for a channel radius $a \cong 9\,\mathrm{cm}$ at the beginning of the contraction stage, and expect a two orders of magnitude increase in the effective viscosity compared to the molecular viscosity.

Figure 4 shows the normalized growth rate $i\omega a / c_i$ versus the normalized wave number ka of the $m = 0$ (continuous line) and $m = 1$ (dashed line) sausage and kink instabilities, respectively (higher order modes are stable), for a discharge with the same conditions as above and assuming $\nu_{turb} \cong 100\nu_i$. On the left-hand side plot it is assumed that $I = 1\,\mathrm{kA}$ and on the right-hand side that $I = 100\,\mathrm{A}$ (approximate values of the current in the time interval $200\mu\mathrm{s} \lesssim t \lesssim 400\,\mu\mathrm{s}$). The characteristic growth time of the interchange instabilities is $\tau \cong 45\,\mathrm{ms}$ for a wavelength of about 25 cm on the left-hand side plot, and $\tau \cong 1.4\,\mathrm{s}$ for a wavelength of 1 m on the right-hand side.

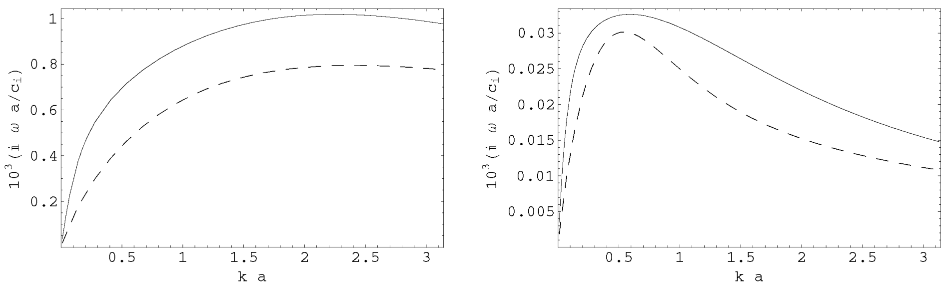

Figure 4: Growth rates of the sausage (continuous) and kink (dashed) instabilities.

A close examination of the beaded structure shown in Fig. 1 indicates that the wavelengths calculated above, where the instabilities are stronger, lie in the correct range to explain the structure. The calculated growth rates are very small, which is consistent with the frozen character of the instability during the pause between successive strokes. However, a much faster growth is needed to explain the appearance of the beaded structure during the short contraction stage (i.e. from $t \cong 300\,\mu\mathrm{s}$ to 1 ms). The problem with the model above is the incorrect representation of the discharge quasi-equilibrium during this stage. It would be necessary to include a transient radial "gravitational" acceleration (the same that is responsible for the short wavelength Rayleigh-Taylor instability) to account for the effect of the contraction. This will be done in a forthcoming paper.

[1] Fieux, R., Gary, C. and Hubert, P., *Nature*, **257**, 212 (1975).

[2] Uman, M. A., *The Lightning Discharge* (Academic, New York, 1987).

[3] Aleksandrov, N. L., Bazelyan, E. M. and Shneider, M. N., *Plasma Physics Reports*, **26**, 893 (2000).

[4] Landau, L. et Lifchitz, E., *Mécanique des Fluids* (MIR, Moscou, 1971).